Wavefield decomposition for viscoelastic anisotropic media

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ABSTRACT

Separating wave modes on seismic records is an essential step in imaging of multicomponent seismic data. Viscoelastic anisotropic models provide a realistic description of subsurface formations that exhibit anisotropy of velocity and attenuation. However, mode separation has not been extended to viscoelastic anisotropic media yet. Here, we propose an efficient approach to wavefield decomposition that takes velocity and attenuation anisotropy into account. Our algorithm operates in the frequency-wavenumber domain and, therefore, is suitable for general dissipative models. We present exact equations for wavefield decomposition in arbitrarily anisotropic attenuative homogeneous media. Then the proposed approach is applied to viscoelastic constant-Q VTI (transversely isotropic with a vertical symmetry axis) models. Numerical examples demonstrate the accuracy and efficiency of our approach for piecewise-homogeneous media characterized by pronounced velocity and attenuation anisotropy.

INTRODUCTION

Decomposing seismic wavefields into the individual P- and S-modes plays a key role in processing and inversion of multicomponent seismic data. Because different wave modes can overlap in space and time, imaging the entire wavefield in reverse time migration can produce corrupted, unphysical images (e.g., Yan and Sava, 2009; Zhang and McMechan, 2010).

P- and S-waves in elastic isotropic media can be separated by simply applying the divergence and curl operators (e.g., Devaney and Oristaglio, 1986). In contrast, computation of the P-, S₁-, and S₂-modes in elastic anisotropic media can be achieved by wave-mode separation or wave-mode decomposition (the latter method is called “wave-vector decomposition” by Zhang and McMechan, 2010). Both approaches are based on the fact that the particle motions of different plane waves (P, S₁, S₂; S₁ and S₂ are the fast and slow shear waves, respectively) propagating in a fixed phase direction are orthogonal to one another (e.g., Tsvankin, 2012). Wave-mode separation uses the divergence and curl operators, but the phase and amplitude of the separated waves differ from those in the complete (original) wavefield (Dellinger and Eigen, 1990; Yan and Sava, 2009, 2011; Zhou and Wang, 2017; Kaur et al., 2021; Wang et al., 2021). In contrast, wave-mode decomposition can produce the individual modes with the correct phase, amplitude, and polarization properties (Dellinger, 1991; Zhang and McMechan, 2010; Cheng and Fomel, 2014; Sripanich et al., 2017; Wang et al., 2018; Yang et al., 2019; Zhang et al., 2022; Chen and Fomel, 2023).

Field observations and rock-physics measurements demonstrate that the magnitude of attenuation anisotropy can exceed that of velocity anisotropy (e.g., Hosten et al., 1987; Zhu et al., 2007a; Clark et al., 2009; Behura et al., 2012; Zhubayev et al., 2016). Viscoelastic anisotropic models are needed to accurately simulate the angle-dependent velocity and attenuation functions of typical subsurface formations. For example, viscoelastic transversely isotropic (TI) media can describe a sequence of thin isotropic layers or intrinsically TI formations (e.g., Zhu et al., 2007b; Bai and Tsvankin, 2016), whereas more complex viscoelastic orthorhombic models are needed to represent fractured TI or isotropic media, such as shales (e.g., Tsvankin and Grechka, 2011). Existing wave-mode separation or decomposition algorithms are limited to purely elastic models. Whereas the particle motion of harmonic plane waves in elastic anisotropic media is linear, it becomes elliptical and frequency-dependent in the presence of attenuation (Červený and Pšencík, 2006; Carcione, 2014). As a result, existing methods for wave-mode separation or decomposition break down for viscoelastic anisotropic models.

The goal of this paper is to generalize wavefield decomposition for attenuative anisotropic media. First, we derive general equations for wavefield decomposition in viscoelastic arbitrarily anisotropic homogeneous models and outline the mode-decomposition procedure. Then the developed approach is applied to separate P- and SV-waves in viscoelastic VTI media described by the “constant-Q” model.
Numerical examples demonstrate the efficiency of the developed methodology for models composed of homogeneous VTI blocks.

**VISCOELASTIC WAVEFIELD DECOMPOSITION**

The wave equation for a homogeneous viscoelastic anisotropic medium can be written as:

\[
\rho \frac{\partial^2 u}{\partial t^2} = \psi_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} + S_i,
\]

where \( \rho \) is the density, \( u = (u_x, u_y, u_z)^T \) is the particle displacement, \( t \) is time, \( x = (x, y, z)^T \) is the Cartesian coordinate vector, and \( S \) is the volume density of the body forces that describe the source. The quantities \( \psi_{ijkl} \) are the so-called relaxation functions, which are time-dependent, and “\( \odot \)” denotes the Riemann-Stieltjes convolution integral (Gurtin and Sternberg, 1962; Apostol, 1974); the symbol \( \odot \) is explicitly described in Hao and Greenhalgh (2021a, 2021b). The repeated indices are used according to the Einstein summation convention.

