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FRACTIONAL INTEGRABLE NONLINEAR SYSTEMS

by

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ABSTRACT

Nonlinear equations and fractional calculus have become important mathematical descriptions of physical applications. We provide context for the intersection of these two mathematical theories with our discovery of integrable, i.e., exactly solvable, fractional nonlinear evolution equations. Using a general method which can be applied to any integrable system with sufficient structure, we derive the first known fractional integrable equations with nonlocal fractional operators, the fractional Korteweg-deVries and fractional nonlinear Schrödinger equations. For these equations, we find the general solution for decaying initial data in terms of a set of linear integral equations. Like other integrable equations, these fractional integrable equations have solitonic solutions, an infinite number of conservation laws, and elastic soliton-soliton interactions. The soliton solutions to these equations have velocities related to their amplitudes by a power law, a simple physical prediction common to these fractional integrable equations known as anomalous dispersion.

The method of finding these fractional integrable equations involves three key mathematical ingredients: power law dispersion, completeness of squared eigenfunctions, and the inverse scattering transform. All of these elements together allow us to define the nonlinear fractional operators underlying these fractional integrable equations through spectral theory, just as Fourier transforms can be used to define fractional derivatives. If any integrable system admits these three structural elements, it will have fractional integrable equations associated to it.

Once these fractional integrable equations are found, we solve them using the inverse scattering transform. The inverse scattering transform is an analytical solution method that linearizes certain nonlinear evolution equations; such equations are called integrable. The method associates an integrable equations to a scattering problem, e.g., the time-independent Schrödinger equation to solve the Korteweg-deVries equation. The solutions to this scattering problem are used to map the nonlinear equation into scattering space, where time evolution is simple. Then, recovering the solution to the nonlinear equation in physical space is performed by inverse scattering which involves solving a set of linear integral equations. The solutions to these integrable problems have radiation and N-soliton components where the solitons have elastic collisions. We linearize the fractional Korteweg-deVries by association to the time-independent Schrödinger equation and the nonlinear Schrödinger equations by the Ablowitz-Kaup-Newell-Segur scattering system.

We then apply this method to derive fractional extensions to the modified Korteweg-deVries, sine-Gordon, and sinh-Gordon equations, reviewing the inverse scattering theory in detail required to

define and solve these equations. The scattering equation associated to these three fractional integrable equations is a scalar reduction of the Ablowitz-Kaup-Newell-Segur scattering system given a certain symmetry condition. We show how completeness for the Ablowitz-Kaup-Newell-Segur scattering system reduces to completeness for the scalar scattering problem and use this to define the relevant nonlinear fractional operators. Then, using this completeness relation, we verify explicitly that the one-soliton solutions to these equations are indeed solutions. As with the fractional Korteweg-deVries and nonlinear Schrödinger equations, these soliton solutions exhibit anomalous dispersion.

We then showed that this method can be applied to discrete systems by defining and solving the fractional integrable discrete nonlinear Schrödinger equation, whose solution is defined on a lattice, i.e., on discrete points on the real line, but still depends continuously on time. As with the other fractional equations, we demonstrated how the three key mathematical ingredients lead to an explicit form for the equation and how the inverse scattering transform can be used to linearize the problem. However, unlike the continuous fractional integrable equations, the one-soliton solution admits a peak velocity related to its amplitude in a much more complicated manner than in anomalous dispersion, allowing for potentially unexpected behavior. In particular, we demonstrate that the velocity can exhibit a turning point where it switches from increasing to decreasing with the fractional parameter at a certain value of the fractional parameter.

We also show how some of the characteristics of these fractional integrable equations reach beyond integrable equations by comparing this discrete integrable equation to the fractional averaged discrete nonlinear Schrödiner equation. This equation is closely related to its integrable counterpart, but has a much simpler mathematical form. Using a Fourier split step method, we numerically integrate the fractional averaged equation and find solitary wave solutions. By studying the emitted radiation, peak position, averaged amplitude, velocity, and form of the solitary waves, we demonstrate that these waves have similar characteristics to the integrable solitons and that these similarities are accentuated for positive fractional parameter and small amplitude waves.

This work invites further research into fractional integrable and nonintegrable nonlinear system and exploration of their many potential applications.

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CHAPTER 1

INTRODUCTION

Our discovery of fractional integrable nonlinear evolution equations connects fractional calculus and nonlinear dynamics in a novel manner. This new class of equations is built on the inverse scattering transformation (IST) and the spectral theory of operators, fundamental ideas from nonlinear dynamics and fractional calculus, in a general way that can be applied to any integrable system with suitable structure. We have so far applied it to the Ablowitz-Kaup-Newell-Segur (AKNS) scattering problem to derive fractional extensions the Korteweg-deVries (KdV), Nonlinear Schrödinger (NLS), modified KdV (mKdV), sine Gordon (sineG), and sinh Gordon (sinhG) equations and the Ablowitz Ladik (AL) scattering problem to derive the fractional integrable discrete NLS (fIDNLS) equation [1–3]. These are the first known fractional integrable nonlinear evolution equations with smooth solutions (with non-local fractional operators). As with other integrable equations, fractional integrable equations have surprising mathematical structure that yields important physical predictions: an infinite set of conservation laws; localized wave solutions which propagate without dispersing, *solitons*; and a general solution expressible in terms of a set of linear integral equations. Further, these fractional integrable equations commonly predict solitons with *anomalous dispersion*; these solitons' velocities are related to their amplitudes by power laws.

Originally, fractional calculus was created to define integrals and derivatives of non-integer order. This theory yielded integral forms of fractional operators such as the Riemann-Liouville and Caputo fractional derivatives, fractional generalizations of the first order derivative [ref]. Fractional calculus was connected to Fourier transforms when Riesz defined the Riesz fractional derivative [4], a fractional generalization of the negative second derivative. Fourier transforms diagonalize fractional derivatives, i.e., in Fourier space fractional derivatives are equivalent to multiplying by a function of the wave-number k; this function is called a *Fourier symbol*. The Fourier symbols of the Riemann-Liouville and Caputo fractional derivatives are both $(ik)^{1+\epsilon}$, while the Fourier symbol for the Riesz fractional derivative is $|k|^{2(1+\epsilon)}$. Thus, the Riesz fractional derivative can be written as $(-\Delta)^{1+\epsilon}f(x) = \mathcal{F}^{-1}(|k|^{2(1+\epsilon)}\mathcal{F}f)$ where $\mathcal{F}$ is the Fourier transform, $\mathcal{F}^{-1}$ is the inverse Fourier transform, and $|k|^{2(1+\epsilon)}$ for $k \in \mathbb{R}$ is the spectrum of the operator. We take this representation in terms of Fourier transforms to be the definition of the Riesz fractional derivative. Here, $0 < \epsilon < 1$ characterizes how far away the fractional derivative is from the integer derivative it is associated to — at $\epsilon = 0$.

Defining linear fractional derivatives using Fourier transforms can be generalized to define fractional versions of more general operators using spectral theory. Spectral theory describes how one can compute the function of an operator, known as a *functional calculus*, using completeness of the operator's eigenfunctions. Specifically, if we have an operator H with eigenfunctions e_k and eigenvalues λ_k, then $\gamma(H)$ acting on e_k is defined by

$$\gamma(H)e_k = \gamma(\lambda_k)e_k. \tag{1.1}$$

The domain of $\gamma(H)$ can be extended to any suitable regular function h if the eigenfunctions are complete, i.e., if we can write h as a linear combination of the e_k's. Notice that the eigenfunctions of $\gamma(H)$ are the same as H, while the eigenvalues of $\gamma(H)$ are $\gamma(\lambda_k)$; taking a function of the operator only changes its spectrum. If we put $\gamma(H) = H^{1+\epsilon}$, we obtain a fractional operator associated to H. For example, if we have $H = -\Delta$, then $\gamma(H) = (-\Delta)^{1+\epsilon}$ is the Riesz fractional derivative introduced in terms of Fourier transforms earlier. The power of this framework is in its generality; we use it because it can be applied to the nonlinear operators in inverse scattering theory to define their corresponding fractional operators.

Just as spectral theory for linear differential operators is described in terms of Fourier transforms, the spectral theory for certain nonlinear operators is given by the Inverse Scattering Transform (IST). The IST is a nonlinear generalization of Fourier transforms which solves nonlinear evolution equations like the well known KdV and NLS equations. Equations solvable by the IST are called *integrable*. The IST solves integrable equations by mapping their initial conditions into *scattering space* using direct scattering, evolving these in time to obtain solutions in scattering space, and then recovering the solutions in physical space using inverse scattering. This is done by taking the solution to the integrable equation to be a potential in a linear eigenvalue problem and then deriving how plane waves are scattered off this potential. Using inverse scattering theory, researchers have developed *completeness of squared eigenfunctions*, which allows a function to be represented as a linear combination of squared eigenfunctions — the eigenfunctions of the nonlinear operators in integrable equations. Completeness of squared eigenfunctions allows us to define fractional versions of these nonlinear operators using spectral theory. For example, Sachs developed completeness of squared eigenfunctions for the KdV equation in 1983 [5], Kaup constructed one for NLS equation in 1976 [6], and Gerdjikov and Ivanov found completeness for the integrable discrete NLS (IDNLS) equation in 1984 [7].

Direct and inverse scattering methods were first applied to solve the KdV equation with decaying initial data in 1967 [8] shortly after the discovery of solitonic solutions in 1965 [9]. This was done by associating the KdV equation to the time-independent Schrödinger equation and represented the first example of a solution method to a higher-order nonlinear dispersive wave equation. These equations and solution methods were shown to be connected when Ablowitz, Kaup, Newell, and Segur, in 1974, discovered that scattering could be applied to a whole family of nonlinear evolution equations which contains the KdV and NLS equations in addition to many others like the mKdV, sineG, and sinhG equations [10]. They called the method the IST and demonstrated that it is a nonlinear generalization of Fourier transforms. Many researchers have sense found other integrable equations including multi-dimensional and discrete systems; inverse scattering has grown to be a vast field of study [11–15].

Although integrable equations are rare in the space of nonlinear evolution equations, their key characteristics and universality make them physically important. Some of these characteristics are solotonic solutions, elastic soliton-soliton interactions, and an infinite number of conservation laws; cf. [11, 12, 16]. Prominent examples of the universality of integrable equations are the KdV equation, which arises whenever weak dispersion interacts with weak nonlinearity and the NLS equation, which describes quasi-monochromatic and weakly nonlinear systems [11, 16]. Fractional equations also arise commonly in physical applications because they are successful effective descriptions of complex processes in biology [17–20], materials science [21–23], porous media [24–26], and many other fields. A prominent example of this is in anomalous diffusion, where the diffusion rate is related to time by a power law instead of being constant [27–30]. Therefore, we expect fractional integrable equations to describe physical applications where fractional calculus and integrable systems overlap.

The fractional nonlinear integrable evolution equations that we develop extend these exactly solvable models to fractional equations, opening a coherent theory connecting nonlinear dynamics and fractional calculus.

1.1 Key Mathematical Elements of Fractional Integrability

Defining fractional integrable nonlinear evolution equations using spectral theory involves three key mathematical ingredients: the IST (1), power law dispersion relations (2), and completeness relations (3). The process we develop throughout this thesis is wrapped in the IST; in fact, ingredients (2) and (3) of our method are only well defined in the context of the IST. Therefore, I will introduce the analog of our method in the simpler context of Fourier transforms for linear evolution equations, and show how we can define and solve linear fractional equations using spectral theory.

Suppose we have the family of linear evolution equations

$$q_t + \gamma(-\Delta) q_x = 0, \tag{1.2}$$

for the solution $q = q(x,t)$ on $x \in \mathbb{R}$, $t \in (0, \infty)$ where $q_t \equiv \frac{\partial q}{\partial t}$, $q_x = \frac{\partial q}{\partial x}$ and γ is an arbitrary function of the negative laplacian $-\Delta \equiv -\frac{\partial^2}{\partial x^2}$. To define $\gamma(-\Delta)$, we develop a functional calculus using the eigenfunctions and eigenvalues of $-\Delta$. Let $e_k(x)$ and λ_k be the eigenfunctions and eigenvalues of the laplacian, i.e.,

$$-\Delta e_k(x) = \lambda_k e_k(x) \tag{1.3}$$

where k indexes the eigenfunctions and eigenvalues and can in principle be discrete or continuous. Then, with $\gamma : \mathbb{R} \to \mathbb{C}$ a sufficiently regular map, the function of an operator $\gamma(-\Delta)$ is defined by

$$\gamma(-\Delta)e_k(x) = \gamma(\lambda_k)e_k(x). \tag{1.4}$$

For equation (1.2), the negative laplacian is defined on the real line, so the eigenfunctions are plane waves, $e_k(x) = e^{ikx}$, and the eigenvalues are $\lambda_k = k^2$ with $k \in \mathbb{R}$ (notice that the negative laplacian was chosen so that λ_k is positive). Due to Fourier analysis, the e_k's are complete — we can write any sufficiently regular function $h(x)$ as

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(y)e^{ik(x-y)}dydk. \tag{1.5}$$

Then, the operation of $\gamma(-\Delta)$ on this function is

$$\gamma(-\Delta)h(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\gamma(k^2)\int_{-\infty}^{+\infty} h(y)e^{ik(x-y)}dydk. \tag{1.6}$$

In particular, if we choose $\gamma(-\Delta) = (-\Delta)^{1+\epsilon}$, we obtain the Riesz fractional derivative

$$(-\Delta)^{1+\epsilon}h(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|k|^{2(1+\epsilon)}\int_{-\infty}^{+\infty} h(y)e^{ik(x-y)}dydk. \tag{1.7}$$

where ϵ can in principle by any complex number so long as the integral above exists, but we take $0 < \epsilon < 1$ throughout. Putting the expansion of $\gamma(-\Delta)$ in equation (1.6) into equation (1.2), we have

$$q_t + \frac{1}{2\pi}\int_{-\infty}^{+\infty} ik\gamma(k^2)e^{ikx}\hat{h}(k)dk. = 0. \tag{1.8}$$

This gives a representation of equation (1.2) in terms of well-understood operations, i.e., Fourier transforms. Notice that to do this we used Fourier completeness relations, the linear analog of ingredient (3) of our method. To pick out a particular equation from all equations of the form (1.2), we specify γ using a dispersion relation. If we put $q = e^{i(kx-w(k)t)}$ into equation (1.2), then we can relate γ to w by

$$w(k) = k\gamma(k^2) \tag{1.9}$$

The dispersion relation tells us how solutions of single frequencies, i.e., $q = e^{i(kx-w(k)t)}$ with wave-number k, are transported. If w is real, then the solution is a traveling wave, while if w is imaginary, then the solution decays or grows in time. Further, the dispersion relation tells us the phase velocity $c_p(k) = w(k)/k$ (how quickly a wave of wave-number k moves) and the group velocity $c_g(k) = w'(k)$ (how quickly the energy of a wave packet moves). Therefore, the relation in equation (1.9) tells us that γ specifies the dynamics of equation (1.2).

To pick out fractional equations from this family of equations, we use element (2) of our method, power law dispersion. Choosing $w(k) = -k^3|k|^{2\epsilon}$ (a power law dispersion relation), we have $\gamma(k^2) = -k^2|k|^{2\epsilon}$ and, thus, equation (1.2) becomes

$$q_t - (-\Delta)^{1+\epsilon} q_x = 0 \tag{1.10}$$

where $(-\Delta)^{1+\epsilon}$ defined in equation (1.7); this is the linear fractional KdV equation. Notice that the phase and group velocity for this equation are

$$c_p(k) = -k^2|k|^{2\epsilon}, \quad c_g(k) = -(3+2\epsilon)k^2|k|^{2\epsilon} \tag{1.11}$$

so both the phase and group velocities of waves predicted by linear KdV are related to wave-number by a power law. We will see a similar characteristic for nonlinear equations, where velocity will be related to amplitude by a power law, i.e., anomalous dispersion.

