Use of Cramer's Rule to Solve Simultaneous Algebraic Equations

Cramer's rule provides a methodology to solve systems of simultaneous algebraic equations. It is most convenient when dealing with systems of 2 or 3 equations. It can be used for more than 3 equations, but since it involves the calculation of determinants, there are other solution methods that are more convenient.

Cramer's Rule for Systems of N Equations

Let's assume we have a system of N simultaneous algebraic equations with N unknowns x_1, x_2, ..., x_N where the equations are in the form:

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,N}x_N = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,N}x_N = b_2$$

and so forth to:

$$a_{N,1}x_1 + a_{N,2}x_2 + \cdots + a_{N,N}x_N = b_N$$

This system of equations can be put into matrix form as:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$

Cramer's rule lets us find the value of each unknown as the ratio of two determinants. In the denominator the determinant is made up of the coefficient matrix (with all of the $a_{i,j}$ terms). In the numerator it is the determinant of the coefficient matrix except the values in the column associated with a particular unknown (the first column for x_1, the second column for x_2, and so forth) is replaced with the column of values from the right-hand side vector.

So, x_1 can be calculated from:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{1,2} & \cdots & a_{1,N} \\ b_2 & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ b_N & a_{N,2} & \cdots & a_{N,N} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{vmatrix}}$$

x_2 can be calculated from:

$$x_2 = \frac{\begin{vmatrix} a_{1,1} & b_1 & \cdots & a_{1,N} \\ a_{2,1} & b_2 & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & b_N & \cdots & a_{N,N} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{vmatrix}}$$

and so forth. Notice that the denominator is the same for each unknown.

Cramer's Rule for Systems of 2 or 3 Equations

Determinants can be tedious to calculate and are most easily done for systems of 2 or 3 equations. For the following system of 2 equations:

$$\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

the determinants can be calculated directly from the definition (i.e., product of the diagonal terms minus the product of the off-diagonal terms):

$$x_1 = \frac{\begin{vmatrix} 10 & 1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{10 \cdot 1 - 0 \cdot 1}{1 \cdot 1 - (-2) \cdot 1} = \frac{10}{3}$$

$$x_1 = \frac{\begin{vmatrix} 1 & 10 \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{1\cdot 0-(-2)\cdot 10}{1\cdot 1-(-2)\cdot 1} = \frac{20}{3}$$

For systems of three equations the determinants can be calculated in a relatively simple manner using augmented arrays. The augmented array looks like this:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix}$$

and you would multiply all of the diagonals together, adding the ones from left to right & subtracting those from right to left. So:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \left(a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2}\right) - \left(a_{3,1}a_{2,2}a_{1,3} + a_{3,2}a_{2,3}a_{1,1} + a_{3,3}a_{2,1}a_{1,2}\right)$$

For the following system of 3 equations:

$$\begin{bmatrix} 1 & -1 & 2 \\ 5 & 2 & 6 \\ -3 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

the determinant for the coefficient matrix (that will be used in the denominator of all Cramer's Rule equations) is:

$$\begin{aligned} \begin{vmatrix} 1 & -1 & 2 \\ 5 & 2 & 6 \\ -3 & 4 & 3 \end{vmatrix} &= \left[1\cdot 2\cdot 3 + (-1)\cdot 6\cdot (-3) + 2\cdot 5\cdot 4\right] - \left[(-3)\cdot 2\cdot 2 + 5\cdot (-1)\cdot 3 + 1\cdot 4\cdot 6\right] \\ &= 6 + 18 + 40 + 12 + 15 - 24 \\ &= 67 \end{aligned}$$

So, the first unknown is:

$$x_1 = \frac{\begin{vmatrix} 10 & -1 & 2 \\ 20 & 2 & 6 \\ 30 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 2 \\ 5 & 2 & 6 \\ -3 & 4 & 3 \end{vmatrix}}$$

$$= \frac{10 \cdot 2 \cdot 3 + (-1) \cdot 6 \cdot 30 + 2 \cdot 20 \cdot 4 - 30 \cdot 2 \cdot 2 - 20 \cdot (-1) \cdot 3 - 10 \cdot 4 \cdot 6}{67}$$

$$= \frac{60 - 180 + 160 - 120 + 60 - 240}{67}$$

$$= -\frac{260}{67}$$

and the other two unknowns can be calculated similarly.