Applying the Fourier transform over time and the spatial coordinates to equation 1 and rewriting the result in matrix form, we obtain:

\[
(G - \rho \omega^2 I) \mathbf{\ddot{u}} = \mathbf{\ddot{S}},
\]

where \( G \) is the Christoffel-type matrix given by \( G_{abc} = M_{ijkl} k_i k_j k_l \) are the wavenumber components, and \( M_{ijkl} = M_{ijkl}^\rho - \text{sgn}(f)M_{ijkl}^\eta \) denote the complex stiffness coefficients for the frequency \( f \) (\( M_{ijkl}^\rho \) and \( M_{ijkl}^\eta \) are real-valued); \( i \) is the imaginary unit. The symbol \( \mathbf{\ddot{S}} \) denotes the matrix of the body-force vector per unit volume in the frequency-wavenumber domain.

The eigendecomposition of \( G \) is given by:

\[
G = \Gamma \Lambda \Gamma^{-1},
\]

where \( \Lambda \) and \( \Gamma \) are the eigenvalue and eigenvector matrices of \( G \), respectively:

\[
\Lambda = \begin{pmatrix}
\phi^{(P)} & 0 & 0 \\
0 & \phi^{(S_1)} & 0 \\
0 & 0 & \phi^{(S_2)}
\end{pmatrix},
\]

\[
\Gamma = \begin{pmatrix}
g_x^{(P)} & g_x^{(S_1)} & g_x^{(S_2)} \\
g_y^{(P)} & g_y^{(S_1)} & g_y^{(S_2)} \\
g_z^{(P)} & g_z^{(S_1)} & g_z^{(S_2)}
\end{pmatrix}.
\]

Here, the superscripts “\( P \),” “\( S_1 \),” and “\( S_2 \)” correspond to the P-wave and the fast and slow S-waves, respectively. The unit eigenvectors \( \mathbf{g}^{(P)} \), \( \mathbf{g}^{(S_1)} \), and \( \mathbf{g}^{(S_2)} \) are complex-valued and orthogonal to each other for a fixed phase direction. The eigenvalues and eigenvectors are determined for the Christoffel matrix \( \mathbf{G} = \{ M_{ijkl} \}_{3 \times 3} \) that describes harmonic plane waves in viscoelastic anisotropic media (see Appendix A), where \( \mathbf{n} \) is the unit vector parallel to the wave vector \( \mathbf{k} \).

Using equation 5, we can rewrite equation 2 as:

\[
\{ \mathbf{g}^{(P)} \cdot [(G - \rho \omega^2 I) \mathbf{\ddot{u}}] \} = \{ \mathbf{g}^{(S_1)} \cdot [(G - \rho \omega^2 I) \mathbf{\ddot{u}}] \}
\]

\[
\{ \mathbf{g}^{(S_2)} \cdot [(G - \rho \omega^2 I) \mathbf{\ddot{u}}] \}
\]

\[
= [g_x^{(P)} \cdot \mathbf{\ddot{S}} g_x^{(P)} + [g_x^{(S_1)} \cdot \mathbf{\ddot{S}}] g_x^{(S_1)} + [g_x^{(S_2)} \cdot \mathbf{\ddot{S}}] g_x^{(S_2)},
\]

where “\( \odot \)” denotes the dot product.

Because \( \mathbf{g}^{(P)} \), \( \mathbf{g}^{(S_1)} \), and \( \mathbf{g}^{(S_2)} \) can be viewed as the base vectors of an orthogonal coordinate system, \( \mathbf{\ddot{u}} \) can be described by:

\[
\mathbf{\ddot{u}} = A_1 \mathbf{g}^{(P)} + A_2 \mathbf{g}^{(S_1)} + A_3 \mathbf{g}^{(S_2)},
\]

where \( A_1, A_2, \) and \( A_3 \) are the projections of \( \mathbf{\ddot{u}} \) onto the base vectors.

Next, we demonstrate that \( A_1, A_2, \) and \( A_3 \) represent the amplitudes of \( P \), \( S_1 \)-, and \( S_2 \)-waves, respectively, in the frequency-wavenumber domain. Substituting equations 3 and 7 into equation 6 yields:

\[
(\phi^{(P)} - \rho \omega^2) A_1 \mathbf{g}^{(P)} + (\phi^{(S_1)} - \rho \omega^2) A_2 \mathbf{g}^{(S_1)} + (\phi^{(S_2)} - \rho \omega^2) A_3 \mathbf{g}^{(S_2)}
\]

\[
= (\mathbf{\ddot{u}}) \mathbf{g}^{(P)} + (\mathbf{\ddot{u}}) \mathbf{g}^{(S_1)} + (\mathbf{\ddot{u}}) \mathbf{g}^{(S_2)},
\]

where we took advantage of the fact that \( \mathbf{g}^{(P)} \), \( \mathbf{g}^{(S_1)} \), and \( \mathbf{g}^{(S_2)} \) are orthogonal to one another for a fixed phase direction.