Just as we can represent the operator $\gamma(-\Delta)$ using Fourier transforms, giving an integral representation of equation (1.2) in (1.8), we can also solve equation (1.2) using Fourier transforms. This process is analogous to how IST is used to solve nonlinear problems. First, we take the Fourier transform to give the equation in Fourier space

$$\hat{q}_t + ik\gamma(k^2)\hat{q}(k,t) = 0. \tag{1.12}$$

Then, we evolve the solution in time from the initial condition by solving this differential equation to give

$$\hat{q}(k,t) = e^{-ik\gamma(k^2)t}\hat{q}(k,0). \tag{1.13}$$

where $\hat{q}(k,0)$ is the Fourier transform of $q(x,t)$ evaluated at $t = 0$. Finally, we map the solution back to physical space using the inverse Fourier transform

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{q}(k,0)e^{ik(x-\gamma(k^2)t)}dk. \tag{1.14}$$

Solving the linear evolution equation using Fourier transforms involves these three steps: mapping the solution into Fourier space, evolving the solution in time, and mapping the solution back into physical space. These steps are analogous to how one solves nonlinear problems with IST. Further, the above discussion of defining linear fractional operators using Fourier transforms and two of our three mathematical ingredients, (2) power law dispersion and (3) completeness, is analogous to how we will define nonlinear fractional operators using the IST.

1.2 Thesis Outline

The principle results of this thesis are the development of a general method of finding fractional nonlinear integrable evolution equations and its application to many different well known integrable systems. The first equation derived with this method, which is also the first known fractional integrable nonlinear evolution equation with smooth solutions and non-local fractional operators, is the fractional KdV (fKdV) equation:

$$q_t + \int_{\Gamma_\infty} dk |4k^2|^\epsilon \frac{\tau^2(k)}{4\pi i k} \int_{-\infty}^{\infty} dy\, G(x,y,k)\,(6qq_y + q_{yyy}) = 0. \tag{1.15}$$

Here $q = q(x,t)$ is the solution and all of the other symbols are defined in chapter 2. In this chapter, the fKdV equation, the fractional NLS (fNLS) equation, and an outline of the general method used to find these equations are given. We also present the one soliton solutions to these equations and demonstrate that they exhibit anomalous dispersion.

Then, in chapter 3, we show how the method can be applied to derive integrable fractional generalizations of the mKdV, sineG, and sinhG equations. After reviewing IST for these equations in detail, we derive the one soliton solutions to the fractional mKdV (fmKdV) and fractional sineG (fsineG) equations using inverse scattering. Using the explicit form of these equations, analogous to equation (1.15), we verify that these solitons are truly solutions of their respective equations.

We extend fractional integrability to discrete problems in chapter 4, where we derive the fractional IDNLS (fIDNLS) equation using our method. We present a one soliton solution to this equation and demonstrate that it exhibits more complicated behavior than the continuous fractional integrable equations studied in previous chapters. We then compare the predictions of this equation to the closely related fractional averaged DNLS (fADNLS) equation, which has a simpler mathematical form but is (likely) not integrable. We find that the soliton solutions of the fIDNLS and fADNLS equations have similar characteristics in the small amplitude and positive ϵ regeme.

This thesis contains the following manuscripts which have been published, are under review, or are to be submitted:

1. Mark J. Ablowitz, Joel B. Been, and Lincoln D. Carr, “Fractional Integrable Nonlinear Soliton Equations,” Phys. Rev. Lett. v. 128 p. 184101 (2022).
2. Mark J. Ablowitz, Joel B. Been, and Lincoln D. Carr, “Integrable Fractional Modified Korteweg-de Vries, Sine-Gordon, and sinh-Gordon Equations,” Journal of Physics A: Mathematical and Theoretical, under review (2022).
3. Mark J. Ablowitz, Joel B. Been, and Lincoln D. Carr, “Fractional Integrable and Related Discrete Nonlinear Schrödinger Equations,” Phys. Lett. A, to be submitted.

CHAPTER 2

FRACTIONAL INTEGRABLE NONLINEAR SOLITON EQUATIONS

Published in *Physical Review Letters* [1].

Mark J. Ablowitz[1,2], Joel B. Been[3,4], Lincoln D. Carr[3,4,5,6]

2.1 Abstract

Nonlinear integrable equations serve as a foundation for nonlinear dynamics, and fractional equations are well known in anomalous diffusion. We connect these two fields by presenting the discovery of a new class of integrable fractional nonlinear evolution equations describing dispersive transport in fractional media. These equations can be constructed from nonlinear integrable equations using a widely generalizable mathematical process utilizing completeness relations, dispersion relations, and inverse scattering transform techniques. As examples, this general method is used to characterize fractional extensions to two physically relevant, pervasive integrable nonlinear equations: the Korteweg–deVries and nonlinear Schrödinger equations. These equations are shown to predict super-dispersive transport of non-dissipative solitons in fractional media.

2.2 Introduction

Fractional calculus is an effective tool when describing physical systems with power law behavior such as in anomalous diffusion, where the mean squared displacement is proportional to t^α, $\alpha > 0$ [27–30]. This form of transport has been observed extensively in biology [17–20], amorphous materials [21–23], porous media [24–26], and climate science [31] amongst others. Equations in multiscale media can express fractional derivatives in any governing term [32, 33], including dispersion, such as found in the 1D nonlinear Schrödinger equation (NLS) in optics [11, 16, 34–38] and the Korteweg-deVries equation (KdV) in water waves [39]. In the case of integer derivatives, NLS and KdV are famously integrable equations, leading to solitonic solutions and an infinite set of conservation laws [12]. Integrable equations are key signposts in nonlinear dynamics as they provide exactly solvable cases and, moreover, are an essential element of Kolmogorov-Arnold-Moser (KAM) theory underlying our understanding of chaos. The

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fundamental solution of 1D dispersive integrable equations is the soliton, a robust nondispersive localized wave. While in the space of possible nonlinear evolution equations integrable cases are extremely rare, they arise frequently in application.

In this Letter, we present a new class of integrable *fractional* nonlinear evolution equations which predict super-dispersive transport in fractional media. Fractional media is "rough" or multiscale media that is neither regular nor random; it includes fractals but is more general as it need not be self-similar. We use the fractional NLS (fNLS) and fractional KdV (fKdV) equations as case studies. We show their integrability, demonstrate exact fractional soliton solutions, and make physical predictions about the speed of these localized waves. To date, to our knowledge, no nonlinear fractional evolution equation has been known to be integrable.

The building blocks of our demonstration are three mathematical ingredients. Two are familiar to physicists as they are well known concepts in physics. They are completeness and the dispersion relations. However, in our case the dispersion relation will use fractional, rather than integer, derivatives. The third building block is the fundamental ingredient of integrability, namely the inverse scattering transform (IST), well known to researchers in nonlinear dynamics.

Different versions of the fNLS equation have been studied in, e.g., [34, 40–42], and soliton type solutions have been found, but unlike the fNLS and fKdV equations that we introduce, none of these are integrable. The fractional operators in the fNLS and fKdV equations are nonlinear generalizations of the Riesz fractional derivative. In fact, the linear limit of the fNLS equation is the well known fractional Schrödinger equation derived using a Feynman path integral over Lévy flights [43, 44]. Fractional equations defined using the Riesz fractional derivative (alternately termed the Riesz transform [4] or fractional Laplacian [45]) are effective tools when describing behavior in complex systems because the Riesz fractional derivative is closely related to non-Gaussian statistics [46]. It has found physical applications in describing movement of water in porous media [47], transport of temperature in fluid dynamics [48], and power law attenuation in materials [49] amongst many others [50–52].

The KdV and NLS equations arise in many physical problems. The KdV equation is applicable in shallow water waves, internal waves, fluid dynamics, plasma physics, and lattice dynamics amongst others [39]. Furthermore, KdV is a universally important equation whenever weak dispersion balances weak quadratic nonlinearity cf. [11, 16]. Similarly, the NLS equation arises in the quasi-monochromatic approximation with dispersion balancing weak nonlinearity and occurs widely in physical applications, e.g. water waves, nonlinear optics, spin waves in ferromagnetic films, plasma physics, Bose-Einstein condensates, etc. [11, 16, 53, 54]. The KdV equation was shown to be solvable using the IST and to admit soliton solutions when associated with the linear time-independent Schrödinger equation in [8]. Then, the

NLS equation with decaying data was solved and shown to possess solitons via the IST in [55]. Soon after, the method was extended to the modified KdV and sine-Gordon equations as well as general classes of equations written in terms of a linearized dispersion relation [10, 16]. IST is now a large field cf. [11–15].

Here we show how to extend this formulation to encompass fractional integrable nonlinear evolution equations. As examples of this technique, we show that fKdV and fNLS are solvable by the IST. These are two examples of many possible fractional integrable equations that can be characterized by this method.

2.3 The IST and Anomalous Dispersion Relations

It is well known that linear evolution equations for $q = q(x,t)$ of the form

$$q_t + \gamma(\partial_x)q_x = 0, \tag{2.1}$$

can be solved by Fourier transforms when $\gamma(\partial_x)$ is a rational function of ∂_x; cf. [16]. We can do this because the completeness of plane waves gives an integral representation of $\gamma(\partial_x)$. The solution to (2.1) is explicitly

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\hat{q}(k,0)e^{ikx-ik\gamma(ik)t}, \tag{2.2}$$

where $\hat{q}(k,0)$ is the Fourier transform of $q(x,t)$ with respect to x evaluated at $t = 0$. However, as Riesz showed [4], the solution (2.2) makes sense for much more general γ. Specifically, Fourier Transforms can be used to solve linear fractional evolution equations, e.g., $\gamma(\partial_x) = |-\partial_x^2|^\epsilon$ with 2ϵ the order of the fractional derivative; we take $0 < \epsilon < 1$ throughout this letter.

Here we show that similar analysis applies to nonlinear evolution equations using the IST. We do this by associating a class of integrable nonlinear equations with a linear scattering problem (ingredient 1, IST), characterizing the fractional equation with an anomalous dispersion relation (ingredient 2, dispersion), and defining the fractional operator associated with this dispersion relation using the completeness of squared eigenfunctions of the scattering equation (ingredient 3, completeness).

We will apply ingredients 1 and 2 to find the fKdV and fNLS equations, and use ingredient 3 to define the fractional operators in these equations. Associated with the non-dimensionalized time-independent Schrödinger equation for $v(x,t)$ with potential $q(x,t)$

$$v_{xx} + \left(k^2 + q(x,t)\right)v = 0, \quad |x| < \infty, \tag{2.3}$$

is the following class of integrable nonlinear equations for $q(x,t)$ [10]

$$q_t + \gamma(L^A)q_x = 0, \quad L^A \equiv -\frac{1}{4}\partial_x^2 - q + \frac{1}{2}q_x\int_x^{\infty} dy. \tag{2.4}$$

where $\int_x^\infty dy$ operates on the function to which L^A is applied by integrating it. Hence, equation (2.4) can be solved by the IST using (2.3). We obtain the fKdV equation by choosing $\gamma(L^A) = -4L^A \left|4L^A\right|^\epsilon$; this will be justified shortly.

Similarly, associated with the following 2×2 scattering problem — termed the Ablowitz-Kaup-Newell-Segur (AKNS) system — for the vector-valued function $\mathbf{v}(x,t) = (v_1(x,t), v_2(x,t))^T$ (T represents transpose)

$$v_x^{(1)} = -ikv^{(1)} + q(x,t)v^{(2)}, \tag{2.5}$$

$$v_x^{(2)} = ikv^{(2)} + r(x,t)v^{(1)}, \tag{2.6}$$

is the set of integrable nonlinear equations [10]

$$\sigma_3 \partial_t \mathbf{u} + 2A_0(\mathbf{L}^A)\mathbf{u} = 0, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.7}$$

where $\mathbf{u} = (r,q)^T$ and the operator

$$\mathbf{L}^A \equiv \frac{1}{2i} \begin{pmatrix} \partial_x - 2rI_- q & 2rI_- r \\ -2qI_- q & -\partial_x + 2qI_- r \end{pmatrix} \tag{2.8}$$

with $I_- = \int_{-\infty}^x dy$. Note that I_- operates both on the function immediately to its right and the functions to which $\mathbf{L}^A$ is applied. Taking $r = \mp q^*$, $*$ the complex conjugate, and $A_0(\mathbf{L}^A) = 2i(\mathbf{L}^A)^2|2\mathbf{L}^A|^\epsilon$ we find fNLS to be the second component of (2.7).

These definitions are justified when we note that $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ can be related to the dispersion relation of the linearization of (2.4) and (2.7). Specifically, if we put $q = e^{i(kx - w(k)t)}$ into the linearizations of (2.4) and (2.7), we have

$$\gamma(k^2) = \frac{w_K(2k)}{2k}, \quad A_0(k) = -\frac{i}{2} w_S(-2k), \tag{2.9}$$

where w_K is the dispersion relation for the linear fKdV equation and w_S is the same for the linear fractional Schrödinger equation. Therefore, $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ for fKdV and fNLS are generated from the dispersion relations for linear fKdV and the linear fractional Schrödinger equation. These equations are, naturally,

$$q_t + \left|-\partial_x^2\right|^\epsilon q_{xxx} = 0, \quad iq_t = \left|-\partial_x^2\right|^{\epsilon/2} q_{xx}, \tag{2.10}$$

where $|-\partial_x^2|^\epsilon$ is the Riesz fractional derivative. So, the corresponding dispersion relations are $w_K(k) = -k^3|k|^{2\epsilon}$ and $w_S(k) = -k^2|k|^\epsilon$ which lead to the aforementioned definitions of $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$.

2.4 Spectral Definitions of fKdV and fNLS by Completeness

To define the fKdV and fNLS equations we need to determine what operating on a function with $\gamma(L^A)$ or $A_0(\mathbf{L}^A)$ means. We do this using ingredient 3, completeness of the associated linear scattering system.

In [10] it was shown that the eigenfunctions of L^A are any of the three functions: $\{\partial_x\varphi^2, \partial_x\psi^2, \partial_x(\varphi\psi)\}$ which we represent generically as Ψ^A, each with eigenvalue $\lambda = k^2$. Here ψ and φ solve the time-independent Schrodinger equation (2.3) subject to appropriate asymptotic boundary conditions at $x = \pm\infty$. Furthermore, the eigenfunctions of $\mathbf{L}^A$ are $\mathbf{\Psi}^A$ and $\overline{\mathbf{\Psi}}^A$ each with eigenvalue $\lambda = k$. These may be written in terms of solutions to equations (2.5) and (2.6) (see Supplemental Material [56]).

Starting from $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ operating on Ψ^A and $\mathbf{\Psi}^A$, we can write

$$\gamma(L^A)\Psi^A = \gamma(k^2)\Psi^A, \tag{2.11}$$

$$A_0(\mathbf{L}^A)\mathbf{\Psi}^A = A_0(k)\mathbf{\Psi}^A. \tag{2.12}$$

To extend this to $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ operating on any function, we need to be able to express any function in terms of Ψ^A and $\mathbf{\Psi}^A$, i.e. we need a completeness relation for each set of eigenfunctions.

In [5] it was shown that the eigenfunctions Ψ^A are complete. Assuming $q(x,t)$ is sufficiently decaying and smooth in x, an arbitrary, and sufficiently regular, function $h(x)$ may be expanded in terms of the eigenfunctions Ψ^A as

$$h(x) = \int_{\Gamma_\infty} dk\ \frac{\tau^2(k)}{4\pi ik} \int_{-\infty}^{\infty} dy\ G(x,y,k)h(y), \tag{2.13}$$

where time is suppressed and $\Gamma_\infty = \lim_{R\to\infty}\Gamma_R$ with Γ_R the semicircular contour in the upper half plane evaluated from $k = -R$ to $k = R$. τ is the transmission coefficient defined by the relation $\varphi(x,k)\tau(k) = \psi(x,-k) + \rho(k)\psi(x,k)$, ρ is the reflection coefficient, and

$$G(x,y,k) = \partial_x(\psi^2(x,k)\varphi^2(y,k) - \varphi^2(x,k)\psi^2(y,k)). \tag{2.14}$$

This completeness relation reduces to Fourier completeness in the linear limit. From (2.11) and (2.13) the operation of $\gamma(L^A)$ on a sufficiently smooth and decaying function h follows as

$$\gamma(L^A)h(x) = \int_{\Gamma_\infty} dk\gamma(k^2)\frac{\tau^2(k)}{4\pi ik} \int_{-\infty}^{\infty} dyG(x,y,k)h(y). \tag{2.15}$$

Hence, equations (2.13)-(2.15) provide an explicit representation of fKdV, i.e. equation (2.4) with $\gamma(L^A) = -4L^A\left|4L^A\right|^\epsilon$, which may be written as

$$q_t + \int_{\Gamma_\infty} dk |4k^2|^\epsilon \frac{\tau^2(k)}{4\pi i k} \int_{-\infty}^{\infty} dy G(x,y,k) \left(6qq_y + q_{yyy}\right) = 0. \tag{2.16}$$

Notice that equation (2.16) is in non-dimensional coordinates x and t. In the linear limit $q \to 0$, we have $\gamma(L^A) \to \gamma\left(-\partial_x^2/4\right)$. So, for fKdV, $\gamma(L^A) \to -\partial_x^2 \left|-\partial_x^2\right|^\epsilon$ which is the Riesz fractional derivative. If we then set $\epsilon = 0$, we recover the KdV equation:

$$q_t + 6qq_x + q_{xxx} = 0. \tag{2.17}$$

We note that $\tau(k,t)$ has a finite number of simple poles along the imaginary axis denoted $k_j = i\kappa_j$ for $j = 1, 2, ..., J$, so the above representation can be evaluated by contour integration (see Supplemental Material [56]).

