GEOPHYSICS, VOL. 59, NO. 8 (AUGUST 1994); P. 1290–1304, 12 FIGS.



# Nonhyperbolic reflection moveout in anisotropic media

Ilya Tsvankin* and Leon Thomsen‡

## ABSTRACT

The standard hyperbolic approximation for reflection moveouts in layered media is accurate only for relatively short spreads, even if the layers are isotropic. Velocity anisotropy may significantly enhance deviations from hyperbolic moveout. Nonhyperbolic analysis in anisotropic media is also important because conventional hyperbolic moveout processing on short spreads is insufficient to recover the true vertical velocity (hence the depth).

We present analytic and numerical analysis of the combined influence of vertical transverse isotropy and layering on long-spread reflection moveouts. Qualitative description of nonhyperbolic moveout on "intermediate" spreads (offset-to-depth ratio $x/z < 1.7$–$2$) is given in terms of the exact fourth-order Taylor series expansion for $P$, $SV$, and $P$-$SV$ traveltime curves, valid for multilayered transversely isotropic media with arbitrary strength of anisotropy. We use this expansion to provide an analytic explanation for deviations from hyperbolic moveout, such as the strongly nonhyperbolic $SV$-moveout observed numerically in the case where $\delta > \varepsilon$. With this expansion, we also show that the weak anisotropy approximation becomes inadequate (to describe nonhyperbolic moveout) for surprisingly small values of the anisotropies $\delta$ and $\varepsilon$.

However, the fourth-order Taylor series rapidly loses numerical accuracy with increasing offset. We suggest a new, more general analytical approximation, and test it against several transversely isotropic models. For $P$-waves, this moveout equation remains numerically accurate even for substantial anisotropy and large offsets. This approximation provides a fast and effective way to estimate the behavior of long-spread moveouts for layered anisotropic models.

## INTRODUCTION

Numerous investigations during the past decade have proven the presence of seismic anisotropy in different geological settings and on various scales. While conventional processing of reflection $P$-wave data is still based on the assumption of isotropy, there is a growing understanding that anisotropy may seriously affect the results of most basic processing and interpretation steps, such as normal moveout (NMO) correction, velocity analysis, migration, dip moveout (DMO) removal, and amplitude-variation with-offset (AVO) analysis (Banik, 1984; Thomsen, 1986; Winterstein, 1986; Wright, 1987; Gonzalez and Lynn, 1991; Larner, 1993; Larner and Cohen, 1993). In general, any process that involves the concept of a scalar velocity field is subject to error if, in fact, the actual velocity is a vector whose magnitude depends upon its direction.

The shape of the moveout curves for reflected waves is of primary importance to most processing and interpretation algorithms. Reflection moveout curves are conventionally approximated by the hyperbolic equation:

$$t^2 \approx t_H^2 \equiv t_v^2 + \frac{x^2}{V_{mo}^2}, \tag{1}$$

where $t_v$ is the approximate vertical (zero-offset) arrival time, $x$ is the source-receiver offset, and $V_{mo}$ is called the "moveout velocity." To invert reflection data for vertical velocities, $V_{mo}$ is often identified with the root-mean-square (rms) velocity $V_{\mathrm{rms}}$ (Taner and Koehler, 1969). This assumption (justified only for short spreads and isotropic, horizon-

Presented at the 62nd Annual International Meeting, Society of Exploration Geophysicists. Manuscript received by the Editor June 4, 1993; revised manuscript received January 14, 1994.
*Formerly Amoco Production Research, Tulsa, OK 74102; presently Department of Geophysics, Colorado School of Mines, Golden, CO 80401-1887.
‡Amoco Production Research, P.O. Box 3385, Tulsa, OK 74102.

tally layered models) makes it possible to recover interval velocities via the Dix (1955) formula. The concept of hyperbolic moveout (and departures from it) becomes unnecessary if prestack depth migration is used. However, the present results form a useful guide for understanding results of anisotropic prestack migration (Gonzalez and Lynn, 1991).

The hyperbolic moveout equation (1) is strictly valid only for a homogeneous isotropic (or elliptically anisotropic) plane layer. This paper is concerned with refinements to this equation caused by multiple layers, or realistic anisotropy, or both.

The presence of anisotropy causes two principal distortions of reflection moveouts. First, the short-spread moveout velocity in the presence of anisotropy is not equal to the rms vertical velocity, even for horizontal layers (Lyakhovitsky and Nevsky, 1971; Thomsen, 1986). The difference between vertical rms and moveout velocities, ignored by conventional techniques, may lead to unacceptable errors in interval velocities and in time-to-depth conversion, even for weak anisotropy (Banik, 1984; Winterstein, 1986). In anisotropic media, it is impossible to obtain the true vertical velocity from the short-spread moveout velocity alone. Recovery of the true vertical velocity from surface data requires, at a minimum, analysis of long-spread (nonhyperbolic) moveouts.

Second, anisotropy leads to nonhyperbolic moveout, even in a homogeneous layer. If not properly corrected for, nonhyperbolic moveout causes distortions in velocity estimation and deteriorates the quality of stacked sections. To determine whether stacking velocities really represent short-spread moveout velocities (which are useful in analysis), it is necessary to understand the character of deviations from hyperbolic moveout in layered anisotropic media.

Most existing work on $P$-wave reflection moveout in anisotropic media has been done for transversely isotropic media, usually with a vertical symmetry axis. This is the simplest common type of anisotropy, and has been frequently observed in both land and marine environments (White et al., 1983; Helbig, 1984; Berge et al., 1991). Among the causes of vertical transverse isotropy are the interleaving of thin horizontal layers, intrinsic anisotropy of preferentially aligned rock-forming minerals, and certain populations of cracks. All formulations in this paper assume transverse anisotropy with a vertical symmetry axis, but are independent of the physical reasons for the anisotropy. For brevity, we will omit the qualifiers in "quasi-$P$-wave" and "quasi-$SV$-wave."

Different aspects of the influence of transverse isotropy on short-spread moveout velocities were studied analytically and numerically by Krey and Helbig (1956), Lyakhovitsky and Nevsky (1971), Levin (1978, 1979, 1989), Thomsen (1986), Seriff and Sriram (1991), among others. Considerably less attention has been devoted to nonhyperbolic (long-spread) moveout for anisotropic models. Radovich and Levin (1982) pointed out that transverse isotropy leads to spread-length-dependent moveout velocity, and introduced the concept of "instantaneous" moveout velocity at different incidence angles. Hake et al. (1984) suggested an approach based on the three-term (quartic) Taylor series expansion of $t^2 - x^2$ curves. Berge (1991) attempted to tie the degree of nonhyperbolic moveout for the $SV$-wave to the curvature of the wavefront near the vertical. Byun et al. (1989) and Byun and Corrigan (1990) suggested a "skewed" hyperbolic formula for long-spread $P$-wave moveout in weakly anisotropic media.

Here we elucidate the dependence of long-spread moveout on the parameters of transversely isotropic media. Generalizing the results by Hake et al. (1984), we first derive the exact quartic Taylor series coefficient for $P$-$SV$ moveout and find the quartic coefficients for the $P$ and $SV$ reflections as special cases. Comparison with formulas obtained in the limit of small anisotropy reveals severe limitations of the weak anisotropy approximation in the description of nonhyperbolic moveout.

The exact Taylor series coefficients are used to give a qualitative description of nonhyperbolic moveout for $P$- and $SV$-waves, and to build a more general analytic approximation for long-spread reflection moveouts, which is numerically accurate even for substantial anisotropy. Although the current treatment is restricted to transversely isotropic media with a vertical symmetry axis, generalization for symmetry planes in azimuthally anisotropic media is straightforward. Implications of these results for the inversion of reflection traveltimes are discussed in a sequel paper (Tsvankin and Thomsen, Inversion of reflection traveltimes for transverse isotropy: submitted to Geophysics).

## SHORT-SPREAD REFLECTION MOVEOUT

In this section, we review the behavior of short-spread moveout and the relation between moveout velocities and parameters of anisotropy. A receiver spread is considered to be short if it does not exceed the reflector depth. To analyze the influence of anisotropy alone on moveout, we consider in detail the case of a single, horizontal, transversely isotropic layer, and generalize later to multiple layers. The model is characterized by the $P$- and $S$-wave vertical velocities ($V_{p0}$ and $V_{s0}$) and three dimensionless parameters ($\varepsilon$, $\delta$, and $\gamma$) introduced by Thomsen (1986).

The most conventional measure of anisotropy ($\varepsilon$) expresses the fractional difference between vertical $[V_p(0)]$ and horizontal $[V_p(90)]$ $P$-wave velocities. It is defined through elastic moduli $C_{\alpha\beta}$, or equivalently through the velocities, as

$$\varepsilon = \frac{C_{11} - C_{33}}{2C_{33}} = \frac{V_p^2(90) - V_p^2(0)}{2V_p^2(0)}. \tag{2}$$

The equivalent measure for the $SH$-wave is

$$\gamma = \frac{C_{66} - C_{44}}{2C_{44}} = \frac{V_{SH}^2(90) - V_{SH}^2(0)}{2V_{SH}^2(0)}. \tag{3}$$

The so-called "strange" anisotropy coefficient $\delta$ is defined as

$$\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})}. \tag{4}$$

The parameter $\delta$ influences velocities of $P$- and $SV$-waves, especially the $P$-wave velocity at near-vertical incidence. All three coefficients reduce to zero in the absence of anisotropy. $P$-$SV$ propagation is described by four coefficients:

$V_{p0}$, $V_{s0}$, $\varepsilon$, and $\delta$, while the $SH$-wave velocity depends on $V_{s0}$ and $\gamma$. Although the parameters $\varepsilon$, $\delta$, and $\gamma$ were originally introduced to simplify velocity equations for weakly anisotropic media, we will show that they also facilitate moveout analysis for transversely isotropic models with arbitrary strength of anisotropy.

The most straightforward approximation for reflection moveouts is the Taylor series expansion of the $t^2(x^2)$ curve near $x^2 = 0$ (Taner and Koehler, 1969):

$$t_T^2 \equiv A_0 + A_2 x^2 + A_4 x^4 + \ldots, \tag{5}$$

$$A_0 = t_0^2, \quad A_2 = \left.\frac{dt^2}{dx^2}\right|_{x=0}, \quad A_4 = \frac{1}{2}\frac{d}{dx^2}\left.\left(\frac{dt^2}{dx^2}\right)\right|_{x=0},$$

where $t_0$ is the true zero-offset arrival time, generally different from the vertical time $t_v$ of the best-fit hyperbola (1). The short-spread moveout velocity is expressed through $A_2$ as $V_2 = 1/\sqrt{A_2}$. In conventional processing, expansion (5) is truncated after the second term, and it is *assumed* that the effective stacking (moveout) velocity $V_{mo}$ [equation (1)] is equal to its short-spread limit $V_2$. The values of $V_2$ for the $P$-, $SV$-, and $SH$-waves reflected from the bottom of a horizontal transversely isotropic layer are (Thomsen, 1986; see also Appendix B):

$$V_2^2(P) = V_{p0}^2(1 + 2\delta), \tag{6}$$

$$V_2^2(SV) = V_{s0}^2(1 + 2\sigma), \tag{7}$$

$$V_2^2(SH) = V_{s0}^2(1 + 2\gamma), \tag{8}$$

where

$$\sigma = \left(\frac{V_{p0}}{V_{s0}}\right)^2 (\varepsilon - \delta) \tag{9}$$

is a useful notation that frequently eliminates the need for the (more intuitive) quantity $\varepsilon$. The anisotrophic parameter $\sigma$ equals zero for elliptical anisotropy ($\varepsilon = \delta$), and in this sense is similar to such parameters, introduced earlier, as $E^2$ (Hake et al., 1984) and $\varepsilon_s$ (Carrion et al., 1992). It is also the most influential parameter in the $SV$-wave velocity [equation (A-10)] and moveout equations. Note that equations (6–8) are valid for transverse isotopy with arbitrary degree of anisotropy.