From equation 8, it follows that:

\[
(\phi^{(P)} - \rho \omega^2) A_1 = \mathbf{\ddot{u}} \mathbf{g}^{(P)}.
\]

\[
(\phi^{(S_1)} - \rho \omega^2) A_2 = \mathbf{\ddot{u}} \mathbf{g}^{(S_1)}.
\]

\[
(\phi^{(S_2)} - \rho \omega^2) A_3 = \mathbf{\ddot{u}} \mathbf{g}^{(S_2)}.
\]

Taking into account the physical meaning of equations 4 and 5, equations 9–11 imply that \( A_1, A_2, \) and \( A_3 \) describe the amplitudes of \( P \), \( S_1 \)-, and \( S_2 \)-waves, respectively, in the frequency-wavenumber domain.

Using equation 7 and equations 9–11, the decomposed displacement takes the form:

\[
\mathbf{\ddot{u}}^{(P)} = (\mathbf{g}^{(P)} \cdot \mathbf{\ddot{u}}) \mathbf{g}^{(P)},
\]

\[
\mathbf{\ddot{u}}^{(S_1)} = (\mathbf{g}^{(S_1)} \cdot \mathbf{\ddot{u}}) \mathbf{g}^{(S_1)},
\]

\[
\mathbf{\ddot{u}}^{(S_2)} = (\mathbf{g}^{(S_2)} \cdot \mathbf{\ddot{u}}) \mathbf{g}^{(S_2)},
\]

where \( \mathbf{\ddot{u}}^{(P)} \), \( \mathbf{\ddot{u}}^{(S_1)} \), and \( \mathbf{\ddot{u}}^{(S_2)} \) are the P-, \( S_1 \)-, and \( S_2 \)-wave particle displacement fields in the frequency-wavenumber domain. Equations 12–14 have the same form as those for the elastic particle-displacement decomposition (e.g., Delsing, 1991; Zhang and McMechan, 2010). However, these existing equations are applied in the time domain, whereas equations 12–14 are supposed to be used in the frequency domain.
Finally, the decomposed wavefields in the time-space domain are given by:

\[
\begin{align*}
\mathbf{u}^{(P)} &= \frac{1}{(2\pi)^4} \iiint_{-\infty}^{\infty} (\mathbf{g}^{(P)} \cdot \mathbf{\hat{u}}) \mathbf{g}^{(P)} e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})} \times \mathbf{dk}, \mathbf{dk}, \mathbf{dk}, \mathbf{do}, \\
\mathbf{u}^{(S_1)} &= \frac{1}{(2\pi)^4} \iiint_{-\infty}^{\infty} (\mathbf{g}^{(S_1)} \cdot \mathbf{\hat{u}}) \mathbf{g}^{(S_1)} e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})} \times \mathbf{dk}, \mathbf{dk}, \mathbf{dk}, \mathbf{do}, \\
\mathbf{u}^{(S_2)} &= \frac{1}{(2\pi)^4} \iiint_{-\infty}^{\infty} (\mathbf{g}^{(S_2)} \cdot \mathbf{\hat{u}}) \mathbf{g}^{(S_2)} e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})} \times \mathbf{dk}, \mathbf{dk}, \mathbf{dk}, \mathbf{do}.
\end{align*}
\]

(15)–(17)

It is noteworthy that the above approach to wavefield decomposition can also be used to compute the point-source solution of the viscoelastic anisotropic wave equation. For example, by setting \(\mathbf{S}\) to represent a point force in equations 9–11, we can obtain the corresponding P-, S1- and S2-waves that comprise the Green’s function for a homogeneous medium. Solving equations 9–11 for \(A_1, A_2, A_3\) and substituting \(\mathbf{g}^{(P)} \cdot \mathbf{\hat{u}} = A_1, \mathbf{g}^{(S_1)} \cdot \mathbf{\hat{u}} = A_2, \) and \(\mathbf{g}^{(S_2)} \cdot \mathbf{\hat{u}} = A_3\) into equations 15–17 yields the P-, S1- and S2-wave components of the point-source solution of the viscoelastic wave equation.

The point-source solution in equations 15–17 is equivalent to equation 2.26 of Tsvankin (2012) derived for purely elastic anisotropic media. The derivation here uses the eigendecomposition technique but does not entail integration over the vertical wavenumber \(k_z\). In contrast, Tsvankin (2012) uses Cauchy’s residue theorem to evaluate the integral over \(k_z\) in equations 15–17 and obtain the plane-wave decomposition of the displacement field. Note that our point-source solution (equations 15–17) cannot be computed directly for elastic anisotropic media because of the poles on the integration path.

When the medium is elastic, the eigenvectors \(\mathbf{g}^{(P)}, \mathbf{g}^{(S_1)},\) and \(\mathbf{g}^{(S_2)}\) become real-valued and frequency-independent. As a result, equations 15–17 reduce to:

\[
\begin{align*}
\mathbf{u}^{(P)} &= \frac{1}{(2\pi)^4} \iiint_{-\infty}^{\infty} (\mathbf{g}^{(P)} \cdot \mathbf{\hat{u}}) \mathbf{g}^{(P)} e^{i\mathbf{k} \cdot \mathbf{x}} \mathbf{dk}, \mathbf{dk}, \mathbf{dk}, \\
\mathbf{u}^{(S_1)} &= \frac{1}{(2\pi)^4} \iiint_{-\infty}^{\infty} (\mathbf{g}^{(S_1)} \cdot \mathbf{\hat{u}}) \mathbf{g}^{(S_1)} e^{i\mathbf{k} \cdot \mathbf{x}} \mathbf{dk}, \mathbf{dk}, \mathbf{dk}, \\
\mathbf{u}^{(S_2)} &= \frac{1}{(2\pi)^4} \iiint_{-\infty}^{\infty} (\mathbf{g}^{(S_2)} \cdot \mathbf{\hat{u}}) \mathbf{g}^{(S_2)} e^{i\mathbf{k} \cdot \mathbf{x}} \mathbf{dk}, \mathbf{dk}, \mathbf{dk}.
\end{align*}
\]