Similarly, the eigenfunctions $\mathbf{\Psi}^A$ are also complete [6]. Thus, we can write the operation of $A_0(\mathbf{L}^A)$ on a sufficiently smooth and decaying vector-valued function $\mathbf{h}(x) = \left(h_1(x), h_2(x)\right)^T$ as

$$\begin{aligned} A_0(\mathbf{L}^A)\mathbf{h}(x) &= \sum_{n=1}^{2} \int_{\Gamma_\infty^{(n)}} dk A_0(k) f_n(k) \int_{-\infty}^{\infty} dy \mathbf{G}_n(x,y,k)\mathbf{h}(y), \\ \mathbf{G}_1(x,y,k) &= \mathbf{\Psi}^A(x,k)\mathbf{\Psi}(y,k)^T, \quad f_1(k) = -\tau^2(k)/\pi, \\ \mathbf{G}_2(x,y,k) &= \overline{\mathbf{\Psi}}^A(x,k)\overline{\mathbf{\Psi}}(y,k)^T, \quad f_2(k) = \overline{\tau}^2(k)/\pi, \end{aligned} \tag{2.18}$$

where $\Gamma_R^{(1)}$ ($\Gamma_R^{(2)}$) is the semicircular contour in the upper (lower) half plane evaluated from $-R$ to $+R$; $\mathbf{\Psi}(x,k)$, $\overline{\mathbf{\Psi}}(x,k)$ are eigenfunctions of $\mathbf{L}$; $\mathbf{\Psi}^A(x,k)$, $\overline{\mathbf{\Psi}}^A(x,k)$ are eigenfunctions of $\mathbf{L}^A$; and $\tau(k)$, $\overline{\tau}(k)$ are transmission coefficients defined similarly to fKdV. Notice that $\mathbf{G}_n$ are 2×2 matrices (see Supplemental Material [56]).

Thus equation (2.18) gives a representation for the fNLS, equation (2.7) with $A_0(\mathbf{L}^A) = 2i(\mathbf{L}^A)^2|2\mathbf{L}^A|^\epsilon$ and $r = \mp q^*$; see the Supplemental Material [56]. In the linear limit, fNLS is represented in terms of the Riesz fractional derivative and for $\epsilon = 0$ we recover NLS:

$$iq_t = q_{xx} \pm 2q^2 q^*. \tag{2.19}$$

With explicit expressions for $\gamma(L^A)$ and $A_0(\mathbf{L}^A)$ in equations (2.15) and (2.18), the fKdV and fNLS equations are characterized. Further, because these equations are inside of the time-independent Schrödinger and AKNS classes of integrable nonlinear equations in (2.4) and (2.7), fKdV and fNLS are solvable by the IST.

2.5 Soliton Solutions of fKdV and fNLS

Given an initial state $q(x,0)$ with sufficient smoothness and decay, we can solve fKdV and fNLS, i.e. obtain $q(x,t)$, using the IST. To do this, we first map the initial state into scattering space, evolve the resulting scattering data in time, and reconstruct the solution in physical space from these data. It turns out that solving fKdV and fNLS are remarkably similar to solving KdV and NLS.

We note that, given the explicit representation of fKdV in equation (2.16), and fNLS in Supplemental Material [56], these equations can also be solved numerically in discrete time by finding the kernels $G/\mathbf{G}_j$ and evaluating the integrals with respect to y and k at each time step.

The fractional soliton solutions of fKdV and fNLS are given in equations (2.20-2.21). These correspond to reflectionless bound states of the Schrödinger and AKNS scattering problems with one complex eigenvalue $k_K = i\kappa$ and $k_S = \xi + i\eta$ respectively:

$$q_K(x,t) = 2\kappa^2 \mathrm{sech}^2(\kappa[(x-x_1) - (4\kappa^2)^{1+\epsilon}t]), \tag{2.20}$$
$$q_S(x,t) = 2\eta e^{-2i\xi x + 4i(\xi^2-\eta^2)|2k_S|^{\epsilon}t} \mathrm{sech}(z_\epsilon(x,t)), \tag{2.21}$$

where $z_\epsilon(x,t) = 2\eta(x - x_0 - 4\xi|2k_S|^\epsilon t)$ and x_0, x_1 can be characterized in terms of scattering data.

It can also be shown that the fractional solitons solve their respective equations by evaluating $\gamma(L^A)\partial_x q_K$ and $A_0(\mathbf{L}^A)\partial_x q_S$ using contour integration methods (this computation for the fKdV equation is given in the Supplemental Material [56].) Further, higher order solitons can be calculated and their interactions are elastic.

2.6 Physical Predictions

The fKdV and the fNLS equations describe the transport of fluid and photons in multiscale fluid channels and laser fiberoptic systems, respectively. The multiscale characteristic of these materials represents a certain "roughness" which is averaged over in fKdV and fNLS. The solitonic solutions of these equations describe how localized waves of fluid/probability are transported in such systems. Both fKdV and fNLS predict solitons with anomalous motion, that is, super-dispersive transport where speeds are larger than expected from regular or ordered systems (note that sub-dispersive transport can also be realized by modifying the dispersion relation). Specifically, the group velocity of fKdV and fNLS and the phase velocity of fNLS are

$$v_K(\epsilon,\kappa) = \left(4\kappa^2\right)^{1+\epsilon} \tag{2.22}$$
$$v_S(\xi,\eta) = 2^{2+\epsilon}\xi(\xi^2+\eta^2)^{\epsilon/2} \tag{2.23}$$
$$v_\theta(\xi,\eta) = 2^{1+\epsilon}\left(\xi^2-\eta^2\right)\left(\xi^2+\eta^2\right)^{\epsilon/2}/\xi \tag{2.24}$$

In a wave tank of height 5 cm we expect solitons with amplitude and KdV speed around 2/3 cm and 0.3 cm/s, respectively. One can similarly associate physical values to solitons in fiberoptics [57], spin waves in ferromagnetic films [58], Bose-Einstein condensates [59], or any of the many other contexts in which NLS is applicable.

Figure 2.1 shows the velocities in equations (2.22-2.24) as they interpolate between KdV (NLS) for $\epsilon = 0$ and $\epsilon = 1$. Notice that fKdV and fNLS predict a power law relationship between the amplitude of the wave, κ^2 and η respectively, and the speed of the wave characterized by ϵ. Experimentally verifying these relations relies on comparing the amplitude of water waves and the amplitude and phase of laser pulses in optical fibers to their speed in multiscale media.

Importantly, the physical properties of fractional solitons, besides the change in velocity described by equations (2.22-2.24), are identical to regular ones. From Figure 2.2, fractional solitons propagate without dissipating or spreading out. An open question is to compare the solitons predicted by fKdV and fNLS to solitary waves predicted by other, non-integrable versions of these equations. This could be done by studying how the velocity of each equation varies with the fractional parameter ϵ and whether soliton-soliton interactions are elastic or inelastic and what the predicted phase shifts are.

2.7 Conclusion

We have demonstrated a new class of integrable equations, namely 1D fractional integrable nonlinear evolution equations, derivable from a general method. As ubiquitous examples of this class we presented integrability and solitonic solutions of the fractional nonlinear Schrödinger and Korteweg-deVries equations. We demonstrated the three basic mathematical ingredients of our procedure: completeness, dispersion relations, and inverse scattering transform techniques. We also gave fractional soliton solutions to these equations and demonstrated super-dispersive transport as a physical implication of the equations. Such fractional equations model multiscale materials and open new directions in integrable nonlinear dynamics for such systems, both artificial and naturally occurring. Our method provides a context for the discovery and understanding of 1D fractional nonlinear evolution equations generally, with integrability acting as a key signpost for fractional nonlinear dynamics.

2.8 Acknowledgements

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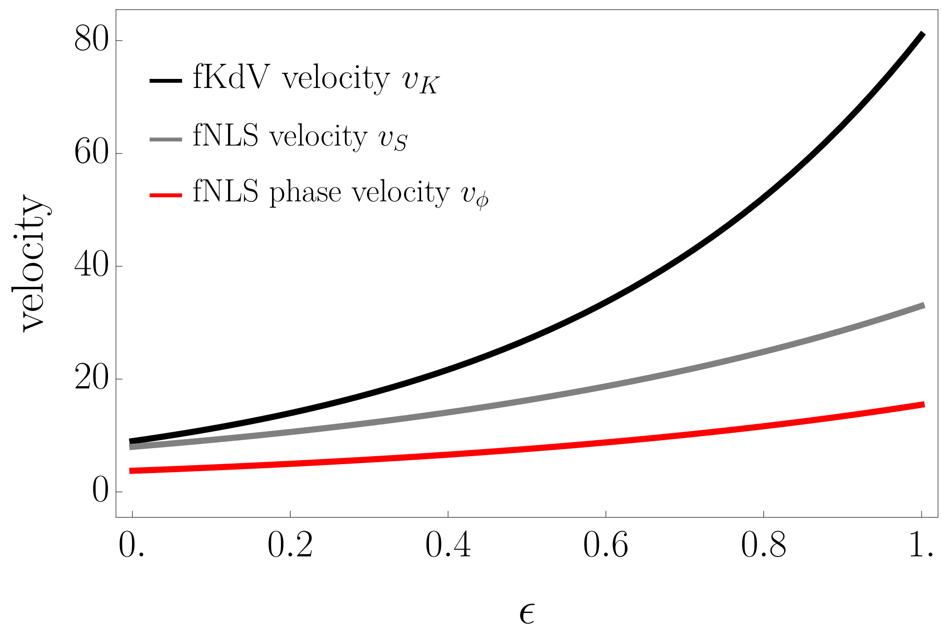

Figure 2.1 Localized waves predicted by the fKdV and fNLS equations, (2.22-2.24), show super-dispersive transport as their velocity increases as ϵ increases from 0 to 1. Like anomalous diffusion where the mean squared displacement is proportional to t^α, the velocity in anomalous dispersion is proportional to A^ϵ, where A is the amplitude of the wave. The parameter values used are $\kappa = 3/2$, $\xi = 2$, and $\eta = 1/2$.

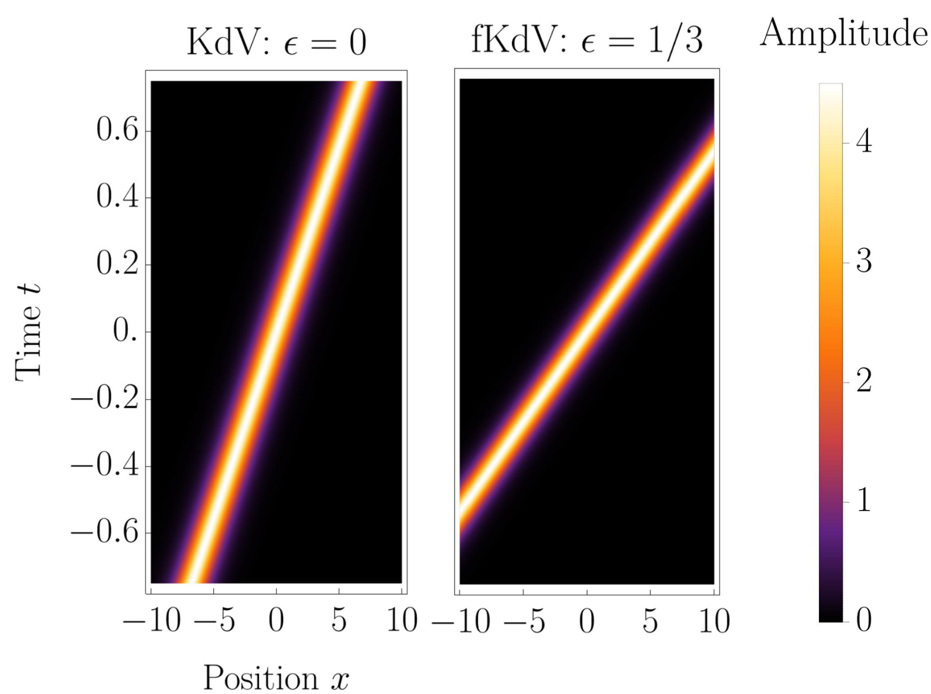

Figure 2.2 Note that soliton solutions to the fKdV equation propagate without dissipating or spreading out. The parameter values used are $\kappa = 3/2$ and $x_0 = 0$.

CHAPTER 3

INTEGRABLE FRACTIONAL MODIFIED KORTEWEG-DE VRIES, SINE-GORDON, AND SINH-GORDON EQUATIONS

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3.1 Abstract

The inverse scattering transform allows explicit construction of solutions to many physically significant nonlinear wave equations. Notably, this method can be extended to fractional nonlinear evolution equations characterized by anomalous dispersion using completeness of suitable eigenfunctions of the associated linear scattering problem. In anomalous diffusion, the mean squared displacement is proportional to t^α, $\alpha > 0$, while in anomalous dispersion, the speed of localized waves is proportional to A^α, where A is the amplitude of the wave. Fractional extensions of the modified Korteweg-deVries (mKdV), sine-Gordon (sineG) and sinh-Gordon (sinhG) and associated hierarchies are obtained. Using symmetries present in the linear scattering problem, these equations can be connected with a scalar family of nonlinear evolution equations of which fractional mKdV (fmKdV), fractional sineG (fsineG), and fractional sinhG (fsinhG) are special cases. Completeness of solutions to the scalar problem is obtained and, from this, the nonlinear evolution equation is characterized in terms of a spectral expansion. In particular, fmKdV, fsineG, and fsinhG are explicitly written. One-soliton solutions are derived for fmKdV and fsineG using the inverse scattering transform and these solitons are shown to exhibit anomalous dispersion.

3.2 Introduction

Fractional calculus has been effectively applied to describe physical systems with anomalous behavior associated with multi-scale media. The underlying fractional mathematical formulation, originally designed to interpolate between integer derivative orders, has been used to describe new phenomena, such as novel forms of transport in biology [17–20], amorphous materials [21–23], porous media [24–26], and climate science [31] amongst others. Fractional equations often predict physically measurable quantities that follow power laws. For example, in anomalous diffusion, the mean squared displacement is related to time by a power law t^α, $\alpha > 0$ [27–30]. Similarly, for integrable soliton equations, fractional generalizations predict

[7]Author contributions: Conceptualization, Methodology, Formal analysis, Writing – Review & Editing
[8]Author contributions: Conceptualization, Writing – Review & Editing

anomalous dispersion where the speed of localized solitonic waves are related to their amplitude by a power law [1].

Integrable evolution equations are key elements in the study of nonlinear dynamics because they have deep mathematical structure and provide exactly solvable models whose results can be compared with experimental and numerical data. Some important and well-known examples of integrable evolution equations are the Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV), nonlinear Schrodinger (NLS), sine-Gordon (sineG), and sinh-Gordon (sinhG) equations. These equations are solvable by the inverse scattering transformation (IST), a nonlinear generalization of Fourier transforms where the nonlinear equation is associated with a linear scattering problem. They also admit an infinite set of conservation laws and have soliton solutions which are robust localized traveling waves [12, 60].

In [1], we obtained and analyzed the integrable fractional Korteweg-de Vries (fKdV) and integrable fractional nonlinear Schrödinger (fNLS) equations. These were two examples of a hierarchy of fractional equations that can be constructed. In the case of NLS, the hierarchy is written in terms of 2×2 matrix operators. In this article, we demonstrate that this process can be applied to define and solve key, physically relevant nonlinear evolution equations — namely the fmKdV, fsineG, and fsinhG equations in terms of scalar operators. Although the fmKdV, fsineG, and fsinhG equations can be written in terms of matrix operators the scalar system is considerably simpler, more compact and, provides a direct analog of the scalar fKdV operator.