Further discussion will be focused mostly on $P$ and $SV$ moveouts. The $SH$-wave, governed by the anisotropic parameter $\gamma$, is completely decoupled from the $P$- and $SV$-waves, whose velocity variations are governed by the parameters $\delta$ and $\sigma$ (or $\delta$ and $\varepsilon$), and the ratio $V_{p0}/V_{s0}$. In a homogeneous transversely isotropic layer, the wavefront of the $SH$-wave is always elliptical, and the $SH$-moveout is purely hyperbolic, with $V_{mo}$ equal to the short-spread value $V_2$ [equation (8)] which, in turn, is equal to the horizontal velocity (Hake et al., 1984). Reflection moveout for the $SH$-wave becomes nonhyperbolic only in stratified media.

For all three waves, the short-spread moveout velocity given by equations (6)–(8) is generally different from the true vertical velocity. Winterstein (1986) *assumed* that for the $P$-wave, $V_2$ is typically close to the vertical velocity (e.g., that $\delta$ is very small) to estimate $\gamma$. However, Banik (1984) found significant differences between the $P$-wave vertical rms and moveout velocities in North Sea shales (effectively estimating $\delta$ of 10 percent or more). If transverse isotropy is a result of periodic interleaving of isotropic layers, which are thin compared to the predominant wavelength, $\delta$ is null only if the velocity ratio $V_p/V_s$ is constant across all layers. There is no doubt that, in many contexts, $\delta$ cannot be ignored.

The $SV$-wave short-spread moveout velocity is determined by the parameter $\sigma$ [equations (7) and (9)], which may be much bigger than the anisotropies $\varepsilon$, $\delta$, or $\gamma$ because of the presence of the squared velocity ratio. Consequently, the short-spread moveout velocity for the $SV$-wave may be more significantly distorted by anisotropy than that for the $P$-wave. Even if $\varepsilon - \delta$ is small (say, of the order of 0.1), $\sigma$ may be of the order of 0.4, implying a difference between $V_2$ $(SV)$ and $V_{s0}$ of 30–40 percent. In terms of phase velocities, this means that the $SV$-wave velocity variations near the vertical [determined by $\sigma$; see equation (A-10)] usually are more pronounced than are those for the $P$-wave.

The $P$ and $SV$ horizontal velocities are:

$$V_h(P) = V_{p0}\sqrt{1 + 2\varepsilon}, \tag{10}$$

$$V_h(SV) = V_{s0}. \tag{11}$$

Comparing equations (6) and (7) with (10) and (11), it is clear that the only transversely isotropic model for which the moveout velocity is equal to the horizontal velocity is elliptical anisotropy ($\varepsilon = \delta$). In this special case, the wavefront is spherical for the $SV$-wave, elliptical for the $P$-wave, and all moveouts are strictly hyperbolic (Levin, 1978). Although the elliptical approximation leads to significant simplifications in all equations of wave propagation, it is hardly more than a mathematical abstraction. According to existing data on crustal rocks (Thomsen, 1986), $\varepsilon$ and $\delta$ almost always differ, with $\varepsilon > \delta$ in most cases, implying that $\sigma > 0$.

Most reflection surveys do not contain offsets long enough to adequately determine the horizontal velocity $V_h$ (reliable estimation of $V_h$ requires incidence angles well beyond 45 degrees). Hence, where $P$-wave time-depth mis-ties are interpreted (Banik, 1984) in terms of horizontal velocities of elliptic media (i.e., in terms of $\varepsilon$), a more conservative interpretation would be to relax the elliptical assumption and conclude that the data determine $\delta$, not $\varepsilon$.

In the absence of anisotropy, reflection moveout for a horizontal homogeneous layer would be hyperbolic, with $V_2$ coinciding with the true velocity in the layer. Although $P$- and $SV$-wave moveout in a transversely isotropic layer is generally nonhyperbolic, it usually remains close to hyperbolic for short spreads (limited by the depth of the boundary, $x_{max}/z \leq 1$). The behavior of $P$- and $SV$-wave moveout is illustrated in Figure 1 for the model of Taylor (no relation to the mathematician) sandstone (Figure 2), taken from Thomsen (1986). The residual moveout in Figure 1 is calculated as the difference between the exact traveltime curve and the best-fit hyperbola (1), found by the least-squares method.

For this model, the residuals on short spreads are small (<2 ms), for both the $P$- and $SV$-wave. Even so, the effective velocity $V_{mo}$ for the best-fit hyperbolas (1) in Figure 1 is not the same as the short-spread velocity $V_2$. This difference is a consequence of nonhyperbolic moveout caused by anisotropy: the effective velocity, obtained by fitting nonhyperbolic data to a hyperbolic function, depends on spread-length $x_{max}$. For the model of Taylor sandstone, the value of the $P$-wave $V_{mo}$ for the best-fit hyperbola on the spread $x_{max} = z$ is 2.6 percent larger than $V_2$. For several other models from Thomsen's (1986) table, this difference for $x_{max} = z$ is generally limited by 2–3 percent. Therefore, the spread length should be *less* than the depth of the boundary for reflection moveout to be well-described by the hyperbolic equation (1) with $V_{mo}$ close to the short-spread velocity (6)–(8). Typical acquisition design violates this criterion everywhere above the target horizon. We return to this point, in the discussion of longer spread lengths (Figure 4, below).

Note also, in Figure 1, that the vertical (zero-offset) arrival time of the best-fit hyperbola [$t_v$, equation (1)] is different

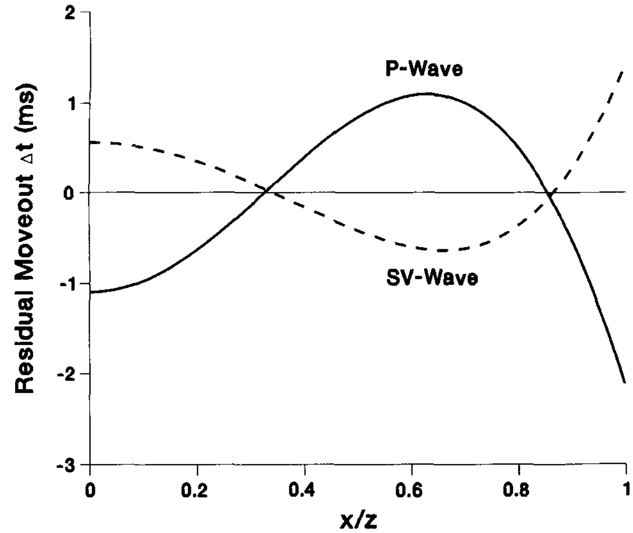


FIG. 1. The difference between the exact traveltimes and best-fit hyperbola $t_H$ (residual moveout) for $P$- and $SV$-reflections in a layer of Taylor sandstone (Figure 2). The spread length is equal to the depth of the reflector, $x_{max} = z = 3$ km.

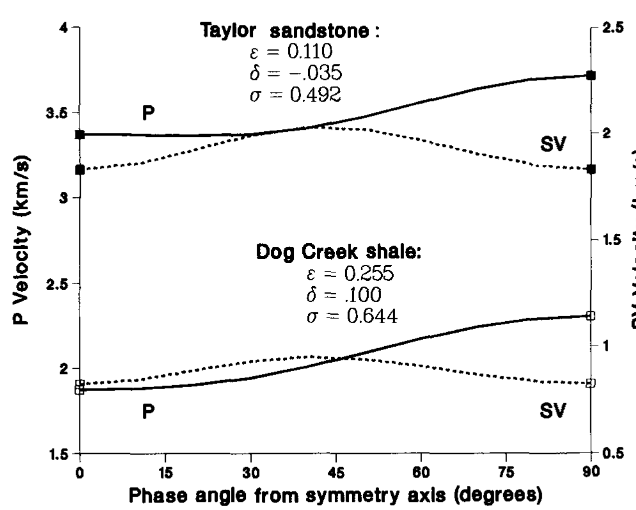


FIG. 2. Phase velocities of $P$- and $SV$-waves for Taylor sandstone, and for Dog Creek shale (from Thomsen, 1986).

from the true vertical time $t_0$. This implies that the apparent arrival time on a typical stacked seismic section differs from the true vertical time.

For a medium with $N$ homogeneous, coarse, constant-velocity layers, the short-spread moveout velocity is given by (Hake et al., 1984; see also Appendix B):

$$V_2^2 = \lim_{x \to 0} \frac{dx^2}{dt^2} = \frac{1}{t_0} \sum_{i=1}^{N} V_{2i}^2 \Delta t_i. \tag{12}$$

Or, using equations (6)–(8)

$$V_2^2 = V_{rms}^2 (1 + 2\zeta), \tag{13}$$

where

$$\zeta = \frac{1}{V_{rms}^2 t_0} \sum_{i=1}^{N} V_{0i}^2 \zeta_i \Delta t_i,$$

with $\zeta_i$ standing for $\delta_i$, $\sigma_i$, or $\gamma_i$ (for the $P$-, $SV$-, or $SH$-wave, respectively) in each layer $i$, and $\Delta t_i$ is the corresponding two-way vertical traveltime. The difference between $V_2$ and $V_{rms}$ is determined by $\zeta$, the weighted average value of the anisotropies $\zeta_i$. If we try to recover interval velocities from equations (12) and (13) by identifying $V_2$ with $V_{rms}$ and applying the Dix (1955) formula, we get the apparent interval velocities $V_{2i} = V_{0i}\sqrt{1 + 2\zeta_i}$, distorted by anisotropy (Appendix C). This happens even though, with the assumed short spread, all rays are close to the vertical. Ignoring this distortion in conventional processing leads to errors in converting reflection time to reflector depth.

The resulting mis-ties can be especially troublesome when drilling a deviated well, so that the drillers are aiming at the wrong target. If the mis-ties vary along a line, they can cause error in the area or degree of trap closure, turning a play into a nonplay, or vice-versa. And, if the erroneous velocities are used for lithology identification, or for pressure prediction, obvious problems arise.

## INTERMEDIATE-SPREAD REFLECTION MOVEOUT

We will generally be interested in spread-lengths $z < x_{max} < 2z$, which are feasible for reflection surveys at the target horizon (although a new approximation, developed in the next section, is applicable to much larger offset-depth ratios). Most examples here are given for Taylor sandstone and/or Dog Creek shale (Figure 2); however, all conclusions have been checked against several other transversely isotropic models from Thomsen (1986). More importantly, the concise analytic formulas developed herein allow one to see intuitively the effects of *any* putative medium.