(18)–(20)

where \(\mathbf{\hat{u}}\) is the particle displacement in the time-wavenumber domain (the bar above \(\mathbf{u}\) denotes the Fourier transform over the spatial coordinates). Equations 18–20 are proposed by Dellinger (1991) for performing elastic anisotropic wavefield decomposition.

The eigenvectors \(\mathbf{g}^{(P)}, \mathbf{g}^{(S_1)},\) and \(\mathbf{g}^{(S_2)}\) also become real-valued and frequency-independent for viscoelastic isotropic media (Hudson, 1980). In this case, the P-wave eigenvector \(\mathbf{g}^{(P)}\) is parallel to the phase propagation direction \(\mathbf{n} = \mathbf{k}/|k|,\) and the eigenvectors of S1- and S2-waves become orthogonal to it, which means that \(\mathbf{g}^{(S_1)}\) and \(\mathbf{g}^{(S_2)}\) are confined to the plane normal to \(\mathbf{n}.\) Hence, P- and S-waves in viscoelastic isotropic media can be separated by applying the Helmholtz decomposition with the divergence and curl operators (Borcherd, 2009).

The main steps of the proposed wavefield decomposition for homogeneous viscoelastic anisotropic media are summarized in the flowchart below.

Flowchart of viscoelastic anisotropic wavefield decomposition.

1. Transform \(\mathbf{u}(x, t)\) to \(\mathbf{\hat{u}}(k, \omega)\)
   - For each \(\omega \in \Omega\)
     - Compute \(M_{ijk} = M_{ijk}(\omega)\)
   - For each \(k \in K\)
     - Compute \(\mathbf{n} = k/|k|, \mathbf{\hat{G}} = \{M_{ijk} n_j\}_{3 \times 3}\)
     - Compute \(\mathbf{g}^{(i)} \cdot \mathbf{\hat{u}}^{(i)} = (\mathbf{g}^{(i)} \cdot \mathbf{\hat{u}})\mathbf{\hat{g}}^{(i)}\) for \(i = P, S_1, S_2\)
   - End for
   - Transform \(\mathbf{\hat{u}}^{(i)}(k, \omega)\) to \(\mathbf{u}^{(i)}(x, t)\)

The algorithm is designed for homogeneous viscoelastic arbitrarily anisotropic media. For heterogeneous models composed of homogeneous blocks, this wavefield-decomposition approach can be applied to each block separately, with subsequent merging of the separated wave modes in all blocks (see a synthetic example below).

Note that the above algorithm may fail to separate S1- and S2-waves near shear-wave singularities, because the eigenvectors of both S-waves at singularity points are not defined (Dellinger, 1991; Tsvankin and Grechka, 2011).

APPLICATION TO CONSTANT-Q VISCOELASTIC VTI MEDIA

In this section, the proposed wavefield-decomposition approach is implemented for constant-Q viscoelastic VTI media discussed in detail by Hao and Tsvankin (2023).

P- and SV-wave eigenvectors

The complex stiffness matrix for attenuative VTI media is given by:

\[
\mathbf{M} = \begin{pmatrix}
M_{11} & M_{12} & M_{13} & 0 & 0 & 0 \\
M_{12} & M_{11} & M_{13} & 0 & 0 & 0 \\
M_{13} & M_{13} & M_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & M_{35} & 0 & 0 \\
0 & 0 & 0 & 0 & M_{35} & 0 \\
0 & 0 & 0 & 0 & 0 & M_{66}
\end{pmatrix}.
\]
For Kjartansson’s (1979) constant-\(Q\) model, the nonzero independent elements of \(M\) in equation 21 are expressed as:

\[
M_{ij} = M_{ij}^{(0)} (\frac{-i f}{f_0})^{2n_j},
\]

where \(Q_{ij} = \frac{M_{ij}}{M_{ij}^{(0)}}\) (\(Q_{ij}\) are frequency-independent), \(f_0\) is the reference frequency, and \(M_{ij}^{(0)}\) denote the real parts of \(M_{ij}\) at \(f_0\); \(M_{ij}^{(0)} = \text{Re}(M_{ij})_{r=f_0}\).

Viscoelastic VTI media can be conveniently described by the Thomsen and Thomsen-type parameters defined at the reference frequency (Zhu and Tsvankin, 2006; Tsvankin and Grechka, 2011). The definitions of these parameters are given in Appendix B. Hao and Tsvankin (2023) show that these parameters and the attenuation coefficients for the constant-\(Q\) model defined by equations 22 and 24 actually vary with frequency, although this frequency dependence is pronounced only for strong attenuation.