The fractional operators of these integrable systems are are nonlinear generalizations of the well-established Riesz fractional derivative. Although there are many fractional derivatives, the Riesz formulation is particularly intuitive and accessible for physicists who do not specialize in this area of mathematics. The Riesz fractional derivative is defined by its Fourier multiplier $|k|^{2\epsilon}$, $|\epsilon| < 1$ (we take this range of values for ϵ throughout the text), and can be understood as the fractional power of $-\partial_x^2$. Fractional equations defined using the Riesz fractional derivative (alternately termed the Riesz transform [4] or fractional Laplacian [45]) are effective tools when describing behavior in complex systems because the Riesz fractional derivative is closely related to non-Gaussian statistics [46]. It has found physical applications in describing movement of water in porous media [47], transport of temperature in fluid dynamics [48], and power law attenuation in materials [49] amongst many others [50–52].

The KdV equation describes quadratic nonlinear waves with weak dispersion; it was discovered in water waves over one hundred years ago [39]. The KdV equation admits solitary wave solutions which are localized waves of permanent form that propagate unidirectionally and whose speed and amplitude are linearly related. Seventy years later, using numerical methods, KdV solitary waves were found to interact elastically; they were termed solitons [61]. Soon afterwards the KdV equation with decaying initial data was linearized and soliton solutions were obtained analytically using inverse scattering methods [62]. A few years later the NLS equation was found to be solvable via inverse scattering and to have soliton solutions [55]. In [10], the linearization procedure was generalized with the NLS, mKdV, and sineG (in light cone coordinates) equations as special cases. The procedure was termed the Inverse Scattering Transform (IST). These equations arise in numerous physical contexts [11, 16]. Remarkably, all of these equations have fractional extensions which pave the way for applications to anomalous dispersion and multi-scale behavior.

In this article, we define and solve the fmKdV, fsineG, and fsinhG equations on the line with suitable initial data using three ingredients: a general nonlinear equation solvable by the IST, a completeness relation for squared eigenfunctions, and an anomalous dispersion relation. We develop a scalar reduction of the Ablowitz-Kaup-Newell-Segur (AKNS) system in which we find the fmKdV, fsineG, and fsinhG equations as special cases using power law dispersion relations. Then, we characterize a completeness relation for squared eigenfunctions of this scalar system. This completeness relation provides a spectral representation for the fractional operators in the fmKdV, fsineG, and fsinhG equations, giving the equations an explicit representation in physical space. From basic IST theory we can derive the general solution to the whole class of nonlinear equations described by the scalar reduction; in particular, we give those for the fmKdV and fsineG equations. This includes the multi-soliton solutions; solitons interact elastically. Unlike standard equations like KdV, we do not know how to integrate even one-soliton solutions directly. But for the one-soliton solutions derived using IST, we check that they are solutions to the fmKdV and fsineG equations. The velocity of these solitons are related to their amplitude by a power law. Therefore, the fmKdV and fsineG equations predict anomalous dispersion. To our knowledge, no nonlinear fractional evolution equations with smooth (physical) solutions have been found to be integrable.

3.3 AKNS Scattering System and Scalar Reduction

The inverse scattering transformation relies on associating the nonlinear problem we want to solve to a linear scattering problem by taking the potential of the linear problem to be the solution of the nonlinear problem. For many nonlinear evolution equations, e.g., the mKdV, sineG, and NLS equations, the associated linear scattering problem is the AKNS system (also often called the AKNS eigenvalue problem). The nonlinear evolution equations are linearized by the scattering problem. We previously demonstrated that the AKNS system can linearize the *fractional* Nonlinear Schrödinger equation [1] via a 2×2 matrix operator.

Here, we will show that given a symmetry reduction, the vector valued nonlinear evolution equation for the solution $\mathbf{u}(x,t) = (r(x,t), q(x,t))^T$ associated to the AKNS scattering problem becomes a scalar nonlinear evolution equation. This family of equations is then shown to contain the mKdV, sineG, and sinhG equations:

$$q_t \mp 6q^2 q_x + q_{xxx} = 0, \tag{3.1}$$
$$u_{xt} = \sin u, \tag{3.2}$$

where $r = \pm q$ for mKdV and $r = -q$, with $u_x/2 = -q$ for sineG with q real and $q_t \equiv \frac{\partial q}{\partial t}$ and $q_x \equiv \frac{\partial q}{\partial x}$. We also note that with $r = q = u_x/2$ and q real we find the sinhG equation:

$$u_{xt} = \sinh u \tag{3.3}$$

Below, we show that this family of scalar evolution equations also contains fmKdV, fsineG, and fsinhG as well as their hierarchies. First, we will outline scattering theory of the AKNS system and show how this leads to the scalar scattering problem.

3.3.1 AKNS Scattering Problem

The Ablowitz-Kaup-Newell-Segur (AKNS) system is the 2×2 scattering problem for the vector-valued function $\mathbf{v}(x) = (v_1(x), v_2(x))^T$ (T represents transpose)

$$v_x^{(1)} = -ikv^{(1)} + q(x,t)v^{(2)}, \tag{3.4}$$
$$v_x^{(2)} = +ikv^{(2)} + r(x,t)v^{(1)}, \tag{3.5}$$

where q and r act as potentials and k is an eigenvalue. We can associate to this scattering problem a vector-valued family of integrable nonlinear equations [10]

$$\sigma_3 \mathbf{u}_t + 2A_0(\mathbf{L}^A)\mathbf{u} = 0, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.6}$$

where $\mathbf{u} = (r, q)^T$ decays sufficiently rapidly at infinity and the 2×2 matrix operator

$$\mathbf{L}^A \equiv \frac{1}{2i} \begin{pmatrix} \partial_x - 2rI_- q & 2rI_- r \\ -2qI_- q & -\partial_x + 2qI_- r \end{pmatrix}, \tag{3.7}$$

with $I_- = \int_{-\infty}^{x} dy$. $\mathbf{L}^A$ is the adjoint of

$$\mathbf{L} \equiv \frac{1}{2i} \begin{pmatrix} -\partial_x - 2qI_+ r & -2qI_+ q \\ 2rI_+ r & \partial_x + 2rI_+ q \end{pmatrix}, \tag{3.8}$$

with $I_+ = \int_x^{\infty} dy$. The function A_0 has been traditionally considered to be meromorphic. The family of equations represented by (3.6) is commonly related to cases when $A_0(\mathbf{L}^A) = \left(\mathbf{L}^A\right)^n$, $n = 1, 2, 3....$ However, using the completeness relation for squared eigenfunctions which is discussed in the next section, it was shown that this can be extended to much more general A_0 [1]. The operator $A_0(\mathbf{L}^A)$ can also be related to the dispersion relation $w(k)$ of the linearization of (3.6). Specifically, if we put $q = e^{i(kx-\omega((k)t)}$ into the linearization of (3.6), we have

$$A_0 \left(\frac{k}{2} \right) = -\frac{i}{2}\omega(-k). \tag{3.9}$$

We can obtain the Nonlinear Schrödinger equation from equation (3.6) by putting $r = \mp q^*$ with its linear dispersion relation $\omega(k) = -k^2$. Similarly, sineG, sinhG, and mKdV follow from $r = -q$, $\omega(k) = k^{-1}$; $r = +q$, $\omega(k) = k^{-1}$; and $r = \pm q$, $\omega(k) = -k^3$, respectively, with q real. In [1], it was shown that fNLS could be obtained from $r = \pm q$, $\omega(k) = -k^2|k|^\epsilon$. The associated hierarchy of integrable equations follows by taking $\omega(k) = -k^n|k|^\epsilon, n = 3, 4, ...$

We will take the linearization of fmKdV, fsineG, and fsinhG to be

$$q_t + (-\partial_x^2)^\epsilon q_{xxx} = 0, \tag{3.10}$$

$$u_{tx} = (-\partial_x^2)^\epsilon u, \quad q_t = \int_{-\infty}^{x} (-\partial_\xi^2)^\epsilon q(\xi, t)\, d\xi, \tag{3.11}$$

where $(-\partial_x^2)^\epsilon$ is the Riesz fractional derivative defined by

$$(-\partial_x^2)^\epsilon q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{q}(k,t)|k|^{2\epsilon} e^{ikx} dk, , \tag{3.12}$$

$$\hat{q}(k,t) = \int_{-\infty}^{\infty} q(x,t) e^{-ikx} dx. \tag{3.13}$$

Notice that both fsineG and fsinhG have the same linear equation (3.11). The linearization of fmKdV has dispersion relation $\omega(k) = -k^3|k|^{2\epsilon}$ and that of fsineG and fsinhG is $\omega(k) = |k|^{2\epsilon}/k$. Therefore, fmKdV can be obtained from (3.6) with $r = \pm q$ and $A_0(k) = -4ik^3|2k|^{2\epsilon}$ and similarly fsineG (sinhG) are (3.6) with $r = -q$ $(r = +q)$ and $A_0(k) = i|2k|^{2\epsilon}/(4k)$.

3.3.2 Scattering Data for the AKNS System

With sufficient decay and smoothness of $\mathbf{u}$, we define eigenfunctions for the AKNS system as solutions to equations (3.4) and (3.5) satisfying the boundary conditions

$$\boldsymbol{\phi}(x;k,t) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad \overline{\boldsymbol{\phi}}(x;k,t) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{+ikx}, \quad x \to -\infty, \tag{3.14}$$

$$\boldsymbol{\psi}(x;k,t) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{+ikx}, \quad \overline{\boldsymbol{\psi}}(x;k,t) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad x \to +\infty. \tag{3.15}$$

As the eigenfunctions $\boldsymbol{\psi}$, $\overline{\boldsymbol{\psi}}$ are linearly independent, we can write $\boldsymbol{\phi}$ and $\overline{\boldsymbol{\phi}}$ as

$$\boldsymbol{\phi}(x;k,t) = b(k,t)\boldsymbol{\psi}(x;k,t) + a(k,t)\overline{\boldsymbol{\psi}}(x;k,t), \tag{3.16}$$
$$\overline{\boldsymbol{\phi}}(x;k,t) = \overline{a}(k,t)\boldsymbol{\psi}(x;k,t) + \overline{b}(k,t)\overline{\boldsymbol{\psi}}(x;k,t). \tag{3.17}$$

Then, we can write the scattering data explicitly in terms of the eigenfunctions as

$$a(k,t) = W(\boldsymbol{\phi},\boldsymbol{\psi}), \quad \overline{a}(k,t) = W(\overline{\boldsymbol{\psi}},\overline{\boldsymbol{\phi}}), \tag{3.18}$$
$$b(k,t) = W(\overline{\boldsymbol{\psi}},\boldsymbol{\phi}), \quad \overline{b}(k,t) = W(\overline{\boldsymbol{\phi}},\boldsymbol{\psi}), \tag{3.19}$$

with the Wronskian given by $W(u,v) = u^{(1)}v^{(2)} - u^{(2)}v^{(1)}$. The transmission and reflection coefficients, $\tau(k,t)$, $\overline{\tau}(k,t)$ and $\rho(k,t)$, $\overline{\rho}(k,t)$, are defined by

$$\tau(k,t) = \frac{1}{a(k,t)}, \quad \rho(k,t) = \frac{b(k,t)}{a(k,t)}, \tag{3.20}$$
$$\overline{\tau}(k,t) = \frac{1}{\overline{a}(k,t)}, \quad \overline{\rho}(k,t) = \frac{\overline{b}(k,t)}{\overline{a}(k,t)}. \tag{3.21}$$

We also define the mixed reflection coefficient by

$$\tilde{\rho}(k,t) = \frac{\overline{b}(k,t)}{a(k,t)}. \tag{3.22}$$

The zeros of a and $\overline{a}$ at $k_j = \xi_j + i\eta_j$, $\eta_j > 0$, $j = 1,2,...,J$ and $\overline{k}_j = \overline{\xi}_j + i\overline{\eta}_j$, $\overline{\eta}_j < 0$, $j = 1,2,...,\overline{J}$, respectively, are eigenvalues of the AKNS system corresponding to bound states. With decaying data, these eigenvalues exist only when $r = -q$. We assume the eigenvalues are 'proper', i.e., they are simple zeros of a or $\overline{a}$, they are not on the real k axis, and $J = \overline{J}$; cf. [60]. The bound state eigenfunctions are related by

$$\boldsymbol{\phi}_j(x,t) = b_j(t)\boldsymbol{\psi}_j(x,t), \quad \overline{\boldsymbol{\phi}}_j(x,t) = \overline{b}_j(t)\overline{\boldsymbol{\psi}}_j(x,t), \tag{3.23}$$

where $b_j(t) = b(k_j,t)$. We also define the norming constants by

$$C_j(t) = b_j(t)/a'_j(t), \quad \overline{C}_j(t) = \overline{b}_j(t)/\overline{a}'_j(t), \tag{3.24}$$
$$\tilde{C}_j(t) = \overline{b}_j(t)/a'_j(t), \tag{3.25}$$

where $a'_j(t) = \partial_k a(k,t)|_{k=k_j}$, etc. When $r = \mp q^*$ in (3.4)-(3.5), we have the symmetry reductions

$$\overline{\boldsymbol{\psi}}(x,k,t) = \sigma\boldsymbol{\psi}^*(x,k^*,t), \quad \overline{\boldsymbol{\phi}}(x,k,t) = \sigma^{-1}\boldsymbol{\phi}^*(x,k^*,t), \tag{3.26}$$

for the eigenfunctions and $\overline{a}(k,t) = a^*(k^*,t)$ and $\overline{b}(k,t) = \mp b^*(k^*,t)$ for the scattering data where

$$\sigma_\pm = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}, \quad \sigma_\pm^{-1} = \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}. \tag{3.27}$$

When $r = \pm q$, q real, we have the symmetry reductions

$$\overline{\boldsymbol{\psi}}(x; k, t) = \sigma_\pm \boldsymbol{\psi}(x; -k, t), \quad \overline{\boldsymbol{\phi}}(x; k, t) = \sigma_\pm^{-1} \boldsymbol{\phi}(x; -k, t), \tag{3.28}$$

for the eigenfunctions and

$$\overline{a}(k, t) = a(-k, t), \quad \overline{b}(k, t) = \pm b(-k, t), \tag{3.29}$$

for the scattering data. From the scattering eigenfunctions $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$, we can construct the eigenfunctions of the operator $\mathbf{L}$, $\boldsymbol{\Psi}(x, k, t)$ and $\overline{\boldsymbol{\Psi}}(x, k, t)$, and its adjoint $\mathbf{L}^A$, $\boldsymbol{\Psi}^A(x, k, t)$ and $\overline{\boldsymbol{\Psi}}^A(x, k, t)$ by

$$\boldsymbol{\Psi}(x, k, t) = \left((\psi^{(1)}(x, k, t))^2, (\psi^{(2)}(x, k, t))^2 \right)^T, \tag{3.30}$$

$$\overline{\boldsymbol{\Psi}}(x, k, t) = \left((\overline{\psi}^{(1)}(x, k, t))^2, (\overline{\psi}^{(2)}(x, k, t))^2 \right)^T, \tag{3.31}$$

$$\boldsymbol{\Psi}^A(x, k, t) = \left((\phi^{(2)}(x, k, t))^2, -(\phi^{(1)}(x, k, t))^2 \right)^T, \tag{3.32}$$

$$\overline{\boldsymbol{\Psi}}^A(x, k, t) = \left((\overline{\phi}^{(2)}(x, k, t))^2, -(\overline{\phi}^{(1)}(x, k, t))^2 \right)^T. \tag{3.33}$$

where $\psi^{(j)}$ and $\phi^{(j)}$ are the jth components of the eigenfunctions $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ (and similarly for $\overline{\boldsymbol{\psi}}$ and $\overline{\boldsymbol{\psi}}$). Notice that these are all written in terms of squared eigenfunctions of the AKNS system. Explicitly, we have

$$\mathbf{L}\boldsymbol{\Psi} = k\boldsymbol{\Psi}, \quad \mathbf{L}\overline{\boldsymbol{\Psi}} = k\overline{\boldsymbol{\Psi}}, \tag{3.34}$$

$$\mathbf{L}^A\boldsymbol{\Psi}^A = k\boldsymbol{\Psi}^A, \quad \mathbf{L}^A\overline{\boldsymbol{\Psi}}^A = k\overline{\boldsymbol{\Psi}}^A. \tag{3.35}$$