The nonhyperbolic moveout, due purely to anisotropy, can be observed in a single-layer model (Figure 3). The error of the hyperbolic approximation (1) for both $P$- and $SV$-waves rapidly grows for $x_{max} > z$. The residual moveout after the hyperbolic correction is especially high for the long-spread $SV$-data. Also, the best-fit moveout velocity $V_{mo}$ departs significantly, for finite spread lengths, from the short-spread moveout velocity $V_2$ (Figure 4). We consider single layers first, in the weak anisotropy approximation, then strong anisotropy, and then generalize to many layers.

### A. Weak anisotropy approximation

The easiest way to give an analytic description of traveltime curves at large offsets is to apply the weak anisotropy approximation (WAA). Following Thomsen (1986), we derive the following expression for reflection moveouts of the $P$- and $SV$-wave in a single layer (Appendix A):

$$t^2 = t_0^2 + A_2^w x^2 + \frac{A_4^w x^4}{1 + \left(\frac{x}{V_0 t_0}\right)^2}. \qquad (14)^*$$

*Equations so marked are valid only for weak anisotropy.

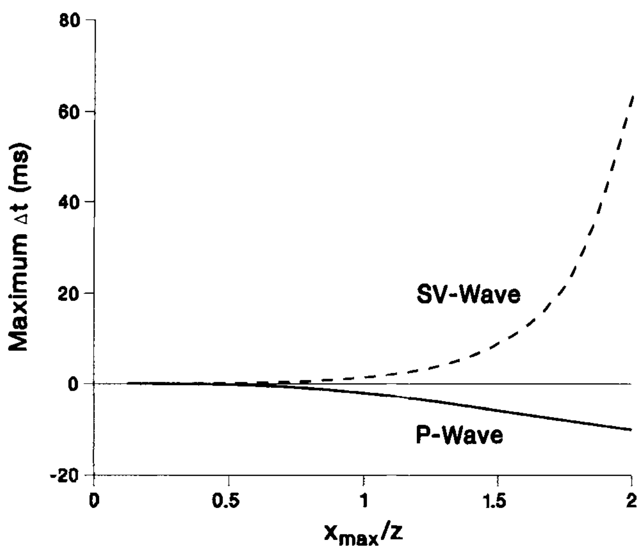


FIG. 3. Maximum difference between the exact traveltimes and the best-fit hyperbola $t_H$ (i.e., maximum residual moveout) as a function of spread-length-to-depth ratio $x_{max}/z$. Model is Taylor sandstone (Figure 2), with $z$ = 3 km.

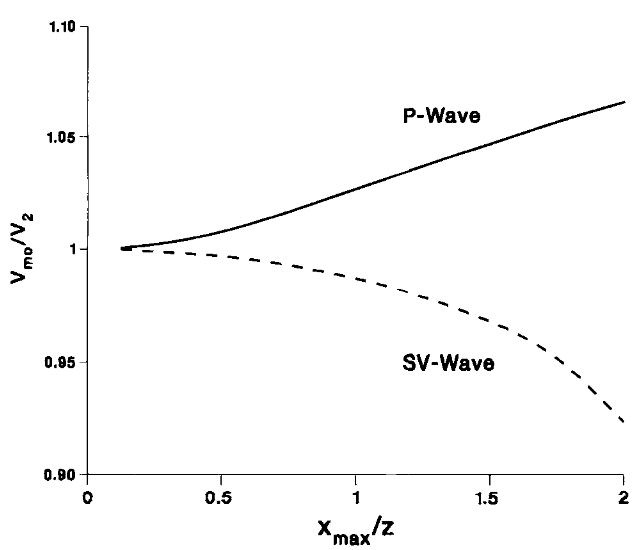


FIG. 4. Effective moveout velocity $V_{mo}$ of the best-fit hyperbola $t_H$ normalized by the short-spread moveout velocity $V_2$ for the model of Taylor sandstone (Figure 2).

The parameters $A_2^w$ and $A_4^w$ are Taylor series coefficients from equation (5) in the limit of weak anisotropy. For the $P$-wave,

$$A_2^w(P) = \frac{1 - 2\delta}{V_{p0}^2}, \qquad (15)^*$$

$$A_4^w(P) = -\frac{2(\varepsilon - \delta)}{t_{p0}^2 V_{p0}^4}. \qquad (16)^*$$

And, for the $SV$-wave

$$A_2^w(SV) = \frac{1 - 2\sigma}{V_{s0}^2}, \qquad (17)^*$$

$$A_4^w(SV) = \frac{2\sigma}{t_{s0}^2 V_{s0}^4} = -\left(\frac{V_{p0}}{V_{s0}}\right)^4 A_4^w(P). \qquad (18)^*$$

Formula (14) is not the only possible weak anisotropy approximation for reflection moveouts. Byun et al. (1989) and Byun and Corrigan (1990) suggested the so-called "skewed" hyperbolic moveout equation for long-spread $P$-wave moveout in a VSP context:

$$t^2 = t_0^2 + \left[\left(\frac{z}{V_\gamma}\right)^2 + \left(\frac{x}{2V_h}\right)^2\right] \frac{x^2}{z^2 + \left(\frac{x}{2}\right)^2}. \qquad (19)^*$$

Here, $z$ is depth, and $V_\gamma$ is an anisotropic parameter introduced by Byun et al. (1989). Originally, the "skewed formula" (19) was suggested as an approximation, with fitted coefficients $V_\gamma$ and $V_h$, to be found for example by least-squares or by a semblance search. Sena (1991) derived analytical expressions for the parameters of equation (19) under the assumption of weak anisotropy (defined slightly differently than here). The quartic Taylor series coefficient for equation (19) for a single-layer model was given by Sena (in the present notation) as

$$A_4(P)(\text{Sena}) = -\frac{2(\varepsilon - \delta)}{t_{p0}^2 V_{p0}^4} \frac{1}{(1 + 2\varepsilon)(1 + 2\delta)}. \qquad (20)^*$$

Because of its algebraic simplicity, the WAA provides a valuable guide to intuition and an explanation of many empirically observed results. However, the WAA cannot explain some existing numerical results, such as the pronounced deviations of the $SV$-wave moveout from hyperbolic for negative $\sigma(\varepsilon < \delta)$ found by Levin (1989). The obvious reason for this inconsistency is a breakdown of the WAA. A principal issue to be addressed in the next section is how weak the anisotropy must be for equations (15)–(18) to be valid.

### B. Exact Taylor series coefficients for $P$, $SV$, and $P$-$SV$ moveouts in a single layer

While concise analytic expressions for traveltime curves [using group (ray) velocities] in transversely isotropic media cannot be obtained without assuming weak or elliptical anisotropy, squared traveltimes $t^2(x^2)$ can be expanded in a Taylor series (Hake et al., 1984). We again analyze the three-term Taylor series near the vertical, equation (5), but with coefficients valid for arbitrary transverse isotropy. Parameters $A_2$ and $A_4$ for a single layer may be written as (Appendix B):

$$V_2^2(P) = \frac{1}{A_2(P)} = V_{p0}^2(1 + 2\delta), \tag{21}$$

$$A_4(P) = \frac{-2(\varepsilon - \delta)}{t_{p0}^2 V_{p0}^4} \frac{1 + \dfrac{2\delta}{1 - V_{s0}^2/V_{p0}^2}}{(1 + 2\delta)^4}, \tag{22}$$

$$V_2^2(SV) = \frac{1}{A_2(SV)} = V_{s0}^2(1 + 2\sigma), \tag{23}$$

$$A_4(SV) = \frac{2\sigma}{t_{s0}^2 V_{s0}^4} \frac{1 + \dfrac{2\delta}{1 - V_{s0}^2/V_{p0}^2}}{(1 + 2\sigma)^4}. \tag{24}$$

The first term in each expression (22) and (24) for the quartic coefficient is the previous (WAA) result, equations (16) and (18), hence the second term may be considered as the correction for "strong" anisotropy. The limits of applicability of WAA (as of any other asymptotic technique) are determined qualitatively: anisotropy should be weak enough for the terms that are quadratic in $\varepsilon$ and $\delta$ to be negligible. In general, one expects that the expressions with an isotropic leading term, equations (15) and (17), will be relatively well-determined, while those with an anisotropic leading term, equations (16) and (18), will be considerably less well-determined. In the formulas for $A_4$, equations (22) and (24), the WAA breaks down for very small $\delta$ and $\sigma$ (say, of the order of 0.05–0.1), because of the influence of the terms $(1 + 2\delta)^4$ and $(1 + 2\sigma)^4$. Since usually $\sigma \gg \delta$, the WAA is less suitable for the $SV$-wave than for the $P$-wave. In fact, even $\varepsilon - \delta = 0.02$ is "strong" anisotropy for the $SV$-wave, since this may mean $\sigma = 0.08$ and $(1 + 2\sigma)^4 = 1.81$! Nevertheless, although these parameters were originally designed for weak anisotropy, they facilitate the moveout analysis for arbitrary transverse isotropy as well.

Note that Sena's (1991) result (20) for $A_4$ is valid only in the WAA; the appearance of quadratic terms in $\delta$ and $\varepsilon$ arises because of his slightly different definition of WAA. By contrast, equations (21)–(24) are valid for any degree of anisotropy.

For $P$-waves, the quadratic coefficient $A_2$, equation (21), depends only on $V_{p0}$ and $\delta$, while the quartic coefficient $A_4$, equation (22), is controlled mostly by three elastic parameters: $V_{p0}$, $\varepsilon$, and $\delta$. This means that the dependence of the $P$-wave intermediate-spread moveout on the $S$-wave vertical velocity $V_{s0}$ is very weak, in accordance with intuition. This result, developed in more detail in the next section, turns out to be valid for long-spread $P$-wave moveout as well.

Equations (21)–(24) enable us to give an analytic description of deviations of $t^2$ from a hyperbola, for offsets where the three-term Taylor series is (at least qualitatively) accurate. The relative magnitude of the quartic term may be considered as a measure of inaccuracy of the conventional hyperbolic moveout equation. Defining this "relative magnitude" as the quartic term normalized by the hyperbolic terms:

$$\bar{A} = \frac{A_4 x^4}{A_0 + A_2 x^2},$$

we have, from equations (21)–(24)

$$\bar{A}(P) = \frac{-2(\varepsilon - \delta)}{(1 + 2\delta)^4} \bar{x}^4 \frac{1 + \dfrac{2\delta}{1 - V_{s0}^2/V_{p0}^2}}{1 + \dfrac{\bar{x}^2}{1 + 2\delta}}, \tag{25}$$

$$\bar{A}(SV) = \frac{2\sigma}{(1 + 2\sigma)^4} \bar{x}^4 \frac{1 + \dfrac{2\delta}{1 - V_{s0}^2/V_{p0}^2}}{1 + \dfrac{\bar{x}^2}{1 + 2\sigma}}, \tag{26}$$

where $\bar{x} \equiv x/2z$ is the normalized offset.