Using the Christoffel matrix for VTI media (e.g., Tsvankin, 2012), we obtain the eigenvectors for P- and SV-waves in the \([x,z]\)-plane. The nonzero (in-plane) components of the P-wave eigenvector are:

\[
g^{(P)}_x = \text{sgn}(n_x) \left[ \frac{M_{55}n_x^2 + M_{33}n_z^2 - \rho v_p^2}{(M_{11} + M_{55})n_x^2 + (M_{33} + M_{55})n_z^2 - 2
\rho v_p^2} \right]^{1/2},
\]

\[
g^{(P)}_z = \text{sgn}(n_z) \left[ \frac{M_{11}n_x^2 + M_{55}n_z^2 - \rho v_p^2}{(M_{11} + M_{55})n_x^2 + (M_{33} + M_{55})n_z^2 - 2
\rho v_p^2} \right]^{1/2}.
\]

The SV-wave eigenvector is described by:

\[
g^{(SV)}_x = \text{sgn}(n_x) \left[ \frac{M_{55}n_z^2 + M_{33}n_x^2 - \rho v_S^2}{(M_{11} + M_{55})n_z^2 + (M_{33} + M_{55})n_x^2 - 2
\rho v_S^2} \right]^{1/2},
\]

\[
g^{(SV)}_z = -\text{sgn}(n_z) \left[ \frac{M_{11}n_z^2 + M_{55}n_x^2 - \rho v_S^2}{(M_{11} + M_{55})n_z^2 + (M_{33} + M_{55})n_x^2 - 2
\rho v_S^2} \right]^{1/2},
\]

where \(v_p\) and \(v_S\) are the P- and SV-wave complex phase velocities:

\[
2\rho v_{P,SV} = (M_{11} + M_{55})n_x^2 + (M_{33} + M_{55})n_z^2 \pm \sqrt{[(M_{11} - M_{55})n_x^2 - (M_{33} - M_{55})n_z^2]^2 + 4(M_{13} + M_{55})n_x^2n_z^2};
\]

Numerical examples

According to Hao and Greenhalgh (2021a) and Hao et al. (2022), the wave equation for constant-\(Q\) VTI models (given in Appendix C) can be expressed as a set of PDEs by using the weighting function method (Hao and Greenhalgh, 2021b), which explicitly involves the quality-factor elements \(Q_{ij}\). We use the finite-difference method to solve that wave equation and model the in-plane-polarized wavefield composed of P- and SV-waves. The source in all examples below represents a vertical force with the time dependence defined by a 50 Hz Ricker wavelet.

First, we test the algorithm on a homogeneous constant-\(Q\) VTI model (see equation 21), with the wavefield excited by the source located at \(x = z = 1\) km. Figure 1 shows a snapshot of the complete wavefield. Application of the elastic anisotropic wavefield-decomposition approach (Dellinger and Eigen, 1990; Zhang and McMechan, 2010) produces what are supposed to be the P- and SV-wave snapshots in Figures 2 and 3. However, as illustrated in Figures 4, 5, and 6, the “P-waveforms” separated by the elastic wavefield-decomposition algorithm include the SV-wave residuals. The P-wave residuals on the separated SV-snapshots are invisible because of the large magnitude of the SV-arrivals. Although the superposition of the P- and SV-waveforms yields the original complete wavefield, it is evident that the elastic decomposition algorithm cannot adequately separate the modes in the presence of attenuation.

In contrast, the approach proposed here accurately separates the P- and SV-waves in this viscoelastic anisotropic medium (Figures 7 and 8). As demonstrated in Figures 9, 10, and 11,
Figure 2. Snapshots of the P-wave displacement components (a) $u_x$ and (b) $u_z$ at $t = 0.2$ s obtained by the elastic wavefield-decomposition method from the complete wavefield in Figure 1. The amplitudes on plots (a) and (b) are displayed on the same scale as in Figure 1. The symbols A, B, and C are observation points located at a distance of 0.75 km from the source O. The source-receiver vectors OA, OB, and OC form angles of 60°, 45°, and 30°, respectively, with the x-axis.

Figure 3. Same as Figure 2 but for the SV-wave displacement components (a) $u_x$ and (b) $u_z$. The amplitude scale is different from that in Figure 1 to make the residual P-wave more visible.

Figure 4. Complete and decomposed displacement components (a) $u_x$ and (b) $u_z$ at point A (see Figures 2 and 3). The P- and SV-modes are obtained by the elastic wavefield-decomposition algorithm. The sum of the decomposed P- and SV-waves is denoted by "P + SV", and "Exact" denotes the original seismogram. The blue dashed ellipses mark the SV-wave residuals on the P-wave seismograms.

Figure 5. Same as Figure 4 but at point B.