3.3.3 Scalar Scattering System

The scattering equations for the scalar system, obtained from the symmetry reduction $r = \pm q$, are

$$v_x^{(1)} = -ikv^{(1)} + q(x, t)v^{(2)}, \tag{3.36}$$

$$v_x^{(2)} = +ikv^{(2)} \pm q(x, t)v^{(1)}. \tag{3.37}$$

To construct a family of nonlinear evolution equations for this system, which is a subset of the class in equation (3.6), we use the eigenvalue relations in equations (3.34) and (3.35) in addition to an orthogonality relation from the AKNS system. Taking $r = \pm q$ and writing out $\mathbf{L}\boldsymbol{\Psi} = k\boldsymbol{\Psi}$ in components, we have

$$2ik(\psi^{(1)})^2 = -\frac{\partial(\psi^{(1)})^2}{\partial x} - 2qI_+\left[q(\pm(\psi^{(1)})^2 + (\psi^{(2)})^2)\right], \tag{3.38}$$

$$2ik(\psi^{(2)})^2 = \frac{\partial(\psi^{(2)})^2}{\partial x} + 2qI_+\left[q((\psi^{(1)})^2 \pm (\psi^{(2)})^2)\right]. \tag{3.39}$$

We define the functions

$$\mu_-(x,k,t) = (\psi^{(1)}(x,k,t))^2 + (\psi^{(2)}(x,k,t))^2, \tag{3.40}$$

$$\mu_+(x,k,t) = (\psi^{(1)}(x,k,t))^2 - (\psi^{(2)}(x,k,t))^2. \tag{3.41}$$

Taking $r = +q$, adding (3.38) to (3.39) and using (3.40) and (3.41) we have

$$2ik\mu_- = -\frac{\partial\mu_+}{\partial x}. \tag{3.42}$$

Subtracting equation (3.39) from (3.38) yields

$$2ik\mu_+ = -\frac{\partial\mu_-}{\partial x} - 4qI_+\left[q\mu_-\right]. \tag{3.43}$$

Putting equation (3.42) into (3.43) we get the following scalar eigenvalue equation

$$L_+\mu_+ = k^2\mu_+, \quad L_+ = -\frac{1}{4}\frac{\partial^2}{\partial x^2} + q^2 + qI_+q_y. \tag{3.44}$$

We can similarly show that

$$L_+^A\nu_+ = k^2\nu_+, \quad L_+^A = -\frac{1}{4}\frac{\partial^2}{\partial x^2} + q^2 + q_xI_-, \tag{3.45}$$

where

$$\nu_+(x,k,t) = (\phi^{(1)}(x,k,t))^2 + (\phi^{(2)}(x,k,t))^2, \tag{3.46}$$

$$\nu_-(x,k,t) = (\phi^{(1)}(x,k,t))^2 - (\phi^{(2)}(x,k,t))^2. \tag{3.47}$$

For $r = -q$, we have

$$L_-\mu_- = k^2\mu_-, \quad L_- = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q^2 - qI_+q_y, \tag{3.48}$$

$$L_-^A\nu_- = k^2\nu_-, \quad L_-^A = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q^2 - q_xI_-q. \tag{3.49}$$

Equations (3.44), (3.45), (3.48), and (3.49) define the operators and squared eigenfunctions of the scalar scattering system. For the AKNS scattering problem, we know that the following orthogonality relation holds [10]

$$\int_{-\infty}^{\infty} \left\{ (r_t + 2\Omega(k) r)\, (\psi^{(1)})^2 + (-q_t + 2\Omega(k) q)\, (\psi^{(2)})^2 \right\} dx = 0, \tag{3.50}$$

where $\Omega(k)$ is a suitable function of k, taken to be meromorphic in [10]. With $r = \pm q$, $\Omega(k) = ik\Theta(k^2)$, and μ_1 and μ_2 in (3.40) and (3.41), we may write

$$\int_{-\infty}^{\infty} \left\{ q_t \mu_\pm + 2qik\Theta(k^2)\mu_\mp \right\} dx = 0. \tag{3.51}$$

Noting that we have

$$2ik\mu_- = -\frac{\partial \mu_+}{\partial x} \text{ for } r = +q, \tag{3.52}$$

$$2ik\mu_+ = -\frac{\partial \mu_-}{\partial x} \text{ for } r = -q, \tag{3.53}$$

and using the extension of equations (3.44) and (3.48)

$$\Theta(L_\pm)\mu_\pm = \Theta(k^2)\mu_\pm, \tag{3.54}$$

we can write

$$\int_{-\infty}^{\infty} \left\{ q_t \mu_\pm + q_x \Theta(L_\pm)\mu_\pm \right\} dx = 0, \tag{3.55}$$

using integration by parts. We can then shift $\Theta(L_\pm)$ from operating on $\mu_\pm$ to q_x using the adjoint of L,

$$L^A_\pm = -\frac{1}{4}\partial_x^2 \pm q^2 \pm q_x I_- q, \tag{3.56}$$

to give

$$\int_{-\infty}^{\infty} \left\{ q_t + \Theta(L^A_\pm) q_x \right\} \mu_\pm dx = 0, \tag{3.57}$$

which implies

$$q_t + \Theta(L^A_\pm) q_x = 0. \tag{3.58}$$

This defines the family of nonlinear evolution equations associated to the scalar scattering system in equations (3.36) and (3.37). Notice that if we take $\Theta(L^A_\pm) = -4L^A_\pm$, equation (3.58) gives mKdV

$$q_t + q_{xxx} \mp 6q^2 q_x = 0. \tag{3.59}$$

We can relate the operator Θ directly to the dispersion relation of the linearization of equation 3.58. As $L_\pm^A \to -\frac{1}{4}\partial_x^2$ implies $\Theta(L_\pm^A) \to \Theta(-\frac{1}{4}\partial_x^2)$, this linearization is

$$q_t + \Theta\left(-\partial_x^2/4\right) q_x = 0. \tag{3.60}$$

Putting $q = e^{i(kx-\omega(k)t)}$ gives

$$\Theta(k^2) = \frac{\omega(2k)}{2k}. \tag{3.61}$$

Therefore, using our definitions of linear fmKdV and linear fsineG and fsinhG, with dispersion relations $\omega(k) = -k^3|k|^{2\epsilon}$ and $\omega(k) = |k|^{2\epsilon}/k$ where $|\epsilon| < 1$, respectively, we have $\Theta(L_\pm^A) = -4L_\pm^A|4L_\pm^A|^\epsilon$, $\Theta(L_-^A) = \frac{|4L_-^A|^\epsilon}{4L_-^A}$, and $\Theta(L_+^A) = \frac{|4L_+^A|^\epsilon}{4L_+^A}$, respectively (recall that fmKdV has $r = \pm q$ while fsineG and fsinhG have $r = -q$ and $r = +q$, respectively). Therefore, we can write fmKdV, fsineG, and fsinhG as

$$q_t - 4L_\pm^A|2L_\pm^A|^\epsilon q_x = 0, \tag{3.62}$$

$$q_t + \frac{|4L_-^A|^\epsilon}{4L_-^A} q_x = 0, \quad u_{tx} + \frac{|4L_-^A|^\epsilon}{4L_-^A} u_{xx}, \tag{3.63}$$

$$q_t + \frac{|4L_+^A|^\epsilon}{4L_+^A} q_x = 0, \quad u_{tx} + \frac{|4L_+^A|^\epsilon}{4L_+^A} u_{xx}. \tag{3.64}$$

Notice that as $L_\pm \to -\partial_x^2/4$, in the linear limit, both fsineG and fsinhG both converge to (3.11). We can also define a hierarchy of fractional equations associated to the fmKdV, fsineG, and fsinhG equations by adding an integer power to the dispersion relation, i.e., $\omega(k) = -k^3|k|^{2(m+\epsilon)}$ and $\omega(k) = |k|^{2(m+\epsilon)}/k$ with $m \in \mathbb{Z}$. This allows us to, in effect, obtain an equation for any fractional order in $\mathbb{R}$. Currently, the meaning of $|L_\pm^A|^\epsilon$ is not clear; it will be defined in the next section using a spectral expansion in terms of the squared eigenfunctions $\mu_\pm$ and $\nu_\pm$.

3.3.4 Completeness of Squared Scalar Eigenfunctions

In [6] it was shown that the eigenfunctions $\mathbf{\Psi}$ and $\mathbf{\Psi}^A$, equations (3.30) and (3.32), of the AKNS system are complete in $L^1(\mathbb{R})$. Specifically, for a sufficiently smooth and decaying vector-valued function $\mathbf{v}(x) = \left(v^{(1)}(x), v^{(2)}(x)\right)^T$, we have

$$\begin{aligned} \mathbf{v}(x) &= \sum_{n=1}^{2} \int_{\Gamma_\infty^{(n)}} dk f_n(k) \int_{-\infty}^{\infty} dy\, \mathbf{G}_n(x,y,k)\mathbf{v}(y), \\ \mathbf{G}_1(x,y,k) &= \mathbf{\Psi}(x,k)\mathbf{\Psi}^A(y,k)^T, \quad f_1(k) = -\tau^2(k)/\pi, \\ \mathbf{G}_2(x,y,k) &= \overline{\mathbf{\Psi}}(x,k)\overline{\mathbf{\Psi}}^A(y,k)^T, \quad f_2(k) = \overline{\tau}^2(k)/\pi, \end{aligned} \tag{3.65}$$

where $\Gamma_R^{(1)}$ ($\Gamma_R^{(2)}$) is the semicircular contour in the upper (lower) half plane evaluated from $-R$ to $+R$ and $\tau(k)$ and $\overline{\tau}(k)$ are transmission coefficients defined in equations (3.16) and (3.17). Time is suppressed throughout this section. Notice that $\mathbf{G}_n$, $n = 1, 2$ are 2×2 matrices. However, here we need to use the adjoint completeness relation, which may be found directly from (3.65) using the inner product $(\mathbf{u}, \mathbf{v}) := \int_{-\infty}^{\infty} \mathbf{u}(x)^T \mathbf{v}(x) dx$ where $\mathbf{u}$, $\mathbf{v}$ are 2×1 column vectors. To do this, we expand $\mathbf{v}$ using the above completeness relation, and then exchange the order of integration to find an expansion for $\mathbf{u}$. This procedure gives us

$$\begin{aligned} \mathbf{v}(x) &= \sum_{n=1}^{2} \int_{\Gamma_\infty^{(n)}} dk f_n(k) \int_{-\infty}^{\infty} dy\, \mathbf{G}_n^A(x, y, k) \mathbf{v}(y), \\ \mathbf{G}_1^A(x, y, k) &= \mathbf{\Psi}^A(x, k) \mathbf{\Psi}(y, k)^T, \quad f_1(k) = -\tau^2(k)/\pi, \\ \mathbf{G}_2^A(x, y, k) &= \overline{\mathbf{\Psi}}^A(x, k) \overline{\mathbf{\Psi}}(y, k)^T, \quad f_2(k) = \overline{\tau}^2(k)/\pi. \end{aligned} \tag{3.66}$$

However, when $r = \pm q$ with q real, the symmetry reductions in equations (3.28) and (3.29) give

$$\overline{\mathbf{\Psi}}(k) = \sigma_+ \mathbf{\Psi}(-k), \quad \overline{\mathbf{\Psi}}^A(k) = -\sigma_+ \mathbf{\Psi}^A(-k). \tag{3.67}$$

Therefore, the adjoint completness relation in (3.66) reduces to

$$\mathbf{v}(x) = -\int_{\Gamma_\infty^{(1)}} dk \frac{\tau^2(k)}{\pi} \int_{-\infty}^{\infty} dy \mathbf{\Psi}^A(x, k) \mathbf{\Psi}(y, k)^T \mathbf{v}(y) \tag{3.68}$$

$$- \int_{\Gamma_\infty^{(2)}} dk \frac{\tau^2(-k)}{\pi} \int_{-\infty}^{\infty} dy \sigma_+ \mathbf{\Psi}^A(x, -k) \mathbf{\Psi}(y, -k)^T \sigma_+ \mathbf{v}(y). \tag{3.69}$$

The second integral may be rewritten with the substitution $\xi = -k$ as

$$- \int_{\Gamma_\infty^{(1)}} d\xi \frac{\tau^2(\xi)}{\pi} \int_{-\infty}^{\infty} dy \sigma_+ \mathbf{\Psi}^A(x, \xi) \mathbf{\Psi}(y, \xi)^T \sigma_+ \mathbf{v}(y). \tag{3.70}$$

Therefore, we have

$$\mathbf{v}(x) = \int_{\Gamma_\infty^{(1)}} d\xi \frac{\tau^2(\xi)}{\pi} \int_{-\infty}^{\infty} dy \left[\mathbf{\Psi}^A(x, k) \mathbf{\Psi}(y, k)^T - \sigma_+ \mathbf{\Psi}^A(x, k) \mathbf{\Psi}(y, k)^T \sigma_+ \right] \mathbf{v}(y). \tag{3.71}$$

If we put $\mathbf{v}(x) = \mathbf{h}(x) = (h(x), \mp h(x))^T$, we can reduce the completeness relation to

$$h(x) = \mp \int_{\Gamma_\infty^{(1)}} dk \int_{-\infty}^{\infty} dy\, g_\pm(x, y, k) h(y), \tag{3.72}$$

for the scalar function $h(x) \in L^1(\mathbb{R})$ where

$$g_{\pm}(x,y,k) = \frac{\tau^2(k)}{\pi}\nu_{\pm}(x,k)\mu_{\pm}(y,k), \tag{3.73}$$

with $\nu_{\pm}(x,k) = (\phi^{(1)}(x,k))^2 \pm (\phi^{(2)}(x,k))^2$ and $\mu_{\pm}(x,k) = (\psi^{(1)}(x,k))^2 \mp (\psi^{(2)}(x,k))^2$ the squared eigenfunctions of the scalar system. Then, the action of the operator $\Theta(L_{\pm}^{A})$ on the function $h(x)$ may be written as

$$\Theta(L_{\pm}^{A})h(x) = \mp \int_{\Gamma_{\infty}^{(1)}} dk\Theta(k^2) \int_{-\infty}^{\infty} dy\, g_{\pm}(x,y,k)h(y), \tag{3.74}$$

and the famility of nonlinear evolution equations in equation (3.58) becomes

$$q_t \mp \int_{\Gamma_{\infty}^{(1)}} dk\Theta(k^2) \int_{-\infty}^{\infty} dy\, g_{\pm}(x,y,k)\partial_y q(y) = 0. \tag{3.75}$$

In particular, fmKdV can be represented as

$$q_t \mp \int_{\Gamma_{\infty}^{(1)}} dk|2k|^{2(1+\epsilon)} \int_{-\infty}^{\infty} dy\, g_{\pm}(x,y,k) \left[q_{yyy} \mp 6q^2\right] = 0, \tag{3.76}$$

and fsineG and fsinhG are given by

$$q_t \mp \int_{\Gamma_{\infty}^{(1)}} dk|2k|^{2(\epsilon-1)} \int_{-\infty}^{\infty} dy\, g_{-}(x,y,k)\partial_y q(y) = 0, \tag{3.77}$$

$$q_t \mp \int_{\Gamma_{\infty}^{(1)}} dk|2k|^{2(\epsilon-1)} \int_{-\infty}^{\infty} dy\, g_{+}(x,y,k)\partial_y q(y) = 0. \tag{3.78}$$

Because the k integral in equation (3.72) is over the semicircle in the upper half plane, it can be expressed instead in terms of an integral along the real line and a sum over the residues using contour integration. This is a useful representation because it explicitly separates the continuous and discrete spectra, where the latter corresponds to bound states at $k = k_j$, $j = 1, 2, ..., J$. Using the closed contour composed of $\Gamma^{(1)}$ and an integral along the real line from ∞ to $-\infty$, we can write

$$\begin{aligned}\Theta(L_{\pm}^{A})h(x) = \mp &\int_{-\infty}^{\infty} dk\Theta(k^2) \int_{-\infty}^{\infty} dy\, g_{\pm}(x,y,k)h(y), \\ &\pm 2\pi i \sum_{j=1}^{J} \mathrm{Res}\left(\Theta(k^2) \int_{-\infty}^{\infty} dy\, g_{\pm}(x,y,k)h(y), k_j\right).\end{aligned} \tag{3.79}$$