For $P$-waves, the "relative magnitude", equation (25), of deviations from hyperbolic moveout increases with increasing $|\varepsilon - \delta|$. For elliptical anisotropy ($\varepsilon = \delta$), the moveout is purely hyperbolic. For fixed $\varepsilon - \delta$ and conventional (small negative or positive) $\delta$, the relative deviation increases with *decreasing* $\delta$; this second effect is missing in the weak anisotropy approximation.

It is interesting to compare the residual $P$-wave moveout (after the hyperbolic correction) for the models of Taylor sandstone and Dog Creek shale (Figure 5), both expressed in percentages (to cancel the difference in vertical velocities). The value of $\varepsilon - \delta$ for both models is almost the same, while the absolute values of $\varepsilon$ and $\delta$ for Dog Creek shale are much higher. Therefore, we would be inclined to describe Dog Creek shale as more "anisotropic" than Taylor sandstone. However, anisotropy-induced deviations from hyperbolic moveout are much more pronounced for Taylor sandstone because of the lower value of $\delta$, and, consequently, the bigger quartic Taylor series term, equation (22). Of course,

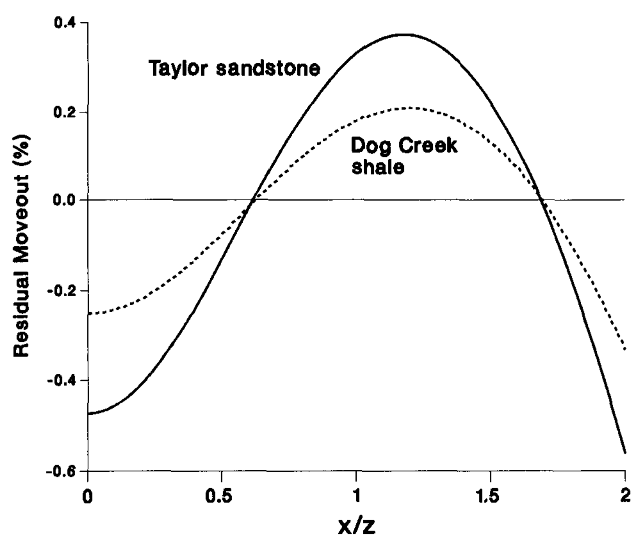


FIG. 5. Comparison of the residual $P$-wave moveout (normalized by the vertical arrival time $t_0$) for the models of Taylor sandstone and Dog Creek shale (Figure 2). The spread-length $x_{max} = 2z$.

its smaller value of δ makes its short-spread velocity $V_{p2}$ closer to its true vertical velocity $V_{p0}$, but this is a separate issue.

This example illustrates the danger of a casual analysis of anisotropic coefficients. Anisotropy is a multidimensional problem, manifesting itself in different ways in different contexts. Concise analytic expressions, such as those discussed here, show these various effects explicitly in terms of the parameters ε and δ (or σ and δ).

For the $SV$-wave, moveout differs in character, depending on the sign of σ. If $\sigma > 0$, the quartic coefficient, equation (24), reaches a maximum near $\sigma = 0.17$ and then decreases at larger σ. According to our estimates, for $\sigma > 0$ the contribution of $A_4$ is often smaller for the $SV$-wave, equation (26), than for the $P$-wave, equation (25). If $\sigma > 0$, nonhyperbolic moveout for $P$- and $SV$-waves is usually moderate and comparable up to $x_{max} = 1.5z$ (Figures 3 and 4).

For longer spreads, not adequately approximated by the Taylor series, equation (5), the $SV$-moveout rapidly becomes increasingly nonhyperbolic. The high residual moveout for the $SV$-wave for $x_{max} > 1.5z$ (Figure 3) is caused by the influence of the velocity maximum (for $\sigma > 0$) at incidence angles of 40–45 degrees (Figure 2). The "instantaneous" $SV$-wave moveout velocity, determined from the first derivative of the $t^2 - x^2$ curve (Radovich and Levin, 1982), sharply decreases from the short-spread velocity $V_2$, equation (7), near the vertical to a much smaller value near 45 degrees, where the phase (and group) velocity (Figure 2) reach their maximum values. As a result, the $SV$-moveout curve rapidly changes its slope near the velocity maximum, at $(x/z)^2 \approx 3 - 4$ (Figure 6). This change cannot be described by the three-term Taylor series equation (5), which reflects only near-vertical velocity variations. We return to the long-spread problem in the next section, where the curve in Figure 6 marked $t_A^2$ is discussed.

When $\sigma < 0$, the influence of the term $(1 + 2\sigma)^4$ in the quartic Taylor series coefficient, equations (24) and (26), leads to strongly nonhyperbolic $SV$-moveout, even at small offsets. This effect was found numerically by Levin (1989) in his study of the relation between $P$, $SV$, and $P$-$SV$ moveout velocities. Figure 7 shows the $SV$-wave traveltime curve for Mesaverde mudshale, which displayed the most anomalous behavior in Levin's calculations. For this model, the critical term in equation (26) is close to zero ($1 + 2\sigma = 0.006$), and the quartic term becomes very large compared to the quadratic term. In fact, the Taylor series expansion practically breaks down, since derivatives of the function $t^2(x^2)$ become almost infinite. The traveltime curve starts out with an extremely low moveout velocity ($V_2$ is almost zero) and then rapidly changes its slope. Thus, our analytic expressions give a clear explanation for the $SV$-wave moveout anomalies discussed by Levin (1989).

The above moveout anomaly is not caused by particularly strong anisotropy: the difference $\varepsilon - \delta = -0.177$. The problem is that σ is negative, implying that the $SV$-wave phase velocity has a maximum at the symmetry axis. If this maximum is strong enough, the $SV$-wave traveltime curve may have a cusp near the vertical (Helbig, 1966).

Thus, the sign of σ is very important for the character of the $SV$-wave moveout. The existing experimental data (Thomsen, 1986) indicate that the values of σ are predominantly positive. For transverse isotropy caused by thin bedding of *isotropic* layers, σ is always positive (Berryman, 1979). However, some rocks from Thomsen's (1986) table exhibit negative σ.

The short-spread moveout velocity for the converted $P$-$SV$ wave is given by (Seriff and Sriram, 1991; see also Appendix B):

$$V_2^2(P - SV) = \frac{1}{A_2(P - SV)}$$

$$= V_2^2(P)\,\frac{t_{p0}}{t_{p0} + t_{s0}} + V_2^2(SV)\,\frac{t_{s0}}{t_{p0} + t_{s0}},$$

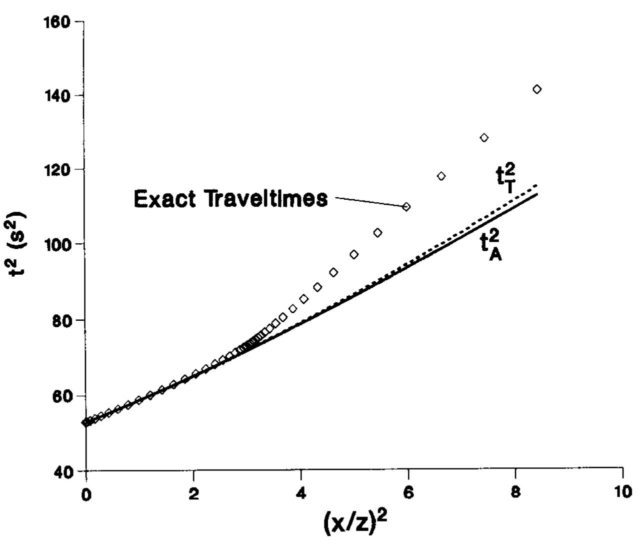


FIG. 6. Long-spread $SV$-wave moveout for the model of Dog Creek shale (Figure 2), with $z = 3$ km. Moveout time $t_T$ is the three-term Taylor series expansion with the coefficients given in equations (22) and (24). Moveout time $t_A$ is the approximation (30).

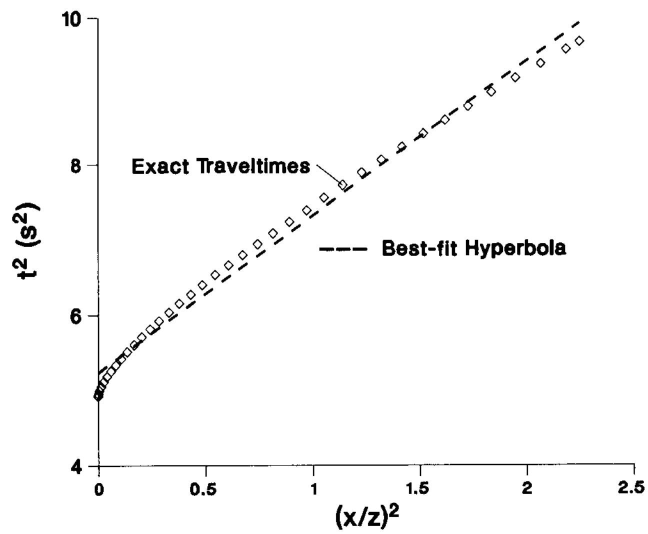


FIG. 7. $SV$-wave moveout with negative σ for the model of Mesaverde mudshale from Levin (1989). $V_{p0} = 4.529$ km/s, $V_{s0} = 2.703$ km/s, $\varepsilon = 0.034$, $\delta = 0.211$ ($\sigma = -0.497$, $1 + 2\sigma = 0.006$), $z = 3$ km.

$$= V_{p0}V_{s0}\left[1 + \frac{\delta V_{p0} + \sigma V_{s0}}{(V_{p0} + V_{s0})/2}\right]. \quad (27)$$

In Appendix B, we derive a corresponding expression (B-15) for the quartic Taylor series coefficient of the $P$-$SV$ wave.

If the Taylor series coefficients are known for any two of the $P$-, $SV$-, or $P$-$SV$-waves, these equations make it possible to find the coefficients for the third one. For instance, if only $P$-waves are excited, the coefficients of the $P$ and $P$-$SV$ reflections can be used to recover $A_2$ and $A_4$ for the $SV$-wave. In isotropic media, equation (27) reduces to a simple relation

$$V_2(SV) = V_2^2(P - SV)/V_2(P). \quad (28)$$

Tessmer and Behle (1988) showed that equation (28) can be effectively used to obtain the $SV$-wave stacking velocity from $P$ and $P$-$SV$ data. Levin (1989) carried out numerical calculations for several transversely isotropic models and demonstrated that equation (28) loses its accuracy for many anisotropic solids. Levin computed all three velocities in equation (28) from the exact traveltimes on finite (but unspecified) spread lengths. The largest errors in $V_2(SV)$ found from equation (28) corresponded to the models with negative $\sigma$, which are characterized by strongly nonhyperbolic $SV$-wave moveout.

These results triggered a lively discussion in the literature (Berge, 1991; Seriff and Sriram, 1991). The latter demonstrated that if the isotropic formula (28) is replaced with the exact expression (27), the $SV$-wave short-spread moveout velocity, derived from the $P$ and $P$-$SV$ data, becomes much closer to the value computed by Levin from the exact traveltimes. Noticeable discrepancies between the value recovered from equation (27) and the measured $SV$-wave moveout velocity remained only for the models with negative $\sigma$.