Figure 6. Same as Figure 4 but at point C.
the P- and SV-waveforms are completely separated by the viscoelastic wavefield-decomposition algorithm, and their superposition closely matches the original complete waveforms. The model in Figure 1 has an uncommonly large positive value of the parameter \( \sigma \equiv (V_{P0}/V_{S0})^2(e - \delta) \), which is primarily responsible for cusps (triplications) on SV-wavefronts (e.g., Tsvankin, 2012). As a result, we observe a large SV-wave cusp at oblique propagation directions. Nevertheless, the proposed algorithm accurately separates the P-wave from the triplicated SV-wave.

Next, we consider a model that includes two constant-\( Q \) VTI halfspaces with the parameters listed in Table 1. The snapshot of the complete wavefield at \( t = 0.18 \) s is shown in Figure 12. The P- and SV-waves overlap in time and space but, as illustrated in Figures 13 and 14, the P- and SV-waves in both halfspaces can be reliably separated by the developed viscoelastic wavefield-decomposition algorithm.

Finally, we test the proposed approach on a more complex constant-\( Q \) VTI model composed of five homogeneous blocks.
(Table 2 and Figure 15). As outlined in the section on methodology, the decomposition is applied to each block separately with subsequent merging of the separated modes. For each block, we first mute the wavefields outside it, then compute the Fourier transforms of the resulting wavefields and apply our decomposition method (designed for a homogeneous medium) to generate the separated modes. Finally, we merge the same-mode wavefields in all blocks to produce the complete decomposed wavefields.
The particle displacement of a harmonic plane wave in viscoelastic media can be defined as:

$$ u = A \hat{g} e^{-\imath \omega (t - \mathbf{k} \mathbf{x})}, $$

(32)

where $A$ is the amplitude, $\hat{g}$ is the polarization vector, and $\mathbf{k} = \mathbf{k}^R + i\mathbf{k}^I$ is the complex wavenumber vector, where $\mathbf{k}^R$ and $\mathbf{k}^I$ are its real and imaginary parts, respectively. The magnitude $|u|$ of the displacement can become infinitely large in the direction opposite to that of $\mathbf{k}^I$, which is unphysical. Although the P-, S1-, and S2-wave polarization vectors are orthogonal to each other for a specified phase direction, viscoelastic plane waves and their superposition do not describe point-source radiation and, therefore, cannot be used for wavefield decomposition. In contrast, plane waves in elastic media have a constant amplitude and are suitable for performing wavefield decomposition.

Our wavefield-decomposition algorithm is applied in the frequency-wavenumber domain. The decomposition involves the Fourier transforms over time and spatial coordinates and the inverse Fourier transforms over frequency and wavenumber. The elastic counterpart of our algorithm can be implemented in the time domain and requires only the direct and inverse Fourier transforms over spatial coordinates (equations 18–20). Clearly, the proposed algorithm requires more computational resources than elastic wavefield decomposition.

Our methodology is designed for homogeneous media but can be used for heterogeneous models composed of homogeneous blocks. The proposed decomposition is also expected to remain sufficiently accurate for weakly heterogeneous models with small spatial parameter gradients. For gridded heterogeneous models, our algorithm can be applied to each grid point separately, but that reduces computational efficiency. For elastic anisotropic media, mode separation can be efficiently implemented using the spatial-filter method (e.g., Yan and Sava, 2009, 2011) or the low-rank approximation method (e.g., Cheng and Fomel, 2014). These techniques could be potentially extended to attenuative media to reduce the computational cost of wavefield decomposition.

All numerical examples in the paper are for viscoelastic VTI media, in which SV- and SH-waves satisfy separate Christoffel equations (e.g., Tsvankin, 2012) and can be easily separated. For lower anisotropic symmetries (e.g., orthorhombic), the S1- and S2-modes are coupled near shear-wave point conical singularities, where the S-wave polarization vectors are undefined (e.g., Dellinger, 1991). Therefore, our algorithm should not be applied close to point singularities in low-symmetry models.

**CONCLUSION**

We developed a wavefield-decomposition method for viscoelastic anisotropic media based on the eigendecomposition of the Christoffel-type matrix in the frequency-wavenumber domain. The wave modes (P, S1, and S2) are separated by projecting the complete displacement field onto the three orthogonal eigenvectors obtained from this eigendecomposition. Because the algorithm operates in the frequency domain, it is suitable for completely general dissipative anisotropic models. The developed method is exact for homogeneous media, which is confirmed by a synthetic example for a strongly anisotropic model with a wide SV-wave triplication (cusp). The algorithm can be also applied to a medium composed of homogeneous blocks by processing each block separately. Then the decomposed
wavefields are computed by combining the results obtained for the individual blocks.

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DATA AND MATERIALS AVAILABILITY

Data associated with this research are available and can be obtained by contacting the corresponding author.

APPENDIX A

EIGENVALUES AND EIGENVECTORS OF THE CHRISTOFFEL-TYPE MATRIX

Here, we present the eigenvalues and eigenvectors of the Christoffel-type matrix $G$ (equation 3) for viscoelastic anisotropic media. According to Červený and Psceník (2005) and Carcione (2014), the displacement of a harmonic plane wave in a viscoelastic anisotropic medium is defined as:

$$u_i = A \tilde{g}_i e^{-i\omega (r - \frac{v_i}{c})},$$  (A-1)

where $A$ is the amplitude, $\tilde{g}_i$ are the components of the polarization vector, and $n$ is the unit vector of the wave-propagation (phase) direction. The complex phase velocity is denoted by $v$, and is related to the complex slowness components $p_i$ as $p_i = n_i/v$. The plane-wave attenuation is described by the imaginary part of $v$.