Because $\nu_{\pm}$ and $\mu_{\pm}$ are all analytic in the upper half plane, the only residues come from the poles of τ^2, or zeros of a^2. These occur at $k_j = \xi_j + i\eta_j$, $j = 1, 2, ..., J$ and are assumed to be simple, meaning τ^2 has a double pole at k_j. Therefore, we can compute the residue at k_j as

$$
\begin{aligned}
\operatorname{Res}\left(\int_{-\infty}^{\infty} dy\, g_{\pm}(x,y,k)h(y), k_j\right) &= \lim_{k\to k_j} \frac{\partial}{\partial k}\left[(k-k_j)^2\Theta(k^2)\int_{-\infty}^{\infty} dy\, g_{\pm}(x,y,k)h(y)\right], \\
&= \frac{\Theta(k_j^2)}{\pi(a_j')^2}\int_{-\infty}^{\infty} dy \left\{\partial_k \nu_{\pm}(x,k)\mu_{\pm}(y,k) + \nu_{\pm}(x,k)\partial_k\mu_{\pm}(x,k)\right\}_{k=k_j} h(y), \\
&+ \left(\frac{2k_j\Theta'(k_j^2)}{\pi(a_j')^2} - \frac{a_j''\Theta(k_j^2)}{\pi(a_j')^3}\right)\int_{-\infty}^{\infty} dy\, \nu_{\pm}(x,k_j)\mu_{\pm}(y,k_j)h(y).
\end{aligned} \tag{3.80}
$$

Defining

$$
g_{\pm,j}^{(1)}(x,y) = \frac{2i}{(a_j')^2}\left\{\partial_k \nu_{\pm}(x,k)\mu_{\pm}(y,k) + \nu_{\pm}(x,k)\partial_k\mu_{\pm}(y,k)\right\}_{k=k_j}, \tag{3.81}
$$

$$
g_{\pm,j}^{(2)}(x,y) = \frac{2i}{(a_j')^2}\nu_{\pm}(x,k_j)\mu_{\pm}(y,k_j), \tag{3.82}
$$

$$
g_{\pm,j}^{(3)}(x,y) = -\frac{2ia_j''}{(a_j')^3}\nu_{\pm}(x,k_j)\mu_{\pm}(y,k_j), \tag{3.83}
$$

we have

$$
\begin{aligned}
h(x) &= \mp\int_{-\infty}^{\infty} dk\Theta(k^2)\int_{-\infty}^{\infty} dy\, g_{\pm}(x,y,k)h(y), \\
&\pm\sum_{j=1}^{J}\int_{-\infty}^{\infty} dy\left\{\Theta(k_j^2)g_{\pm,j}^{(1)}(x,y) + 2k_j\Theta'(k_j^2)g_{\pm,j}^{(2)}(x,y) + \Theta(k_j^2)g_{\pm,j}^{(3)}(x,y)\right\} h(y).
\end{aligned} \tag{3.84}
$$

Notice that if we take $\Theta(k^2) = 1$, then we have the identity in (3.72) written with continuous and discrete spectra separated.

3.4 The IST for Fractional Modified KdV, SineG, and SinhG

Solving nonlinear evolution equations with the IST is analogous to solving linear evolution equations with Fourier transforms. To solve linear problems, the Fourier transform is taken to map the problem into Fourier space where the time evolution is described by a simple set of differential equations. These equations are then solved to give the solution at any time t in Fourier space. Finally, the solution is mapped back to physical space using the inverse Fourier transform, which amounts to evaluating an integral. Mapping the initial condition into scattering space via direct scattering is analogous to taking the Fourier transform, time evolution in scattering space is nearly identical to that in Fourier space, and inverse scattering maps the solution to the nonlinear problem back into physical space just as the inverse Fourier transform does. The major difference between Fourier transforms and the IST is that performing integrals for the Fourier transform and inverse Fourier transform is replaced by solving linear integral equations for direct scattering and inverse scattering. In the following we, outline direct scattering, time evolution, and inverse scattering for the scalar scattering system.

3.4.1 Direct Scattering

To solve the nonlinear evolution equation, equation (3.58), by the inverse scattering transform, we first map the initial condition into scattering space; this is analogous to taking the Fourier transform of a linear partial differential equation. This process involves analyzing linear integral equations for the eigenfunctions, determining their analytic properties, and then obtaining the scattering data using Wronskian relations.

Eigenfunctions of the scalar scattering problem are precisely $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ of the AKNS system with the symmetry reduction in equations (3.28) and (3.29). They are solutions to equations (3.36) and (3.37) subject to the boundary conditions in equations (3.14) and (3.15) with the scattering data defined by equation (3.16). It is convenient to express the scattering functions in terms of *Jost solutions* by taking

$$\mathbf{M}(x,k,t) = e^{ikx}\boldsymbol{\phi}(x,k,t), \qquad \mathbf{N}(x,k,t) = e^{-ikx}\boldsymbol{\psi}(x,k,t). \tag{3.85}$$

$$\overline{\mathbf{M}}(x,k,t) = \sigma_\pm^{-1}\mathbf{M}(x,-k,t), \quad \overline{\mathbf{N}}(x,k,t) = \sigma_\pm\mathbf{N}(x,-k,t), \tag{3.86}$$

where the symmetry reductions for $r = \pm q$ are in terms of

$$\sigma_\pm = \begin{pmatrix} 0 & 1 \\ \pm1 & 0 \end{pmatrix}, \quad \sigma_\pm^{-1} = \begin{pmatrix} 0 & \pm1 \\ 1 & 0 \end{pmatrix}. \tag{3.87}$$

Then, the boundary conditions become constant

$$\mathbf{M}(x,k,t) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x \to -\infty, \quad \mathbf{N}(x,k,t) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x \to \infty, \tag{3.88}$$

and the scattering equation, where either $\mathbf{M}$ or $\mathbf{N}$ is represented generically as $\boldsymbol{\chi} = \boldsymbol{\chi}(x,k,t)$, becomes

$$\partial_x\boldsymbol{\chi} = ik\mathbf{B}\boldsymbol{\chi} + \mathbf{Q}\boldsymbol{\chi}, \tag{3.89}$$

where

$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \tag{3.90}$$

This differential equation can be converted to an integral equation for $\mathbf{M}$ and $\mathbf{N}$, cf. [16],

$$\mathbf{M}(x,k,t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{\infty} \mathbf{G}(x-\xi,k,t)\mathbf{Q}(\xi,t)\mathbf{M}(\xi,k,t)\,d\xi, \tag{3.91}$$

$$\mathbf{N}(x,k,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{\infty} \overline{\mathbf{G}}(x-\xi,k,t)\mathbf{Q}(\xi,t)\mathbf{N}(\xi,k,t)\,d\xi, \tag{3.92}$$

where

$$\mathbf{G}(x,k,t) = \theta(x)\begin{pmatrix} 1 & 0 \\ 0 & e^{2ikx} \end{pmatrix}, \tag{3.93}$$

$$\overline{\mathbf{G}}(x,k,t) = -\theta(-x)\begin{pmatrix} e^{-2ikx} & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.94}$$

with $\theta(x)$ the Heaviside function defined by

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{3.95}$$

So long as $q, r \in L^1(\mathbb{R})$, these Volterra integral equations have absolutely and uniformly convergent Neumann series in the upper half k-plane [60]. Therefore, the functions $\mathbf{M}$ and $\mathbf{N}$ are analytic functions of k for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k \geq 0$. This also implies that $\overline{\mathbf{M}}$ and $\overline{\mathbf{N}}$ are analytic for $\operatorname{Im} k < 0$ and continuous for $\operatorname{Im} k \leq 0$ from their relations in equation (3.86). Using these integral equations and the intial condition at $t = 0$, the Jost solutions $\mathbf{M}$ and $\mathbf{N}$ can be constructed at $t = 0$ and, subsequently, the scattering functions from the relations in equation (3.85). Then, the initial scattering data may be derived from the Wronskian relations in equations (3.18) and (3.19).

We will also need the asymptotic properties of $\mathbf{N}$ and $\mathbf{M}$ to reconstruct the solution in inverse scattering; so, expanding equations (3.91) and (3.92) in large k, after integrating by parts, we have

$$\mathbf{M}(x,k,t) = \begin{pmatrix} 1 - \frac{1}{2ik}\int_{-\infty}^{x} q(\xi,t)r(\xi,t)\,d\xi \\ -\frac{1}{2ik}r(x,t) \end{pmatrix} + \mathcal{O}(k^{-2}) \tag{3.96}$$

$$\mathbf{N}(x,k,t) = \begin{pmatrix} \frac{1}{2ik}q(x,t) \\ 1 - \frac{1}{2ik}\int_{x}^{\infty} q(\xi,t)r(\xi,t)\,d\xi \end{pmatrix} + \mathcal{O}(k^{-2}) \tag{3.97}$$

3.4.2 Time Evolution

After the initial condition is projected into scattering space by reconstructing the scattering functions and scattering data, the data is evolved in time by solving a simple set of ordinary differential equations. The scattering functions evolve in time according to

$$\mathbf{v}_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\mathbf{v} \tag{3.98}$$

where A, B, C are functions of x, k, t which cannot be represented generally. However, their asymptotic properties can be used to characterize the time evolution of the scattering data [16] as

$$a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)e^{-2ik\Theta(k^2)t}, \tag{3.99}$$

$$\rho(k,t) = \rho(k,0)e^{-2ik\Theta(k^2)t}, \quad C_j(t) = C_j(0)e^{-2ik_j\Theta(k_j^2)t}, \tag{3.100}$$

for $j = 1, 2, ..., J$. Further, the mixed reflection coefficient and norming constants, defined in equation (3.101), evolve according to

$$\tilde{\rho}(k,t) = \tilde{\rho}(k,0)e^{2ik\Theta(k^2)t}, \quad \tilde{C}_j(t) = \tilde{C}_j(0)e^{2i\overline{k}_j\Theta(\overline{k}_j^2)} \tag{3.101}$$

We recall that $\Theta(k^2)$ is related to the linear dispersion relation as in equation (3.61). To characterize the spectral expansion in equation (3.75), we need to be able to compute the time evolution of the scattering functions $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$. Although equation (3.98) does not give this in a simple way, the scattering functions can be evolved in time using inverse scattering, which is discussed next.

3.4.3 Inverse Scattering

Inverse scattering allows the construction of the solution to the nonlinear evolution equation $q(x,t)$ and the scattering functions $\boldsymbol{\psi}(x,k,t)$ and $\boldsymbol{\phi}(x,k,t)$ from the scattering data, $a(k,t)$ and $b(k,t)$ obtained from equation (3.99). For the Jost solutions (3.85), we can write the scattering data in equation (3.16), using (3.20), as

$$\boldsymbol{\mu}(x,k,t) = \overline{\mathbf{N}}(x,k,t) + \rho(k,t)e^{2ikx}\mathbf{N}(x,k,t), \quad \boldsymbol{\mu}(x,k,t) = \mathbf{M}(x,k,t)/a(k,t), \tag{3.102}$$

where $\mathbf{M}$ is analytic in the upper half plane, $\boldsymbol{\mu}$ is meromorphic with simple poles at the zeros of a, and $\overline{\mathbf{N}}$ is analytic in the lower half plane (ρ is not analytic in general). Therefore, equation (3.102) defines the "jump" condition of a Riemann-Hilbert problem which we will transform onto an integral equation for $\mathbf{N}$. This equation will allow the construction of the scattering function $\boldsymbol{\psi}$ and the solution $q(x,t)$. We will also outline how the same method can be used to derive an equation for $\mathbf{M}$.

We assume that a has simple zeros, and hence $\boldsymbol{\mu}$ has simple poles in the upper half plane at k_j for $j = 1, 2, ..., J$ with no zeros along the real line. Then, as $\boldsymbol{\mu}$ has only simple poles, we can represent it as

$$\boldsymbol{\mu}(x,k,t) = \mathbf{h}(x,k,t) + \sum_{j=1}^{J} \frac{\mathbf{A}_j(x,t)}{k - k_j}, \tag{3.103}$$

where $\mathbf{h}$ is analytic in k for $\operatorname{Im} k > 0$. By integrating in a small neighborhood around each k_j, and using equation (3.102), we find that $\mathbf{A}_j$ is given by

$$\mathbf{A}_j(x,t) = C_j(t)e^{2ik_jx}\mathbf{N}_j(x,t), \text{ for } j = 1, 2, ..., J, \tag{3.104}$$

where $C_j(t) = b_j(t)/a'_j(t)$ and $\mathbf{N}_j(x,t) = \mathbf{N}(x,k_j,t)$. We then define the projection operators

$$P^{\pm}[f](k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - (k \pm i0)} d\xi. \tag{3.105}$$

If f_+ (f_-) is analytic in the upper (lower) half plane and $f_\pm(k) \to 0$ as $|k| \to \infty$ for $\operatorname{Im} k > 0$ ($\operatorname{Im} k < 0$), then

$$P^{\pm}[f_\pm] = \pm f_\pm, \quad P^{\pm}[f_\mp] = 0. \tag{3.106}$$

Taking equation (3.102) and subtracting $(1,0)^T$ and the simple poles of $\boldsymbol{\mu}$, we have

$$\mathbf{h}(x,k,t) - \begin{pmatrix}1\\0\end{pmatrix} = \overline{\mathbf{N}}(x,k,t) - \begin{pmatrix}1\\0\end{pmatrix} - \sum_{j=1}^{J} \frac{\mathbf{A}_j(x,t)}{k-k_j} + \rho(k,t)e^{2ikx}\mathbf{N}(x,k,t). \tag{3.107}$$

The left side is analytic in the upper half plane and approaches zero as $|k| \to \infty$; $\overline{\mathbf{N}} - (1,0)^T$ is also analytic in the lower half plane and vanishes asymptotically. Therefore, applying P^- to (3.107) gives

$$\overline{\mathbf{N}}(x,k,t) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j=1}^{J} \frac{\mathbf{A}_j(x,t)}{k-k_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\xi,t)e^{2i\xi x}}{\xi - (k - i0)}\mathbf{N}(x,\xi,t)\, d\xi. \tag{3.108}$$

Noting that $\overline{\mathbf{N}}(x,k,t) = \sigma\mathbf{N}(x,-k,t)$ and using the expression for $\mathbf{A}_j$ in equation (3.104), we find the integral equation for $\mathbf{N}$

$$\mathbf{N}(x,k,t) = \begin{pmatrix}0\\1\end{pmatrix} - \sum_{j=1}^{J} \frac{C_j(t)e^{2ik_jx}}{k+k_j}\sigma^{-1}\mathbf{N}_j(x,t) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\xi,t)e^{2i\xi x}}{\xi + k + i0}\sigma^{-1}\mathbf{N}(x,\xi,t)\, d\xi. \tag{3.109}$$

Evaluating this at k_ℓ for $\ell = 1, 2, ..., J$, we obtain an equation for $\mathbf{N}_\ell(x,t)$.

$$\mathbf{N}_\ell(x,t) = \begin{pmatrix}0\\1\end{pmatrix} - \sum_{j=1}^{J} \frac{C_j(t)e^{2ik_jx}}{k_\ell+k_j}\sigma^{-1}\mathbf{N}_j(x,t) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\xi,t)e^{2i\xi x}}{\xi + k_\ell}\sigma^{-1}\mathbf{N}(x,\xi,t)\, d\xi. \tag{3.110}$$

We can similarly show that $\mathbf{M}$ and $\mathbf{M}_j$ solve

$$\mathbf{M}(x,k,t) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j=1}^{J} \frac{\tilde{C}_j(t)e^{-2ik_jx}}{k+k_j}\sigma\mathbf{M}_j(x,t) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\tilde{\rho}(\xi,t)e^{-2i\xi x}}{\xi + k + i0}\sigma\mathbf{M}(x,\xi,t)\, d\xi, \tag{3.111}$$

$$\mathbf{M}_\ell(x,t) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j=1}^{J} \frac{\tilde{C}_j(t)e^{-2ik_jx}}{k_\ell+k_j}\sigma\mathbf{M}_j(x,t) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\tilde{\rho}(\xi,t)e^{-2i\xi x}}{\xi + k_\ell}\sigma\mathbf{M}(x,\xi,t)\, d\xi. \tag{3.112}$$

These equations can then be expanded for large k and, by comparing these expansions to those for the direct scattering problem in equations (3.96) and (3.97), we can recover the solution at any time $q(x,t)$ from $\mathbf{N}(x,k,t)$ as

$$q(x,t) = \mp 2i \sum_{j=1}^{J} e^{2ik_j x} C_j(t) N_j^{(2)}(x,t) \pm \frac{1}{\pi} \int_{-\infty}^{\infty} \rho(\xi,t) e^{2i\xi x} N^{(2)}(x,\xi,t)\, d\xi. \tag{3.113}$$

Notice that just as the spectral expansion of the $\Theta(L_{\pm}^{A})$ split into discrete and continuous spectra in equation (3.84), the solution q is composed of a sum over discrete contributions and an integral over continuous contributions. These equations can be converted into the following GLM type integral equations [10]

$$\mathbf{K}(x,y;t) \pm \begin{pmatrix}1\\0\end{pmatrix} F(x+y;t) + \int_{x}^{\infty} \sigma^{-1}\mathbf{K}(x,s;t)F(s+y;t)\, ds = 0, \tag{3.114}$$

$$\mathbf{L}(x,y;t) \pm \begin{pmatrix}0\\1\end{pmatrix} G(x+y;t) + \int_{-\infty}^{x} \sigma^{-1}\mathbf{L}(x,s;t)G(s+y;t)\, ds = 0, \tag{3.115}$$

where

$$F(x;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\xi,t) e^{+i\xi x} d\xi - i \sum_{j=1}^{J} C_j(t) e^{ik_j x}, \tag{3.116}$$

$$G(x;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\rho}(\xi,t) e^{-i\xi x} d\xi - i \sum_{j=1}^{J} \tilde{C}_j(t) e^{ik_j x}, \tag{3.117}$$

and the Jost eigenfunctions are related to the triangular kernel by

$$\mathbf{N}(x;k,t) = \begin{pmatrix}0\\1\end{pmatrix} + \int_{x}^{\infty} \mathbf{K}(x,s;t) e^{-ik(x-s)} ds, \ \operatorname{Im} k > 0, \tag{3.118}$$

$$\mathbf{M}(x;k,t) = \begin{pmatrix}1\\0\end{pmatrix} - \int_{-\infty}^{x} \mathbf{L}(x,s;t) e^{+ik(x-s)}\, ds, \ \operatorname{Im} k > 0. \tag{3.119}$$

The solution of the nonlinear scalar equation can then be obtained from

$$q(x,t) = -2K^{(1)}(x,x;t), \tag{3.120}$$

where $K^{(1)}$ denotes the 1st component of the vector $\mathbf{K}$. If we partition F and G into continuous and discrete parts, then we can show that the continuous part (radiation) goes to zero as $t \to \infty$, leaving just the discrete part; i.e., the N-soliton solution.