To clarify the point of the above discussion, it is helpful to distinguish between two different questions:

1) Is it possible to accurately find the $SV$-wave short-spread moveout velocity $V_2$ from equation (27) using the $P$ and $P$-$SV$ moveout velocities measured on conventional short (but finite) spreads, i.e.,

$$V_2^2(SV)? = ?V_{mo}^2(P - SV)\,\frac{t_{p0} + t_{s0}}{t_{s0}} - V_{mo}^2(P)\,\frac{t_{p0}}{t_{s0}}.$$

2) Is the value of $V_2(SV)$ close to the $V_{mo}(SV)$ measured on conventional short spreads, i.e.,

$$V_{mo}(SV)? = ?V_2(SV).$$

To answer the first question, we have to determine whether the $P$ and $P$-$SV$ moveouts are close enough to hyperbolas, so that $V_{mo}$ may be used in place of $V_2$ in equation (27) (the second question is not relevant to this problem). As shown above, the short-spread $P$-wave moveout usually does not diverge much from hyperbolic, whether $\sigma$ is positive or negative. From equations (27) and (B-15), it is also clear that the converted $P$-$SV$ wave does not exhibit the same extreme anomalous behavior for negative $\sigma$, as does the $SV$-wave. Therefore, in most cases the values of $V_2(P)$ and $V_2(P - SV)$ can in fact be extracted from short-spread moveout curves. This means that equation (27) does represent a viable tool for recovering the short-spread $SV$ moveout velocity $V_2(SV)$ from $P$ and $P$-$SV$ data.

The second question was answered in the previous section. For short spreads, the $SV$-wave moveout will be approximately hyperbolic if $\sigma > 0$, so there the answer is yes. But if $\sigma < 0$, the short-spread velocity $V_2(SV)$ may be far different from the measured moveout velocity $V_{mo}(SV)$, even for $x_{max} < z$, so the answer is no. This concisely explains the results of Seriff and Sriram (1991).

### C. Exact quartic Taylor series coefficients in multilayered media

In a stratified medium, the quartic Taylor series coefficient reflects the combined influence of layering and anisotropy. In Appendix B, we derive the quartic coefficient for the $P$-$SV$ waves, equation (B-15), and obtain the quartic coefficients for pure $P$- and $SV$-waves as special cases:

$$A_4(P \text{ or } SV) = \frac{(\Sigma_i V_{2i}^2 \Delta t_i)^2 - t_0 \Sigma_i V_{2i}^4 \Delta t_i}{4(\Sigma_i V_{2i}^2 \Delta t_i)^4} + \frac{t_0 \Sigma_i A_{4i} V_{2i}^8 \Delta t_i^3}{(\Sigma_i V_{2i}^2 \Delta t_i)^4}, \quad (29)$$

$V_{2i}$ and $A_{4i}$ for individual layers are given by equations (21)–(24). Hake et al. (1984), who derived an expression equivalent to this, attributed the first term in equation (29) to the effect of inhomogeneity, noting that it formally reduces to the expressions for isotropic media first found by Taner and Koehler (1969), and in the present form by Al-Chalabi (1974). However, in transversely isotropic media, the hyperbolic coefficients $V_{2i}$ in the first term are affected already by anisotropy. The second term is caused entirely by anisotropy.

According to the Cauchy-Schwartz inequality, the first term is always nonpositive (Hake et al., 1984). The second term may be either positive or negative depending on the wave type and the signs of $A_{4i}$, i.e., of $\sigma_i$. If the $\sigma_i$ are predominantly positive, the two terms have the same sign (negative) for the $P$-wave and different signs for the $SV$-wave.

It is easy to show, using the quartic expansion (5), that the spread-length dependence of moveout velocity, $V_{mo}(x_{max}/z)$, should be approximately quadratic (Figure 4), as described by Al-Chalabi (1974) without explanation in principle. The quartic parameter $A_4$ derived here provides a way to interpret that quadratic variation in terms of layer properties.

The coefficient $A_4$ for the $SV$-wave does not become infinite even if $1 + 2\sigma_i = 0$ ($V_{2i} = 0$) in some of the layers. This means that the impact of layers with negative $\sigma$ on deviations from hyperbolic moveout is considerably dampened by the layered structure. Unless a layer with $\sigma < 0$ happens to be directly below the free surface, we should not expect to encounter such an anomalously nonhyperbolic moveout curve as the one in Figure 7.

In Appendix C, we show how to recover interval values of the quartic terms $A_{4i}$, hence a measure of layer anisotropy, from seismic data, using a Dix-type differentiation of equation (29).

**Long-spread reflection moveout**

Although the three-term Taylor series provides valuable analytical insight into peculiarities of nonhyperbolic moveout, it loses accuracy rapidly with increasing offset (Figure 8). It is interesting that, for a single layer with $\sigma > 0$, the exact traveltimes for both $P$- and $SV$-waves exceed the value given by the three-term Taylor series. Therefore (since $A_4$ has different signs for two wave-types), the estimates of deviations from hyperbola based on equations (21)–(24) are overstated for the $P$-wave, and understated for the $SV$-wave (if $\sigma > 0$).

The increasing error of the three-term Taylor series for $x_{max} > 1.5z$ is not surprising, since this expansion reflects the shape of the velocity curve at small incidence angles and therefore cannot account for changes in velocity at larger angles. A better analytic approximation may be obtained by combining the functional form of $t^2$, found for weak anisotropy, equation (14), with the exact Taylor series coefficients:

$$t_A^2 \equiv t_0^2 + A_2 x^2 + \frac{A_4 x^4}{1 + A^* x^2}. \tag{30}$$

$A_2$ and $A_4$ are given by equations (21)–(24) for a single layer, and by equations (13)–(29) for multilayered media. The parameter $A^*$ is introduced in the denominator to ensure the correct behavior of moveout at large $x$. The far-offset limit of equation (30) is

$$t_A^2|_{x\to\infty} = t_0^2 - \frac{A_4}{(A^*)^2} + x^2\left(A_2 + \frac{A_4}{A^*}\right) + \dots,$$

therefore

$$A^* = \frac{A_4}{\dfrac{1}{V_h^2} - A_2}, \tag{31}$$

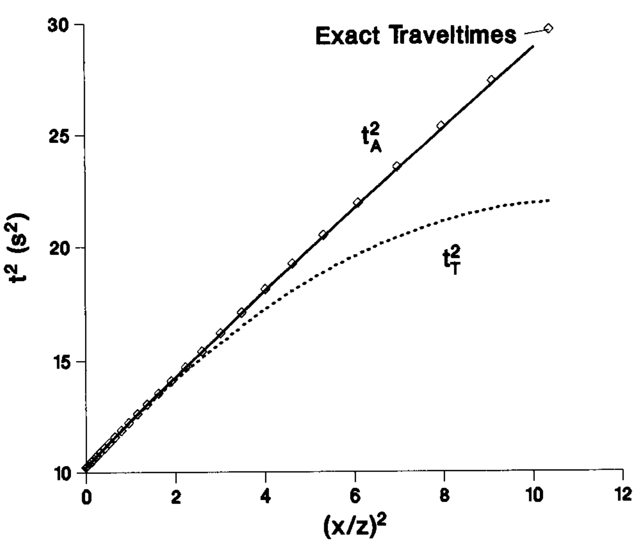


FIG. 8. Three-term Taylor series $t_T$ and approximation $t_A$ for $P$-wave moveout in a layer of Dog Creek shale (Figure 2), with $z$ = 3 km.

where $V_h$ is the horizontal velocity. Although $t_A$ inherits its functional form from the weak anisotropy approximation, it is based on the exact Taylor series coefficients, and converges at large $x$ as well.

For $P$-waves, the new approximation provides an excellent fit to the exact traveltimes, even for long spreads and substantial anisotropy (Figure 8). The results for a more complicated, four-layered transversely isotropic model (Figure 9) are presented in Figure 10. The horizontal velocity $V_h$ used in equation (31) is the root-mean-square of the horizontal velocities in the individual layers. In spite of pronounced anisotropy in the two bottom layers and strongly nonhyper-

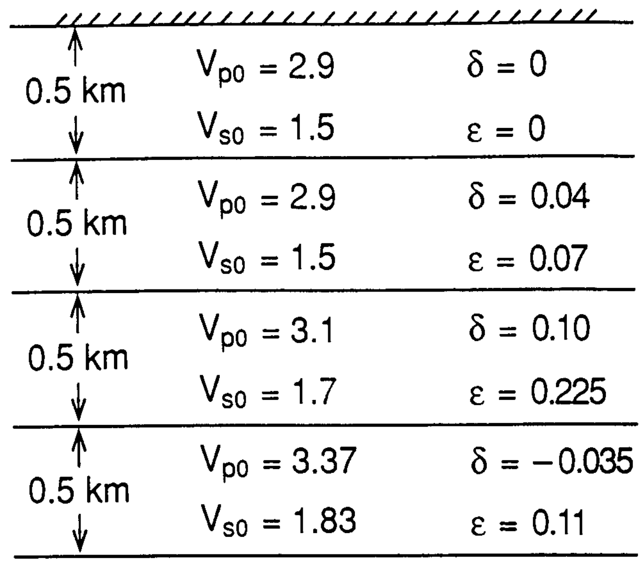


FIG. 9. Layered transversely isotropic model used to test analytical approximation $t_A$.

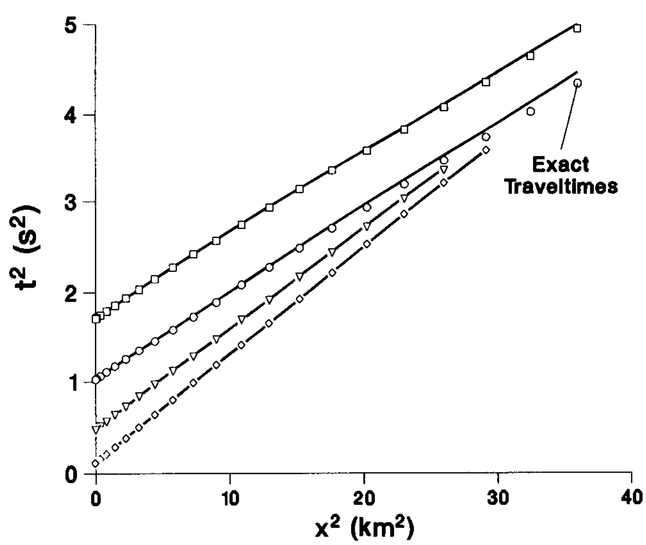


FIG. 10. Comparison between the exact $P$-wave traveltimes and approximation $t_A$ [equation (30), solid curves] for the primary reflections from each boundary of the model in Figure 9. The subsurface layer is isotropic, so the first reflection has purely hyperbolic moveout. The two lowest layers have significant anisotropy.

bolic moveout of the corresponding reflections, $t_A$ remains numerically accurate at large horizontal offsets.

The new approximation represents a fast and efficient way to estimate deviations from hyperbolic moveout for layered transversely isotropic models without doing actual ray tracing. Also, it can be applied in moveout correction on long-spread gathers, refining the more conventional way to correct for nonhyperbolic moveout, using the quartic Taylor series (Gidlow and Fatti, 1990). In the sequel paper mentioned in the Introduction, we use moveout equation (30) to analyze the ambiguity in deducing the parameters of anisotropy from long-spread data.