Substituting the plane-wave displacement from equation A-1 into the viscoelastic anisotropic wave equation 1 without the source term $S_i$, we obtain the complex Christoffel equation (Hudson, 1980; Červený and Psceník, 2005):

$$(\tilde{G} - \rho v^2 I)\tilde{g} = 0,$$  (A-2)

where $\tilde{G}$ is the Christoffel matrix defined as $\tilde{G}_{ik} = M_{jkl}n_in_l$, and $\rho v^2$ is the eigenvalue of $\tilde{G}$. For general anisotropic media, there are three (usually distinct) eigenvalues corresponding to P-, S1-, and S2-waves.

As follows from equation 3, the eigenvalues and eigenvectors of the Christoffel-type matrix $G$ can be found from:

$$(G - \phi I)g = 0,$$  (A-3)

where $G_{ik} = M_{jkl}k_jk_l$ and $\phi$ is an eigenvalue of $G$. Note that $k_i$ here do not represent the complex wavenumber components but are the real wavenumbers in the Fourier transform.

Equations A-2 and A-3 have the same form. Comparing the above expressions for $G$ and $\tilde{G}$, we obtain:

$$\phi = \rho v^2 |n|^2,$$  (A-4)

$$g = \tilde{g}.$$  (A-5)

with

$$n = k/|k|.$$  (A-6)

According to equations A-4–A-6, the eigenvalues and eigenvectors of $G$ can be expressed through the complex velocity and polarization vectors of harmonic plane waves.

APPENDIX B

PARAMETERIZATION FOR VISCOELASTIC VTI MEDIA

Here, we present the Thomsen-type parameterization for viscoelastic VTI media. The complex stiffness coefficients in the Voigt notation are denoted by $M_{ij} = M^R_{ij} - i\text{sgn}(f)M^I_{ij}$, where $f$ is the frequency and $i$ is the imaginary unit. The minus sign in front of $\text{sgn}(f)M^I_{ij}$ follows from the definition of the Fourier transform used by Červený (2001) and Červený and Psceník (2009). The $Q$-matrix is defined as $Q_{ij} = M^R_{ij}/M^I_{ij}$, where there is no summation over $i$ and $j$.

Viscoelastic VTI media can be conveniently described by the Thomsen-type velocity parameters (Thomsen, 1986; Tsvankin, 2012) and the Thomsen-type attenuation parameters (Zhu and Tsvankin, 2006; Tsvankin and Grechka, 2011). The velocity parameters are defined in the reference nonattenuative VTI medium. The attenuation parameters can be used to concisely describe the normalized phase attenuation coefficients for P-, SV-, and SH-waves. Following Tsvankin (2012) and Tsvankin and Grechka (2011), the full parameter set is defined below.

The parameter $V_{P0}$ is the vertical velocity of P-waves:

$$V_{P0} \equiv \sqrt{\frac{M^R_{33}}{\rho}},$$  (B-1)

where $\rho$ denotes density.

The parameter $V_{S0}$ is the vertical velocity of S-waves:

$$V_{S0} \equiv \sqrt{\frac{M^R_{55}}{\rho}}.$$  (B-2)

The parameter $\epsilon$ is approximately equal to the fractional difference between the horizontal and vertical velocities of P-waves:

$$\epsilon \equiv \frac{M^R_{33} - M^R_{11}}{2M^R_{33}}.$$  (B-3)

The parameter $\delta$ determines the second derivative of the P-wave phase velocity at vertical incidence and is given by:

$$\delta \equiv \frac{(M^R_{33} + M^R_{55})^2 - (M^R_{33} - M^R_{55})^2}{2M^R_{33}(M^R_{33} - M^R_{55})}.$$  (B-4)

The parameter $\gamma$ is approximately equal to the fractional difference between the horizontal and vertical velocities of SH-waves:

$$\gamma \equiv \frac{M^R_{66} - M^R_{55}}{2M^R_{55}}.$$  (B-5)
The parameter $A_{P0}$ is the vertical attenuation coefficient of P-waves:

$$A_{P0} \equiv Q_{33} \left( \sqrt{1 + \frac{1}{Q_{33}^2}} - 1 \right) \approx \frac{1}{2Q_{33}}. \quad (B-6)$$

The parameter $A_{S0}$ is the vertical attenuation coefficient of S-waves:

$$A_{S0} \equiv Q_{55} \left( \sqrt{1 + \frac{1}{Q_{55}^2}} - 1 \right) \approx \frac{1}{2Q_{55}}. \quad (B-7)$$