3.5 The One Soliton Solution

Pure soliton solutions of the scalar general evolution equation (3.58) are reflectionless, i.e., $\rho(k,t) = 0$ on the real line. They are also bound states corresponding to the discrete eigenvalues at the zeros of a. We note that soliton solutions for $r = q$ do not exist when we assume q and r vanish sufficiently rapidly as $|x| \to \infty$. Because of this, we will only consider $r = -q$ for the remainder of our study. For a given initial condition, the number of discrete eigenvalues $k_j = \xi_j + i\eta_j$ $j = 1, 2, ..., J$ gives the number of solitons. The general J-soliton solution can be reduced to solving a linear algebraic system. For simplicity we consider the one-soliton solution, $J = 1$, although the argument we lay out can be used also for larger J. We find the one-soliton solution by constructing $\mathbf{N}(x, k, t)$ from equation (3.109) and (3.110) and then recovering $q(x, t)$ using (3.113). We will then explicitly verify that this solution is in fact a solution of the general evolution equation characterized in physical space by the completeness relation, equation (3.75), using complex variable methods. This will require that we construct $\mathbf{N}(x, k, t)$ and $\mathbf{M}(x, k, t)$ (thereby $\mu_\pm(x, k, t)$, $\nu_\pm(x, k, t)$ and $\tau(k, t)$).

3.5.1 Deriving the One Soliton from Inverse Scattering

Putting $\rho = 0$ and $J = 1$ into equation (3.110) and taking $k_1 = i\eta$ and $C_1(0) = -2i\eta e^{2\eta x_0}$ such that $C_1(t) = -2i\eta e^{2\eta x_0 + 2\eta\Theta(-\eta^2)t}$ gives an algebraic equation for $\mathbf{N}_1(x, t)$

$$\mathbf{N}_1(x,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{-z_t(x)}\sigma^{-1}\mathbf{N}_1(x,t), \tag{3.121}$$

where $z_t(x) = 2\eta(x - x_0) - 2\eta\Theta(-\eta^2)t$. The eigenvalue k_1 is imaginary because for $r = -q$, all discrete eigenvalues come in pairs $\{k_j, -k_j^*\}_{j=1}^J$. Solving this gives

$$N_1^{(1)}(x,t) = -\frac{1}{2}\text{sech}\{z_t(x)\}, \quad N_1^{(2)}(x,t) = \frac{1}{2}\left(1 + \tanh\{z_t(x)\}\right), \tag{3.122}$$

and then putting these components into equation (3.113) yields

$$q(x,t) = 2\eta\,\text{sech}\{z_t(x)\}. \tag{3.123}$$

3.5.2 Verifying the One Soliton

To verify that the one soliton given in equation is truly a solution to the general evolution equation in physical space (3.58), we evaluate the spectral expansion of $\Theta(L_\pm^A)$ in equation (3.74) at time t; this means we need to know $\mu_\pm$, $\nu_\pm$ and τ at time t. These can be recovered from the scattering functions $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ which are related to $\mathbf{N}$ and $\mathbf{M}$ by equation (3.85). We first recover the Jost functions. The $\mathbf{N}$ function can be constructed from $\mathbf{N}_1$ using equation (3.109) which gives

$$\mathbf{N}(x,k,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{2i\eta e^{-z_t(x)}}{k + i\eta}\sigma^{-1}\mathbf{N}_1(x,t) = \left(-i\eta\frac{\text{sech}\{z_t(x)\}}{k + i\eta}, \frac{k + i\eta\tanh\{z_t(x)\}}{k + i\eta}\right)^T. \tag{3.124}$$

We can find the $\mathbf{M}$ Jost solutions in a similar manner to $\mathbf{N}$ using equations (3.111) and (3.112) noting that $\tilde{C}_1(0) = 2i\eta e^{-2\eta x_0}$, $C_1(0)$ with η replaced by $-\eta$, and so $\tilde{C}_1(t) = 2i\eta e^{-2\eta x_0 + 2\eta\Theta(-\eta^2)t}$. We find

$$M_1^{(1)}(x,t) = \frac{1}{2}\left(1 - \tanh\{z_t(x)\}\right), \quad M_1^{(2)}(x,t) = -\frac{1}{2}\text{sech}\{z_t(x)\}, \tag{3.125}$$

$$\mathbf{M}(x,k,t) = \left(\frac{k - i\eta\tanh\{z_t(x)\}}{k+i\eta}, -i\eta\frac{\text{sech}\{z_t(x)\}}{k+i\eta}\right)^T. \tag{3.126}$$

From these, we can construct $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ using equation (3.85). Using the wronskian relation in equation (3.18) and the fact that $\tau = 1/a$, we have

$$\tau(k) = \frac{k+i\eta}{k-i\eta}. \tag{3.127}$$

We then build μ_- and ν_-, the squared eigenfunctions for the scalar system. These are

$$\mu_-(x,k,t) = e^{2ikx}\frac{\left(\eta^2 - k^2 + 2ik\eta\tanh\{z_t(x)\}\right)}{(k+i\eta)^2}, \tag{3.128}$$

$$\nu_-(x,k,t) = e^{-2ikx}\frac{\left(k^2 - \eta^2 - 2ik\eta\tanh\{z_t(x)\} + 2\eta^2\text{sech}^2\{z_t(x)\}\right)}{(k+i\eta)^2}. \tag{3.129}$$

Therefore, we can construct the kernel $g_-(x,y,k,t) = \tau^2(k)\nu_-(x,k,t)\mu_-(y,k,t)/\pi$ inside of the spectral definition of $\Theta(L_-^A)$ in equation (3.74). We can now show that the soliton solution for q given in equation (3.123) is truly a solution to the general evolution equation (3.58) by explicitly demonstrating that equation (3.75) holds. We will evaluate the operator

$$\Theta(L_-^A)\partial_x q(x,t) = \int_{\Gamma_\infty^{(1)}} dk\Theta(k^2)\int_{-\infty}^{\infty} dy\, g_-(x,y,k,t)\partial_y q(y,t), \tag{3.130}$$

and show that it is equivalent to $-q_t$ where q is defined by equation (3.123). It is most prudent to split this into its continuous and discrete parts as in equation (3.84). As we will see, the portion of the operator related to the continuous spectra will vanish while that associated to the single eigenvalue k_1 will satisfy equation (3.75). The continuous part vanishing comes from the fact that

$$I(k,t) = \int_{-\infty}^{\infty} \mu_-(y,k,t)\partial_y q(y,t)\, dy = 0, \tag{3.131}$$

for all k. Looking at equation (3.84), we can see that this statement implies that the continuous portion, the integral of $g_-(x,y,k)$ over the real line, vanishes. It also implies that the integrals associated to the discrete kernels $g_{-,1}^{(2)}(x,y)$ and $g_{-,1}^{(3)}(x,y)$ and the first half of the $g_{-,1}^{(1)}(x,y)$ integral are zero. The single term that does not vanish is

$$\Theta(L_-^A)\partial_x q(x,t) = -\frac{2i}{(a_1')^2}\Theta(-\eta^2)\nu_-(x,i\eta,t)\int_{-\infty}^{\infty}\partial_k\mu_-(y,k,t)|_{k=i\eta}\partial_y q(y)\,dy. \tag{3.132}$$

By making the change of variables $\xi = z_t(y)$, $d\xi = z_t'(y)dy = 2\eta dy$ where $z_t(x) = 2\eta(x-x_0) - 2\eta\Theta(-\eta^2)t$, the above integral becomes

$$\int_{-\infty}^{\infty}\partial_k\mu_-(y,k,t)|_{k=i\eta}\partial_y q(y)\,dy = -ie^{z_t(x)-2\eta x}\int_{-\infty}^{\infty}(2\eta x - z_t(x) + \xi)\,\mathrm{sech}^2\xi\tanh\xi\,d\xi. \tag{3.133}$$

Of the three terms in the integral, the first two vanish because the function is odd. The final term, i.e., $\xi\mathrm{sech}^2\xi\tanh\xi$, can be evaluated using integration by parts and the fundamental theorem of calculus to give

$$\int_{-\infty}^{\infty}\xi\mathrm{sech}^2\xi\tanh\xi\,d\xi = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{sech}^2\xi\,d\xi = 1 \tag{3.134}$$

Therefore, evaluating $\nu_-(x,k,t)$ in equation (3.129) at $i\eta$ and using $(a_1')^2 = -4\eta^2$ we find

$$\Theta(L_-^A)\partial_x q(x,t) = -4\eta^2\Theta(-\eta^2)\mathrm{sech}\{z_t(x)\}\tanh\{z_t(x)\}. \tag{3.135}$$

Comparing this to q_t

$$q_t(x,t) = 4\eta^2\Theta(-\eta^2)\mathrm{sech}\{z_t(x)\}\tanh\{z_t(x)\}, \tag{3.136}$$

we notice that equation (3.130) is true provided equation (3.131) holds, which we will now confirm. Again, we make the change of variables $\xi = z_t(y)$ so that the integral $I(k,t)$ becomes

$$I(k,t) = A(k,t)\int_{-\infty}^{\infty}\tilde{\mu}_-(\xi,k)q'(\xi)\,d\xi \tag{3.137}$$

where $A(k,t) = e^{2ikx_0 + 2ik\Theta(-\eta^2)t}$ and

$$\tilde{\mu}_-(\xi,k) = e^{ik\xi/\eta}\frac{\left(k^2 - \eta^2 + 2ik\eta\tanh\xi\right)}{(k+i\eta)^2}, \tag{3.138}$$

$$q'(\xi) = -2\eta\,\mathrm{sech}\xi\tanh\xi. \tag{3.139}$$

If we introduce

$$I_1(k) = \int_{-\infty}^{\infty} e^{ik\xi/\eta}\mathrm{sech}\xi\tanh\xi\,d\xi, \tag{3.140}$$

$$I_2(k) = \int_{-\infty}^{\infty} e^{ik\xi/\eta}\mathrm{sech}\xi\tanh^2\xi\,d\xi, \tag{3.141}$$

$I(k,t)$ can be written as

$$I(k,t) = -A(k,t)\frac{2\eta}{(k+i\eta)^2}\left[(k^2-\eta^2)I_1(k) + 2ik\eta I_2(k)\right]. \tag{3.142}$$

Then, using a tricky manipulation, I_2 can be written in terms of I_1 as

$$I_2(k) = \left[\frac{ik}{2\eta} + \frac{\eta}{2ik}\right] I_1(k). \tag{3.143}$$

Putting equation (3.143) into (3.142), we find that

$$I(k,t) = 0. \tag{3.144}$$

Therefore,

$$\Theta(L_-^A)\partial_x q(x,t) = -q_t(x,t), \tag{3.145}$$

and the one soliton in (3.123) is a solution to the general evolution equation in (3.58).

3.5.3 The One Soliton for fmKdV and fSG

The general nonlinear evolution equation becomes the fmKdV equation when we put $\Theta(L_-^A) = -4L_-^A|4L_-^A|^\epsilon$ and it becomes the fsineG equations with $\Theta(L_-^A) = \frac{|4L_-^A|^\epsilon}{4L_-^A}$ where $|\epsilon| < 1$. Therefore, for these two equations, the one-soliton solution given in (3.123) becomes

$$q_m(x,t) = 2\eta\,\mathrm{sech}\left\{2\eta(x-x_0) - (2\eta)^{3+2\epsilon}t\right\}, \tag{3.146}$$
$$q_{SG}(x,t) = 2\eta\,\mathrm{sech}\left\{2\eta(x-x_0) + (2\eta)^{-1+2\epsilon}t\right\}. \tag{3.147}$$

We also find the "kink" solution u from $q_{SG} = u_x/2$ to be

$$u(x,t) = \arctan\sinh\left\{2\eta(x-x_0) + (2\eta)^{2\epsilon-1}t\right\}. \tag{3.148}$$

Both solutions are traveling waves which propagate without dissipating. The peak velocity of the two solitons are given by

$$v_m(\eta) = (2\eta)^{2+2\epsilon} \tag{3.149}$$
$$v_{SG}(\eta) = (2\eta)^{-2+2\epsilon} \tag{3.150}$$

Notice that the fractional equations predict power law relationships between the speed of the wave and the amplitude of the wave η as shown in Figure 3.1. Therefore, the fmKdV and fsineG equations predict anomalous dispersion, showing that this is a common characteristic of fractional nonlinear systems.

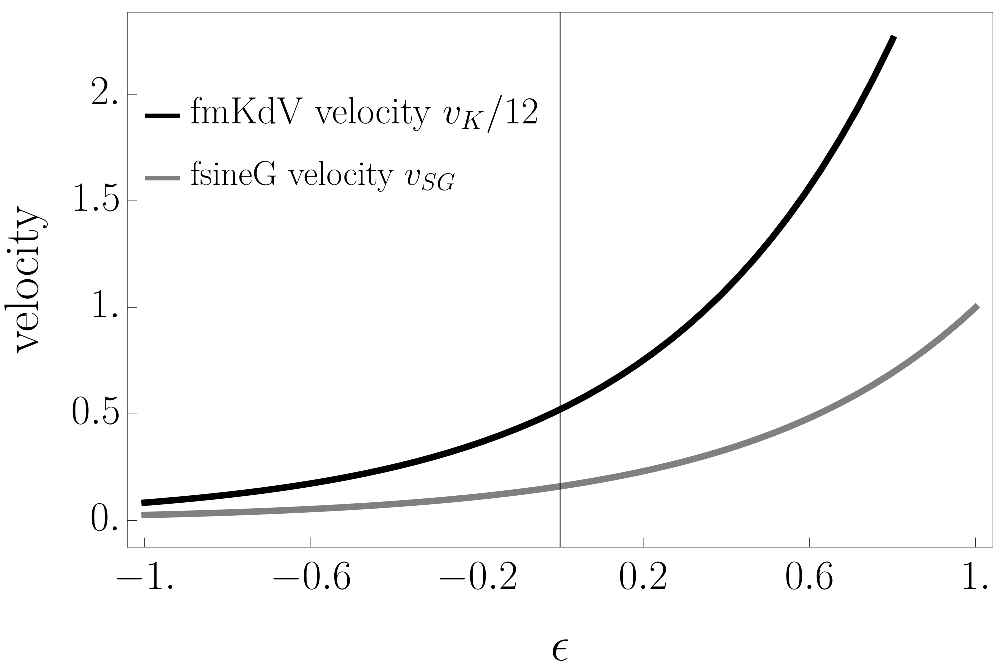

Figure 3.1 Localized waves predicted by the fmKdV and fsineG equations, equations (3.149-3.150), show super-dispersive transport as their velocity increases for $-1 < \epsilon < 1$. Notice that the fmKdV velocity is scaled by 1/12. Also, at $\epsilon = 1$, the resulting nonlinear equations are described by integer operators. Just as in anomalous diffusion where the mean squared displacement is proportional to t^α, the velocity in anomalous dispersion is proportional to A^ϵ, where A is the amplitude of the wave. Here $\eta = 3/2$.