Equation (30) can be extended directly to symmetry planes of azimuthally anisotropic media, provided the appropriate Taylor series coefficients are substituted. It can even be applied outside symmetry planes, although it will not account properly for significant out-of-plane energy propagation.

The exact equations for P-wave phase and group velocities contain four coefficients: $V_{p0}$, $V_{s0}$, $\varepsilon$, and $\delta$. However, in the previous section it was shown that the $P$-wave intermediate-spread moveout is practically independent of the $S$-wave vertical velocity $V_{s0}$. The same is true for $P$-wave traveltimes on long spreads, even when the anisotropy is strong. From equations (30)–(31), it is clear that $V_{s0}$ can affect $P$-wave moveout slightly through the quartic coefficient $A_4$, but this influence is very weak.

Thomsen (1986) showed that the $P$-wave phase velocity for weak transverse isotropy is determined just by the $P$-wave vertical velocity $V_{p0}$ and the anisotropies $\varepsilon$ and $\delta$. It turns out that although Thomsen's (1986) velocity equations linearized in $\varepsilon$ and $\delta$ become inaccurate for strong anisotropy, the overall influence of $V_{s0}$ on $P$-wave velocity and traveltimes remains insignificant. This conclusion is supported by the exact numerical calculations discussed in more detail by Tsvankin ("Normal moveout from dipping reflectors in an isotropic media," accepted for publication in GEOPHYSICS). The possibility to reduce the number of independent parameters affecting $P$-wave traveltimes from four to three ($V_{p0}$, $\varepsilon$, and $\delta$) represents one of the most important advantages of the present notation over the conventional elastic moduli $C_{ij}$. This advantage is especially valuable in the inversion of $P$-wave traveltimes for the anisotropic coefficients.

For $SV$-waves, the area of validity of approximation is much more limited (Figures 6 and 11). The accuracy of $t_A$ decreases at $x = (1.7 - 2)z$ as the result of the sharp changes in the slope of the $SV$-moveout curve near the velocity maximum (Figure 6). The model of Dog Creek shale used in Figure 6 is characterized by strong anisotropy-induced nonhyperbolic moveout on long spreads, with the residual moveout after the hyperbolic correction reaching almost 200 ms at $x_{max} = 2z$ (2.75 percent of $t_0$). For more mild $SV$-wave anisotropy, $t_A$ is much closer to the exact moveout curve, but still diverges (toward smaller times) near the velocity maximum.

The accuracy of the moveout approximation (30) for the $SV$-wave may be significantly increased if analytically derived parameters $A^*$, $A_2$, and $A_4$ are replaced with fitted coefficients determined by the least-squares method (Figure 11). For models with $\sigma < 0.5$, approximation (30) with fitted coefficients practically eliminates $SV$-wave residual moveout for spread lengths up to at least $x_{max} = 2z$.

The difficulties with an analytical description of the $SV$-moveout reflect a general difference between the character of long-spread $P$- and $SV$-wave moveouts for transverse isotropy. The $P$-wave moveout curve on long spreads ($x_{max} = 2z$ or more) is usually much smoother and more hyperbolic, because the phase (and group) velocities for the $P$-wave rarely have pronounced minima or maxima between 0 degrees and 90 degrees (Figure 2). For the most common case, $\sigma > 0$ ($\varepsilon > \delta$), the $P$-wave phase velocity contains a minimum if $\delta < 0$ (assuming $\varepsilon > 0$), but since $\delta$ is usually small, this minimum is shallow. For the $SV$-wave, the phase velocity always has either a minimum (if $\sigma < 0$) or a maximum (if $\sigma > 0$) near an incidence angle of 45 degrees, which might be relatively sharp since $\sigma$ is often large. Therefore, deviations from hyperbolic moveout for spread lengths of about $x_{max} = 2z$ are usually more significant for the $SV$-wave, although the overall $P$-wave velocity anisotropy may be stronger. Refraction of rays in multilayered media is likely to make the $SV$-wave moveout anomaly near 45 degrees less pronounced and extend it over a wide range of incidence angles.

Ultimately, the shape of the $SV$-wavefront near the velocity maximum may lead to a cusp on the traveltime curve. In Figure 12, the cusp occupies a range of offsets starting at $x = 1.48z$ (group velocity angle 36.5 degrees). The presence of a cusp, though diagnostic of anisotropy, may seriously impede the analysis of reflection moveout. Our approximations will properly describe only the first branch of a cusp up to the first turning point. However, the cusp will only occur if the anisotropy is sufficiently strong (Helbig, 1966; Musgrave, 1970).

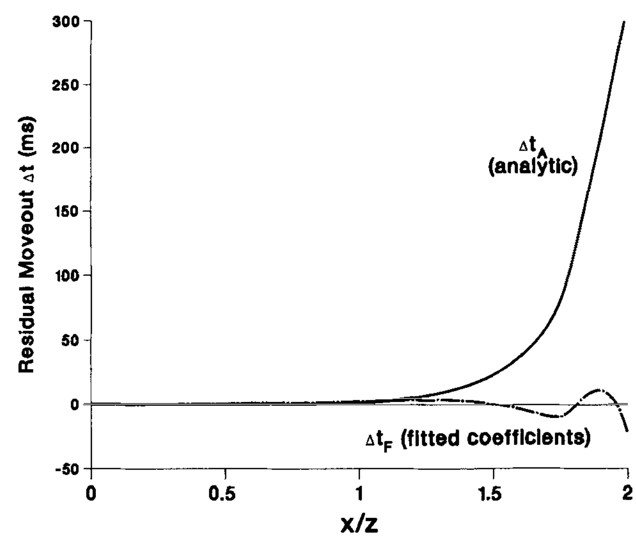


FIG. 11. Accuracy of approximation $t_A$ for the $SV$-wave. $\Delta t_A$ is the residual moveout after application of the analytical approximation (30) ($\Delta t_A$ = exact traveltime $t - t_A$; $\Delta t_F$ is the residual moveout for the same approximation, but with fitted coefficients). Model is Dog Creek shale (Figure 2), with $z = 3$ km.

## DISCUSSION AND CONCLUSIONS

We have given a description of long-spread reflection moveout of $P$- and $S$-waves in multilayered transversely isotropic media. Two analytic approximations were used to describe deviations from the conventional hyperbolic moveout equation. The first one is the three-term (quartic) Taylor series for $t^2 - x^2$ moveout curves. We derived the quartic Taylor series coefficient for the converted $P$-$SV$ wave in multilayered transversely isotropic media with arbitrary strength of anisotropy and obtained the quartic coefficients for $P$- and $SV$-waves, found earlier by Hake et al. (1984), as special cases. The expressions for the Taylor series coefficients, formulated in terms of $\varepsilon$ and $\delta$ (or $\sigma$ and $\delta$), give a clear analytic explanation for deviations from hyperbolic moveout caused by anisotropy.

We also suggested a new, more general moveout equation, which inherits its form from the weak anisotropy approximation but is based on the exact Taylor series coefficients and converges at large offsets as well. For the $P$-wave, the new approximation provides a very good fit to the exact traveltimes, even for relatively strong anisotropy and long spreads. For the $SV$-wave, the accuracy of the new equation decreases in the area of sharp moveout changes near the velocity maximum, at $x = (1.7 - 2)z$. To obtain an accurate result for the $SV$-wave in this offset range, the analytic parameters should be replaced with fitted coefficients derived either by the least-squares method or by semblance search.

Comparison with the exact solution shows that the weak anisotropy approximation breaks down, in the description of nonhyperbolic moveout, for surprisingly small values of anisotropic coefficients. The characterization of anisotropic media and effects in terms of suitably defined parameters is crucial in this analysis. For application to $SV$-wave moveout and velocity equations, we introduced a dimensionless parameter $\sigma$, equation (9), which reduces to zero for isotropy or elliptical anisotropy. Coefficient $\sigma$ often replaces the more intuitive quantity $\varepsilon$ as a more suitable parameter in the moveout problem for transverse isotropy. The sign of $\sigma$ determines the sign of the quartic Taylor series coefficient in the single-layer model.

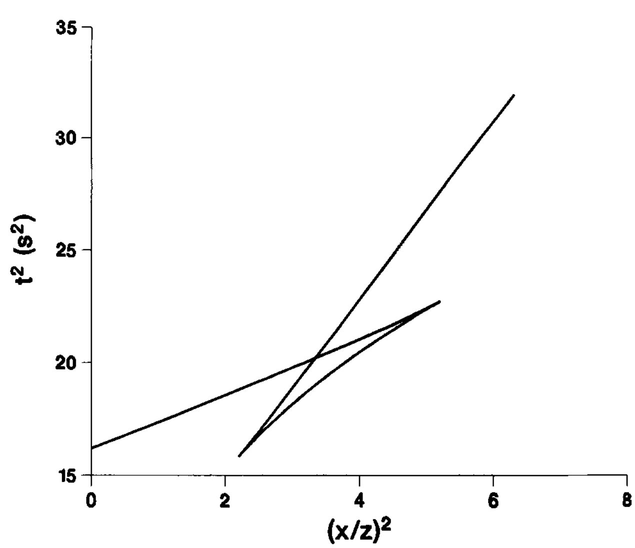


FIG. 12. $SV$-wave moveout curve with a cusp. Model parameters are (from Thomsen, 1986): $V_{p0}$ = 3.048 km/s, $V_{s0}$ = 1.490 km/s, $\varepsilon$ = 0.255, $\delta$ = −0.05 ($\sigma$ = 1.276), $z$ = 3 km.

The results of our analysis of reflection moveout may be summarized as follows:

1) On short spreads, limited by the reflector depth, the $P$-wave moveout remains close to hyperbolic even in the presence of anisotropy.
2) The difference between the short-spread moveout velocity and the vertical rms velocity, caused by anisotropy, is much more significant for the $SV$-wave (40 percent and more for models considered here) than for the $P$-wave.
3) $P$-wave moveout is almost entirely controlled by the $P$-wave vertical velocity $V_{p0}$, $\delta$, and $\varepsilon$. The dependence of $P$-wave traveltimes on the $S$-wave vertical velocity $V_{s0}$ is very weak, even for long spreads and strong anisotropy.
4) For a single transversely isotropic layer, the $P$-wave moveout diverges from hyperbolic with increasing $|\varepsilon - \delta|$ and (for fixed $\varepsilon - \delta$) with *decreasing* $\delta$.
5) The character of the $SV$-wave moveout is strongly dependent on the sign of $\sigma$. If $\sigma > 0$ (the most common case), deviations from a hyperbola for $SV$-wave are moderate and comparable to those for the $P$-wave up to $x = 1.5z$. The $SV$-wave moveout at small offsets can become strongly nonhyperbolic if $\sigma$ is negative ($\varepsilon < \delta$).
6) While the $P$-wave moveout is usually relatively smooth up to at least $x = 2z$, the $SV$-wave traveltime curve exhibits a sharp change in moveout velocity at $x = (1.7 - 2)z$ caused by the influence of the velocity maximum (for $\sigma > 0$) located at incidence angles near 45 degrees. For $x_{max} = 2z$ the $SV$-wave residual moveout, after the hyperbolic correction, may reach several percent of $t_0$.
7) The $P$-$SV$ wave does not exhibit the same anomalous nonhyperbolic moveout at very small offsets for $\sigma < 0$, as does the $SV$-wave. Therefore, the short-spread $P$ and $P$-$SV$ moveouts usually can be used to recover the analytical value of the $SV$-wave short-spread moveout velocity.
8) In multilayered media, anisotropy may either enhance or weaken the influence of layering on deviations from hyperbolic moveout. The contribution of layering is determined by contrasts in short-spread moveout velocities rather than true vertical velocities.