The parameter $\epsilon_Q$ is close to the fractional difference between the horizontal and vertical attenuation coefficients of P-waves:

$$\epsilon_Q \equiv \frac{Q_{33} - Q_{11}}{Q_{11}}. \quad (B-8)$$

The parameter $\delta_Q$ controls the second derivative of the P-wave attenuation coefficient at incident angle and is expressed as:

$$\delta_Q \equiv \left( \frac{Q_{33} - Q_{55}}{Q_{55}} \right) \left( \frac{M_{55}^R}{M_{33}^R} + \frac{M_{33}^R}{M_{55}^R} \right) \left( \frac{M_{55}^R + M_{33}^R}{M_{33}^R - M_{55}^R} \right) - 2 \left( \frac{Q_{33} - Q_{13}}{Q_{13}} \right) \left( \frac{M_{13}^R}{M_{33}^R} + \frac{M_{13}^R}{M_{55}^R} \right) \left( \frac{M_{55}^R + M_{33}^R}{M_{33}^R - M_{55}^R} \right). \quad (B-9)$$

The parameter $\gamma_Q$ is close to the fractional difference between the horizontal and vertical attenuation coefficients of SH-waves:

$$\gamma_Q \equiv \frac{Q_{55} - Q_{66}}{Q_{66}}. \quad (B-10)$$

**APPENDIX C**

**WAVE EQUATIONS FOR THE CONSTANT-Q VISCOELASTIC VTI MODEL**

Here, we present the “nearly constant-Q” viscoelastic VTI wave equations, which are used in the section “Numerical examples” to synthesize 2D wavefields. According to Hao and Greenhalgh (2021a) and Hao et al. (2022), the 2D wave equations for the constant-Q viscoelastic VTI model (equation 22) can be obtained by the weighting-function method (Hao and Greenhalgh, 2021b):

$$\frac{\partial^2 u_x}{\partial t^2} = b^{(0)}_{11} \frac{\partial^2 u_x}{\partial x^2} + b^{(0)}_{33} \frac{\partial^2 u_x}{\partial z^2} + (b^{(0)}_{13} + b^{(0)}_{35}) \frac{\partial^2 u_z}{\partial x \partial z} - \sum_{\ell=1}^L r^{(\ell)}_x S_x,$$

$$\frac{\partial^2 u_z}{\partial t^2} = b^{(0)}_{55} \frac{\partial^2 u_z}{\partial x^2} + b^{(0)}_{33} \frac{\partial^2 u_z}{\partial z^2} + (b^{(0)}_{13} + b^{(0)}_{35}) \frac{\partial^2 u_x}{\partial x \partial z} - \sum_{\ell=1}^L r^{(\ell)}_z S_z,$$  

with

$$\frac{\partial r^{(\ell)}_x}{\partial t} = s^{(\ell)} \left[ b^{(1)}_{11} \frac{\partial^2 u_x}{\partial x^2} + b^{(1)}_{35} \frac{\partial^2 u_x}{\partial z^2} + (b^{(1)}_{13} + b^{(1)}_{35}) \frac{\partial^2 u_z}{\partial x \partial z} - \sum_{\ell=1}^L r^{(\ell)}_x \right] - \frac{1}{\tau^{(\ell)}_x} r^{(\ell)}_x,$$

$$\frac{\partial r^{(\ell)}_z}{\partial t} = s^{(\ell)} \left[ b^{(1)}_{55} \frac{\partial^2 u_z}{\partial x^2} + b^{(1)}_{33} \frac{\partial^2 u_z}{\partial z^2} + (b^{(1)}_{13} + b^{(1)}_{35}) \frac{\partial^2 u_x}{\partial x \partial z} - \sum_{\ell=1}^L r^{(\ell)}_z \right] - \frac{1}{\tau^{(\ell)}_z} r^{(\ell)}_z.$$  

Here $b^{(0)}_{ij}$, $b^{(1)}_{ij}$, and $b^{(2)}_{ij}$ are given by:

$$b^{(0)}_{ij} = \frac{M_{ij}}{\rho} \left( 1 + g \frac{Q_{ij}}{2Q_{ij}} \right),$$

$$b^{(1)}_{ij} = \frac{M_{ij}}{\rho Q_{ij}} \left( 1 + g \frac{Q_{ij}}{Q_{ij}} \right),$$

$$b^{(2)}_{ij} = \frac{M_{ij}}{2\rho Q_{ij}^2}$$

and

$$g = \sum_{\ell=1}^L \frac{r^{(\ell)} \tau^{(\ell)}_x - 1}{1 + \omega_Q^2 \left( \frac{s^{(\ell)}}{\tau^{(\ell)}} \right)^2},$$

$$s^{(\ell)} = \frac{1}{\tau^{(\ell)}_x} \left( \frac{r^{(\ell)}_x}{\tau^{(\ell)}_x} - 1 \right).$$

The quantities $r^{(\ell)}_x$ and $r^{(\ell)}_z$ are the strain and stress coefficients of the $\ell$th standard-linear-solid element in the weighting function (Hao and Greenhalgh, 2021a, 2021b). These coefficients are independent of the medium parameters and are defined for a specified frequency range of interest. Their optimized values for various frequency ranges can be found in Hao and Greenhalgh (2021b).

**REFERENCES**

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