Although the analytic developments in this paper are strictly valid for transversely isotropic models with a vertical symmetry axis, they can be directly extended to symmetry planes of azimuthally anisotropic media.

## ACKNOWLEDGMENTS

We thank K. Helbig, K. Larner, F. Levin, F. Lyakhovitsky, M. Meadows, and M. Schoenberg for useful comments. We thank Amoco Production Company, where both authors were employed during this work, for permission to publish it.

## REFERENCES


Al-Chalabi, M., 1974, An analysis of stacking, RMS, average, and interval velocities over a horizontally layered ground: Geophys. Prosp., **22**(3), 458–475.

Banik, N. C., 1984, Velocity anisotropy of shales and depth estimation in the North Sea basin: Geophysics, **49**, 1411–1419.

Berge, P. A., 1991, Estimating $SV$-stacking velocities for transversely isotropic solids: Geophysics, **56**, 1596–1602.

Berge, P. A., Mallick, S., Fryer, G. J., Barstow, N., Carter, J. A., Sutton, G. H., and Ewing, J. I., 1991, In-situ measurement of transverse isotropy in shallow-water marine sediments: Geophys. J. Int., **104**, 241–254.

Berryman, J. G., 1979, Long-wave elastic anisotropy in transversely isotropic media: Geophysics, **44**, 896–917.

Byun, B. S., and Corrigan, D., 1990, Seismic traveltime inversion for transverse isotropy: Geophysics, **55**, 192–200.

Byun, B. S., Corrigan, D., and Gaiser, J. E., 1989, Anisotropic velocity analysis for lithology discrimination: Geophysics, **54**, 1564–1574.

Carrion, P., Costa, J., Ferrer Pinheiro, J. E., and Schoenberg, M., 1992, Cross-borehole tomography in anisotropic media: Geophysics, **57**, 1194–1198.

Dix, C. H., 1955, Seismic velocities from surface measurements: Geophysics, **20**, 68–86.

Gidlow, P. W., and Fatti, J. L., 1990, Preserving far offset seismic data using nonhyperbolic moveout correction: 60th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1726–1729.

Gonzalez, A., and Lynn, W., 1991, Prestack frequency-wavenumber ($f$-$k$) migration in a transversely isotropic medium, EAEG Ann. Mtg., Extended Abstracts, 132–133.

Hake, H., Helbig, K., and Mesdag, C. S., 1984, Three-term Taylor series for $t^2 - x^2$ curves over layered transversely isotropic ground: Geophys. Prosp., **32**, 828–850.

Helbig, K., 1966, A graphical method for the construction of rays and traveltimes in spherically layered media, part 2: Anisotropic case, theoretical considerations: Bull Seis. Soc. Am., **56**, 527–559.

——— 1984, Transverse isotropy in exploration seismics: Geophys. J. Roy. Astr. Soc., **76**, 79–88.

Krey, Th., and Helbig, K., 1956, A theorem concerning anisotropy of stratified media and its significance for reflection seismics: Geophys. Prosp., **4**, 294–302.

Larner, K., 1993, Dip-moveout error in transversely isotropic media with linear velocity variation in depth: Geophysics, **58**, 1442–1453.

Larner, K., and Cohen, J., 1993, Migration error in transversely isotropic media with linear velocity variation in depth: Geophysics, **58**, 1454–1467.

Levin, F. K., 1978, The reflection, refraction, and diffraction of waves in media with an elliptical velocity dependence: Geophysics, **43**, 528–537.

——— 1979, Seismic velocities in transversely isotropic media: Geophysics, **44**, 918–936.

——— 1989, $SV$-wave velocities from $P$-$P$ and $P$-$SV$ data for transversely isotropic solids: Geophysics, **54**, 1336–1338.

Lyakhovitsky, F., and Nevsky, M., 1971, The traveltime curves of reflected waves or a transversely isotropic medium: Dokl. Akad. Nauk SSSR, **196**, 327–330 (in Russian).

Musgrave, M. J. P., 1970, Crystal acoustics: Holden Day.

Radovich, B. J., and Levin, F. K., 1982, Instantaneous velocities and reflection times for transversely isotropic solids: Geophysics, **47**, 316–322.

Sena, A. G., 1991, Seismic traveltime equations for azimuthally anisotropic and isotropic media: Estimation of interval elastic properties: Geophysics, **56**, 2090–2101.

Seriff, A. J., and Sriram, K. P., 1991, $P$-$SV$ reflection moveouts for transversely isotropic media with a vertical symmetry axis: Geophysics, **56**, 1271–1274.

Taner, M. T., and Koehler, F., 1969, Velocity spectra-digital computer derivation and applications of velocity functions: Geophysics, **34**, 859–881.

Tessmer, G., and Behle, A., 1988, Common reflection point data-stacking technique for converted waves: Geophys. Prosp., **36**, 671–688.

Thomsen, L. A., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954–1966.

White, J. E., Martineau-Nicoletis, L., and Monash, C., 1983, Measured anisotropy in Pierre Shale: Geophys. Prosp., **31**, 709–725.

Winterstein, D. F., 1986, Anisotropy effects in $P$-wave and $S$-wave stacking velocity contain information on lithology: Geophysics, **51**, 661–672.

Wright, J., 1987, The effects of transverse isotropy on reflection amplitude versus offset: Geophysics, **52**, 564–567.


## APPENDIX A

### TRAVELTIME CURVES OF *P*- AND *SV*-WAVES IN THE WEAK ANISOTROPY APPROXIMATION

In this Appendix, traveltimes of reflected $P$- and $SV$-waves are derived using the weak anisotropy approximation, as described by Thomsen (1986). We consider the reflections from the bottom of a horizontal homogeneous transversely isotropic layer. The $P$-wave traveltime is

$$t(\psi) = \frac{V_{p0} t_{p0}}{V_g(\psi) \cos \psi}, \qquad \text{(A-1)}$$

$V_g(\psi)$ is the group (ray) velocity for propagation at ray angle $\psi$ (measured from the vertical), and $t_{p0}$ is the (two-way) vertical arrival time. Assuming that anisotropy is weak, so that terms quadratic in $\varepsilon$ and $\delta$ may be dropped, we can equate the group velocity $V_g$ at $\psi$ with the phase velocity $V_{ph}$ at phase angle $\theta$, and write (Thomsen, 1986):

$$V_g(\psi) = V_{ph}(\theta) = V_{p0}(1 + \delta \sin^2 \theta \cos^2 \theta + \varepsilon \sin^4 \theta), \qquad \text{(A-2)*}$$

$$\tan \theta = \frac{\tan \psi}{1 + 2\delta + 4(\varepsilon - \delta) \sin^2 \theta}. \qquad \text{(A-3)*}$$

Substituting $\psi$ for $\theta$ and retaining only terms linear in $\varepsilon$ and $\delta$, we get from (A-2) and (A-3):

$$V_g(\psi) = V_{p0}(1 + \delta \sin^2 \psi \cos^2 \psi + \varepsilon \sin^4 \psi). \qquad \text{(A-4)*}$$

Now for $t^2$, we have from equation (A-1)

$$t^2(\psi) = \frac{t_{p0}^2}{\cos^2 \psi(1 + 2\delta \sin^2 \psi \cos^2 \psi + 2\varepsilon \sin^4 \psi)}. \qquad \text{(A-5)*}$$

Expressing the group angle $\psi$ through $x$ and $z$, and linearizing further in $\delta$ and $\varepsilon$, we get

$$t^2(\bar{x}^2) = t_{p0}^2 \left[1 + \bar{x}^2\left(1 - \frac{2\delta}{1 + \bar{x}^2}\right) - \bar{x}^4 \frac{2\varepsilon}{1 + \bar{x}^2}\right], \qquad \text{(A-6)*}$$

where the normalized offset is

$$\bar{x} = \frac{x}{2z} = \frac{x}{V_{p0} t_{p0}}.$$

Equation (A-6) may be expanded as a Taylor series in $x^2$, as in equation (5), with coefficients $A_2^w(P)$ and $A_4^w(P)$ (bearing superscript $^w$ to denote the WAA):

$$A_2^w(P) = \frac{1 - 2\delta}{V_{p0}^2}, \qquad \text{(A-7)*}$$

*Equations so marked are valid only for weak anisotropy.

$$A_4^w(P) = -\frac{2(\varepsilon - \delta)}{t_{p0}^2 V_{p0}^4}.$$

Then, equation (A-6) may be rewritten as

$$t^2 = t_0^2 + A_2^w x^2 + \frac{A_4^w x^4}{1 + \left(\frac{x}{V_0 t_0}\right)^2}. \quad \text{(A-8)*}$$

For large offsets ($x \to \infty$), equation (A-6) becomes

$$t^2 = t_{p0}^2(1 + 2\varepsilon - 2\delta) + \frac{x^2}{V_h^2}, \quad \text{(A-9)*}$$

where $V_h^2 = V_{p0}^2(1 + 2\varepsilon)$ is the square of the horizontal velocity.

The $SV$-wave phase velocity in the weak anisotropy approximation is (Thomsen, 1986):

$$V_{ph}(\theta) = V_{s0}(1 + \sigma \sin^2 \theta \cos^2 \theta). \quad \text{(A-10)*}$$

Following the derivation above, the $SV$-wave traveltime curve may be represented by equation (A-8), with coefficients:

$$A_2^w(SV) = \frac{1 - 2\sigma}{V_{s0}^2}, \quad \text{(A-11)*}$$

$$A_4^w(SV) = \frac{2\sigma}{t_{s0}^2 V_{s0}^4}.$$

In the limit of large offsets, the traveltime is given by

$$t^2 = t_{s0}^2(1 - 2\sigma) + \frac{x^2}{V_{s0}^2}. \quad \text{(A-12)*}$$

## APPENDIX B

### EXACT TAYLOR SERIES EXPANSION OF REFLECTION MOVEOUTS

Here, generalize the results of Hake et al. (1984) and derive the fourth-order Taylor series coefficient for $P$-$SV$ reflections in multilayered transversely isotropic media with a vertical symmetry axis, not restricted to weak anisotropy. The corresponding coefficients for $P$-$P$ and $SV$-$SV$ waves are then obtained by inspection.

Let us consider the Taylor series expansion (5) of the squared arrival time $t^2(x^2)$. Our goal is to express the parameters $A_2$ and $A_4$ through the parameters of a transversely isotropic medium. The horizontal offset for the $P$-$SV$ reflection in $N$-layered transversely isotropic media may be written as

$$x = p\left[\sum_{i=1}^{N} U_{pi}\Delta t_{pi} + \sum_{i=1}^{N} U_{si}\Delta t_{si}\right], \quad \text{(B-1)}$$

where

$$U_i = \frac{V_{g1i}}{p}. \quad \text{(B-2)}$$

Here, $p$ is the horizontal component of the slowness vector (the ray parameter, $dt/dx$) for the ray emerging at offset $x$, $\Delta t_i$ is the oblique one-way traveltime in the $i$th layer. $V_{g1}$ is the horizontal component of the group velocity vector; subscript "p" refers to the $P$-wave, subscript "s" to the $SV$-wave. The first derivative of $t^2$, needed for $A_2$, is

$$\frac{dx^2}{dt^2} = \frac{x}{t}\frac{dx}{dt} = \frac{x}{tp} = \frac{\sum_{i=1}^{N} U_{pi}\Delta t_{pi} + \sum_{i=1}^{N} U_{si}\Delta t_{si}}{\sum_{i=1}^{N} \Delta t_{pi} + \sum_{i=1}^{N} \Delta t_{si}}. \quad \text{(B-3)}$$

Evidently, for the pure $P$-$P$ reflection

$$\frac{dx^2}{dt^2}(P) = \frac{\sum_{i=1}^{N} U_{pi}\Delta t_{pi}}{\sum_{i=1}^{N} \Delta t_{pi}}, \quad \text{(B-4)}$$

and for the $SV$-$SV$ reflection

$$\frac{dx^2}{dt^2}(SV) = \frac{\sum_{i=1}^{N} U_{si}\Delta t_{si}}{\sum_{i=1}^{N} \Delta t_{si}}. \quad \text{(B-5)}$$

Using $V_{2i}^2 = \lim_{x \to 0} U_i$, the second-order parameters are:

$$V_2^2(P) = \frac{1}{A_2(P)} = \lim_{x \to 0} \frac{dx^2}{dt^2}(P) = \frac{1}{t_{p0}} \sum_{i=1}^{N} U_{pi}\Delta t_{pi}$$

$$= \frac{1}{t_{p0}} \sum_{i=1}^{N} V_{2pi}^2 \Delta t_{pi}, \quad \text{(B-6)}$$

$$V_2^2(SV) = \frac{1}{A_2(SV)} = \lim_{x \to 0} \frac{dx^2}{dt^2}(SV) = \frac{1}{t_{s0}} \sum_{i=1}^{N} U_{si}\Delta t_{si}$$

$$= \frac{1}{t_{s0}} \sum_{i=1}^{N} V_{2si}^2 \Delta t_{si}, \quad \text{(B-7)}$$

where the times are either one- or two-way vertical traveltimes. For the $P$-$SV$ converted wave we find from equations (B-3), (B-6), and (B-7) (Seriff and Sriram, 1991):

$$V_2^2(P-SV) = \frac{1}{A_2}(P-SV) = V_2^2(P)\frac{t_{p0}}{t_{p0}+t_{s0}} + V_2^2(SV)\frac{t_{s0}}{t_{p0}+t_{s0}}. \quad \text{(B-8)}$$

For the quartic coefficient $A_4$, we need the second derivative of $t^2$, c.f., equation (5) of the main text. Differentiating equation (B-3), we get

$$\frac{1}{2}\frac{d}{dx^2}\left(\frac{dt^2}{dx^2}\right) = \frac{G - (\Sigma_i \Delta t_{pi} + \Sigma_i \Delta t_{si})\dfrac{1}{p}\dfrac{dG}{dx}}{4G^3}, \quad \text{(B-9)}$$

where

$$G = \frac{x}{p} = \Sigma_i U_{pi}\Delta t_{pi} + \Sigma_i U_{si}\Delta t_{si}.$$

To find $A_4$, we only need the limit of equation (B-9) at $x = 0$. Following the derivation in Hake et al. (1984), in terms of one-way traveltimes

$$\left.\frac{1}{p}\frac{dG}{dx}\right|_{\substack{x=0\\p=0}} = \frac{\Sigma_i(H_{pi}+V_{2pi}^4)\Delta t_{pi} + \Sigma_i(H_{si}+V_{2si}^4)\Delta t_{si}}{\Sigma_i V_{2pi}^2\Delta t_{pi} + \Sigma_i V_{2si}^2 \Delta t_{si}}, \quad \text{(B-10)}$$

where

$$H_i = \lim_{x\to 0}\frac{1}{p}\frac{dU_i}{dp}, \quad \text{(B-11)}$$

for each wave type ($P$ or $S$). Also, for conciseness, here we write $V_{2pi} = V_2(P)$ for the $i$th layer etc.

Taking the limit of equation (B-9) at $x = 0$ and substituting equation (B-10) gives

$$A_4(P-SV) = \frac{(\Sigma_i V_{2pi}^2\Delta t_{pi} + \Sigma_i V_{2si}^2\Delta t_{si})^2 - (t_{p0}+t_{s0})[\Sigma_i(H_{pi}+V_{2pi}^4)\Delta t_{pi} + \Sigma_i(H_{si}+V_{2si}^4)\Delta t_{si}]}{(\Sigma_i V_{2pi}^2\Delta t_{pi} + \Sigma_i V_{2si}^2\Delta t_{si})^4}, \quad \text{(B-12)}$$

where all times are now two-way vertical traveltimes. For the pure $P$-$P$ and $SV$-$SV$ reflections, (B-12) reduces (Hake et al., 1984) to:

$$A_4(P) = \frac{(\Sigma_i V_{2pi}^2\Delta t_{pi})^2 - t_{p0}\Sigma_i(H_{pi}+V_{2pi}^4)\Delta t_{pi}}{4(\Sigma_i V_{2pi}^2\Delta t_{pi})^4}, \quad \text{(B-13)}$$

$$A_4(SV) = \frac{(\Sigma_i V_{2si}^2\Delta t_{si})^2 - t_{s0}\Sigma_i(H_{si}+V_{2si}^4)\Delta t_{si}}{4(\Sigma_i V_{2si}^2\Delta t_{si})^4}. \quad \text{(B-14)}$$

Now we can represent the quartic coefficient for the $P$-$SV$ waves, equation (B-12), through the coefficients of the $P$-$P$ and $SV$-$SV$ reflections. First we define quartic parameters that are independent of the (two-way) traveltimes:

$$\tilde{A}_4(P) = A_4(P)t_{p0}^2,$$

$$\tilde{A}_4(SV) = A_4(SV)t_{s0}^2,$$

$$\tilde{A}_4(P-SV) = A_4(P-SV)(t_{p0}+t_{s0})^2/4.$$

Finally, using equations (B-6), (B-8), and (B-12)–(B-14), the desired coefficient is

$$\tilde{A}_4(P-SV) = \tilde{A}_4(P)\frac{t_{p0}}{t_{p0}+t_{s0}}\left[\frac{A_2(P-SV)}{A_2(P)}\right]^4 + \tilde{A}_4(SV)\frac{t_{s0}}{t_{p0}+t_{s0}}\left[\frac{A_2(P-SV)}{A_2(SV)}\right]^4 - \frac{t_{p0}t_{s0}}{(t_{p0}+t_{s0})^2}\frac{A_2^4(P-SV)[A_2(SV)-A_2(P)]^2}{4A_2^2(P)A_2^2(SV)}. \quad \text{(B-15)}$$

Equation (B-15) is valid for layered transversely isotropic models with arbitrary strength of the anisotropies. Unlike $P$ and $SV$ moveouts, the moveout curve for the converted $P$-$SV$ reflection is nonhyberbolic even in a single isotropic layer; the fourth-order coefficient $A_4$ in this case is determined by the last term in equation (B-15).

To express the Taylor series coefficients through the parameters of transversely isotropic media, it is necessary to determine the values of $V_2$ and $H$ for a single layer. Hake et al. (1984) found $V_2$ and $H$ as functions of the elastic moduli $C_{\alpha\beta}$ and density $\rho$. In the present notation, their results for $V_2$ are given as equations (6) and (7) in the main text, and those for $H$ are given by

$$H(P) = 8V_{p0}^4(\varepsilon-\delta)\left(1 + \frac{2\delta}{1-\dfrac{V_{s0}^2}{V_{p0}^2}}\right), \quad \text{(B-16)}$$

$$H(SV) = -8V_{s0}^4\sigma\left(1 + \frac{2\delta}{1-\dfrac{V_{s0}^2}{V_{p0}^2}}\right). \quad \text{(B-17)}$$

For the quartic Taylor series coefficient in a single layer we get, from equations (B-13) and (B-14) for either $P$ or $SV$:

$$A_4 = -\frac{H}{4t_0^2V_2^8}. \quad \text{(B-18)}$$

Then, using equations (B-16)–(B-18), we get the single-layer equations (22) and (24) in the main text. Substitution of $V_2$ and $H$, with subscripts $i$ on all quantities, into equations (B-6)–(B-8), (B-13), and (B-14) permits the calculation of the Taylor series coefficients $A_2$ and $A_4$ for $P$- and $SV$-waves in multilayered media.

Equation (29) of the main text expresses $A_4$($P$ or $SV$) from equations (B-13) and (B-14) through the coefficients $V_{2i}$ and $A_{4i}$ of the individual layers. The Taylor series coefficients for the $P$-$SV$ converted wave, in one or many layers, are given by using these quantities $V_2(A_2)$ and $A_4$ in formulas (B-8) and (B-15).

## APPENDIX C

### DIX-TYPE FORMULAS FOR THE TAYLOR SERIES COEFFICIENTS

Here, we derive interval properties from global parameters by Dix-type differentiation. The Dix (1955) formula makes it possible to recover the interval velocity for any particular layer from the short-spread moveout velocity, in flat-layered isotropic media. In flat-layered transversely isotropic media, the interval velocity $V_{2N}$ for $P$-$P$ or $SV$-$SV$ waves in the $N$-th layer may be recovered similarly, using the short-spread moveout velocities for the reflections from the top $[V_2(N - 1)]$ and bottom $[V_2(N)]$ of the layer, equations (B-6) and (B-7):

$$V_{2N}^2 = \frac{V_2^2(N)t_0(N) - V_2^2(N-1)t_0(N-1)}{t_0(N) - t_0(N-1)}. \tag{C-1}$$

The expressions for the quartic coefficient are more complicated. Equations (B-13) and (B-14) may be rearranged using (B-6) and (B-7), as

$$\frac{1}{t_0(N)}\sum_{i=1}^{N}(V_{2i}^4 + H_i)\Delta t_i = V_2^4(N)[1 - 4A_4(N)t_0^2(N)V_2^4(N)].$$

We denote the right-hand side as

$$F(N) = V_2^4(N)[1 - 4A_4(N)t_0^2(N)V_2^4(N)].$$

$F(N)$ is thus a known function of the Taylor series coefficients for the reflection from the $N$th boundary. Having found $F(N)$ and $F(N - 1)$ using the reflections from the top and bottom of the $N$th layer, we may obtain

$$H_N = \frac{F(N)t_0(N) - F(N-1)t_0(N-1)}{t_0(N) - t_0(N-1)} - V_{2N}^4. \tag{C-2}$$

This expression determines $H_N$ and hence, using equation (B-18), $A_{4N}$. Thus, Dix-type formulas are valid for both the second-order and fourth-order Taylor series coefficients in transversely isotropic media.