3.6 Conclusion

We developed fractional extensions of the modified KdV, sine-Gordon, and sinh-Gordon equations on the line with decaying data. This process requires three key steps: a general evolution equation solvable by the inverse scattering transformation, completeness of squared eigenfunctions, and an anomalous dispersion relation. We demonstrated these three elements by developing a scalar general evolution equation using a symmetry reduction of the AKNS scattering system. Then, we found the fmKdV, fsineG, and fsinhG equations as a special case of this general evolution equation using the anomalous dispersion relations of the linear fmKdV, fsineG, and fsinhG equations, respectively. From scattering theory for the AKNS system, we found squared eigenfunctions and their associated operators for the scalar scattering problem. We then re-expressed completeness of the AKNS system in terms of these scalar squared eigenfunctions to give a spectral representation of the operator $\Theta(L_{\pm})$ in the general evolution equation. We developed the direct scattering, time evolution, and inverse scattering for the scalar scattering system and used these to derive the one-soliton solution for fmKdV and fsineG. We used the completeness relation to verify that these one-soliton solutions were truly solutions of fmKdV and fsineG. Finally, we showed that the one-soliton solutions of fmKdV and fsineG have power law relationships between the soliton's amplitude and velocity. This super-dispersive transport is an experimentally testable prediction of this theory.

3.7 Acknowledgements

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CHAPTER 4

FRACTIONAL INTEGRABLE AND RELATED DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS

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4.1 Abstract

Integrable fractional equations such as the fractional Korteweg-deVries and nonlinear Schrödinger equations are key to the intersection of nonlinear dynamics and fractional calculus. In this manuscript, we discover the first discrete/differential difference equation of this type is found, the fractional integrable discrete nonlinear Schrödinger equation. This equation is linearized; special soliton solutions are found whose peak velocities exhibit more complicated behavior than other previously obtained fractional integrable equations. This equation is compared with the closely related fractional *averaged* discrete nonlinear Schrödinger equation which has simpler structure than the integrable case. For positive fractional parameter and small amplitude waves, the soliton solutions of the integrable and averaged equations have similar behavior.

4.2 Introduction

Integrable systems play a central role in nonlinear dynamics because they provide exactly solvable models for important physical systems. Notable examples of integrable equations are the Korteweg-deVries (KdV), applicable to shallow water waves, plasma physics, and lattice dynamics among others [11, 16, 39], and the nonlinear Schrödinger (NLS) equation, which finds applications in nonlinear optics, Bose-Einstein condensates, spin waves in ferromagnetic films, plasma physics, water waves, etc. [11, 16, 53, 54]. These integrable nonlinear evolution equations have an infinite number of conservation laws and soliton solutions [12]. Solitons, the fundamental solutions of such equations, are stable, localized nonlinear waves which propagate without dispersing and interact elastically with other solitons. Nonlinear integrable evolution equations have these surprising properties because of their deep mathematical structure described by the inverse scattering transform (IST).

IST is a method of solving nonlinear equations which generalizes Fourier transforms. It solves these equations in three steps: mapping the initial condition into scattering space, evolving the intial data in

[9]Author contributions: Conceptualization, Methodology, Formal analysis, Writing – Review & Editing
[10]Author contributions: Conceptualization, Methodology, Writing – Review & Editing

scattering space in time, and mapping the evolved scattering data back to physical space; i.e., inverse scattering. This process gives the solution to nonlinear equations solvable by IST in terms of linear integral equations; such nonlinear equations are called integrable. Recently, we used the mathematical structure of IST associated with for the Korteweg-deVries (KdV) and nonlinear Schrödinger (NLS) equations to develop a method of finding and solving the fractional KdV (fKdV) and fractional NLS (fNLS) equations [1]. We also showed that this method could be applied to find fractional extensions of the modified KdV, sine-Gordon, and sinh-Gordon equations [2]. These equations represent the first known fractional integrable nonlinear evolution equations with smooth (physical) solutions and deeply connect the fields of nonlinear dynamics and fractional calculus.

Fractional calculus is a mathematical structure originally designed to define non-integer derivatives and integrals. It has sense become an effective way of modeling many physical processes that exist in multi-scale media [32, 33] or exhibit non-Gaussian statistics or power law behavior [28, 45, 46]. A particularly important example is anomalous diffusion, where the mean squared displacement is proportional to t^α, $\alpha > 0$ [27–30]. This form of transport has been observed extensively in biology [17–20], amorphous materials [21–23], porous media [24–26, 47], climate science [31], and attenuation in materials [49] amongst others. As we have shown, the merger of fractional and nonlinear characteristics in integrable equations such as fKdV and fNLS predict *anomalous dispersion*, where the velocity and amplitude of solitonic solutions are related by a power law [1].

In this article, we demonstrate how the method introduced in Ref. [1] can be applied to discrete (or differential-difference) systems to define integrable discrete fractional nonlinear evolution equations by presenting a fractional generalization of the integrable discrete nonlinear Schrödinger (IDNLS) equation. We do this by demonstrating the three key mathematical ingredients of our method — IST, power law dispersion relations, and completeness relations — for the Ablowitz-Ladik (AL) discrete scattering problem.

The KdV equation was the first equation shown to be solvable by IST in Ref. [8]; it was soon followed by the NLS equation in Ref. [55]. These two equations were then found to be contained in a general class of equations solvable by IST when associated to the Ablowitz-Kaup-Newell-Segur (AKNS) system [10, 16]. Shortly thereafter IST was used to solve families of discrete (or differential difference) problems like the self-dual network [63]. In particular, it was discovered that the AKNS system could be discretized while maintaining integrability, leading to the AL scattering problem which was used to solve a family of discrete nonlinear evolution equations [64]. This family contained important discrete evolution equations — continuous in time but discretized in space — such as integrable discretizations of the nonlinear Schrödinger, KdV, modified KdV, and sine Gordon equations. Further, this family of equations was shown to have soliton solutions and an infinite number of conservation laws [64].

We derive the fIDNLS equation from the AL scattering problem using three key components: linear dispersion relations, completeness relations, and IST. IST is used to linearized the equation and obtain special soliton solutions.

We also show how the characteristics of the fractional IDNLS (fIDNLS) equation reach beyond integrability by comparing the one-soliton solution of the fIDNLS equation to the solitary wave solution of the fractional averaged discrete nonlinear Schrödinger (fADNLS) equation. This equation is a different fractional generalization of the IDNLS equation in which the linear second order difference is replaced by the discrete fractional Laplacian [65–68]. The fADNLS equation can be understood as a discretization of a fractional NLS equation involving the Riesz derivative which has been shown to have soliton type solutions [34, 40–42]; it is also is also closely related to the (likely) non-integrable fractional DNLS equation, recently studied in [65, 69]. Though the fADNLS equation is likely not integrable to our knowledge (apart from the limiting case when fADNLS reduces to IDNLS), the similarity between the two equations suggests that some of the physical predictions of fractional integrable equations are shared by equations which are simpler to realize computationally.

4.3 The Discrete Fractional Linear Schrödinger Equation

$$\partial_t q_n + \gamma(-\Delta_n) q_n = 0 \tag{4.1}$$

for the function $q_n(t)$ which depends on the discrete variable $n \in \mathbb{Z}$ and the continuous variable $t \in \mathbb{R}$. Here, γ is a sufficiently regular function of the discrete laplacian, $-\Delta_n$, defined by

$$(-\Delta_n) q_n(t) = \frac{1}{h^2}\big(- q_{n+1}(t) + 2q_n(t) - q_{n-1}(t)\big) \tag{4.2}$$

where h is the distance between lattice sites. Using the Z-transform, which is equivalent to the discrete Fourier transform, the solution to Eq. (4.1) can be explicitly written as

$$q_n(t) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} dk \hat{q}(k,0) e^{iknh - \gamma(4\sin^2(kh/2)/h^2)t} \tag{4.3}$$

where $\hat{q}(k,0) = h\sum_{n=-\infty}^{\infty} q_n(0)e^{-iknh}$ is the Z-transform of $q_n(t)$ at $t = 0$ and $4\sin^2(kh/2)/h^2$ is the Fourier symbol of $-\Delta_n$. Note that the Z-transform is often written in terms of z with the substitution $z = e^{-ikh}$ where integration in k becomes integration with respect to z on the unit circle. If we choose γ to be power law, then Eq. (4.1) becomes a fractional discrete equation in terms of the discrete fractional laplacian. For example, if we put $\gamma(-\Delta_n) = -i(-\Delta_n)^{1+\epsilon}$, $|\epsilon| < 1$, then we obtain the linear fractional discrete Schrödinger equation

$$i\partial_t q_n + (-\Delta_n)^{1+\epsilon} q_n = 0. \tag{4.4}$$

Here, $(-\Delta_n)^{1+\epsilon}$ is the discrete fractional laplacian of order $1+\epsilon$ which is defined in terms of its Fourier symbol $[4\sin^2(hk/2)/h^2]^{1+\epsilon}$ and the Z-transform/discrete Fourier transform as

$$(-\Delta_n)^{1+\epsilon} q_n = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} dk \hat{q}(k) e^{iknh} [4\sin^2(kh/2)/h^2]^{1+\epsilon}. \tag{4.5}$$

Notice that the k integral above can be evaluated to express the discrete fractional laplacian as a summation over m of q_m multiplied by a weight vector. The solution to Eq. (4.4) can still be written in the form Eq. (4.3) with

$$\gamma(4\sin^2(kh/2)/h^2) = -i[4\sin^2(kh/2)/h^2]^{1+\epsilon}$$

and, because $4\sin^2(kh/2)/h^2$ is real and positive, the solution to equation (4.1) with this choice of γ is well posed. In defining and solving the linear fractional discrete Schrödinger equation, we used a power law dispersion relation, ingredient 1 of our method, and we defined the fractional operator using completeness of the discrete Fourier transform/Z-transform, ingredient 2. Then we solve the equation by the inverse discrete Fourier transform, the analog of ingredient 3.

4.4 The Fractional Integrable Discrete Schrödinger Equation

To develop the fIDNLS equation, the integrable nonlinear analog of Eq. (4.4), and solve it, we apply the three key ingredients of our method, starting with writing the equation in terms of a linear dispersion relation. Note that $h = 1$ is taken in this section without loss of generality; to recover the scaling factor for $h \neq 1$, replace q_n by hq_n and r_n by hr_n.

As in the linear case, Eq. (4.1), we have a family of nonlinear evolution equations for the solutions $q_n(t)$ and $r_n(t)$ [7], see also [70],

$$\sigma_3 \frac{d\mathbf{u}_n}{dt} + \gamma(\Lambda_+)\mathbf{u}_n = 0, \quad \mathbf{u}_n = (q_n, -r_n)^T \tag{4.6}$$

where T represents transpose, $\sigma_3 = \mathrm{diag}(1,-1)$, and Λ_+ is

$$\Lambda_+\mathbf{x}_n = h_n \begin{pmatrix} E_n^+ & 0 \\ 0 & E_n^- \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \tag{4.7}$$

$$+ \begin{pmatrix} q_n \sum_{n-1}^+ r_{k-1} & q_n \sum_{n-2}^+ q_{k+1} \\ -r_n \sum_{n-1}^+ r_{k-1} & -r_n \sum_{n-2}^+ q_{k+1} \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \tag{4.8}$$

$$+ h_n \begin{pmatrix} q_{n+1} \sum_{n+1}^+ \frac{r_k}{h_k} & q_{n+1} \sum_{n+1}^+ \frac{q_k}{h_k} \\ -r_{n-1} \sum_n^+ \frac{r_k}{h_k} & -r_{n-1} \sum_n^+ \frac{q_k}{h_k} \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \tag{4.9}$$

where $h_n = 1 - r_n q_n$, $\sum_n^+ = \sum_{k=n}^{\infty}$, $I_n x_k^{(q)} = x_n^{(q)}$, and $E_n^{\pm} x_k^{(q)} = x_{n\pm1}^{(q)}$ with $q = 1, 2$. The inverse of this operator is

$$\Lambda_+^{-1}\mathbf{x}_n = h_n \begin{pmatrix} E_n^- & 0 \\ 0 & E_n^+ \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \tag{4.10}$$

$$+ \begin{pmatrix} -q_n \sum_n^+ r_{k+1} & -q_n \sum_{n+1}^+ q_{k-1} \\ r_n \sum_{n-1}^+ r_{k+1} & r_n \sum_n^+ q_{k-1} \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix} \tag{4.11}$$

$$+ h_n \begin{pmatrix} -q_{n-1} \sum_n^+ \frac{r_k}{h_k} & -q_{n-1} \sum_n^+ \frac{q_k}{h_k} \\ r_{n+1} \sum_{n+1}^+ \frac{r_k}{h_k} & -r_{n+1} \sum_{n+1}^+ \frac{q_k}{h_k} \end{pmatrix} \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \end{pmatrix}. \tag{4.12}$$

Here, γ is a sufficiently regular function of the operator Λ_+ and is connected with the linearized dispersion relation. Specifying this dispersion relation, or γ directly, picks out particular equations from this family. For example, if we take

$$\gamma(\Lambda_+) = -i(2 - \Lambda_+ - \Lambda_+^{-1})$$

and let $r_n = \mp q_n^*$, then we obtain the IDNLS equation

$$i\partial_t q_n + \Delta_n q_n \pm |q_n|^2 (q_{n+1} + q_{n-1}) = 0. \tag{4.13}$$

We can relate γ to the dispersion relation of the linearization of (4.6) by considering the linear limit $q_n \to 0$. In this limit, we have

$$\Lambda_+ \to \begin{pmatrix} E_n^+ & 0 \\ 0 & E_n^- \end{pmatrix} \equiv \mathbf{D}_n, \tag{4.14}$$

so the linearization of the nonlinear evolution equation is

$$\sigma_3 \frac{d\mathbf{u_n}}{dt} + \gamma(\mathbf{D}_n)\mathbf{u}_n = 0. \tag{4.15}$$

Because $\mathbf{D}_n$ is a diagonal matrix, we have

$$\gamma(\mathbf{D}_n) = \begin{pmatrix} \gamma(E_n^+) & 0 \\ 0 & \gamma(E_n^-) \end{pmatrix}. \tag{4.16}$$

Taking the first component of (4.15) with

$$q_n = z^{2n} e^{-i\omega(z)t}$$

gives

$$\gamma(z^2) = i\omega(z). \tag{4.17}$$

Therefore, by specifying the linear limit of the nonlinear evolution equation, we obtain the nonlinear equation itself. To define the fIDNLS equation, we choose the linear limit to be the discrete linear fractional Schrödinger equation in (4.4), which gives the dispersion relation $\omega(z) = -(2 - z^2 - z^{-2})^{1+\epsilon}$ and, hence, $\gamma(z^2) = -i(2 - z^2 - z^{-2})^{1+\epsilon}$. So, the fIDNLS equation is

$$i\partial_t \mathbf{u}_n + (2 - \Lambda_+ - \Lambda_+^{-1})^{1+\epsilon} \mathbf{u}_n(t) = 0. \tag{4.18}$$

In fact, by choosing $\gamma(z^2) = -i(2 - z^2 - z^{-2})^{m+\epsilon}$, for integer m, we generate a hierarchy of fractional equations

$$i\partial_t \mathbf{u}_n + (2 - \Lambda_+ - \Lambda_+^{-1})^{m+\epsilon} \mathbf{u}_n(t) = 0. \tag{4.19}$$

It can be shown that the limit of (4.18) as $\epsilon \to 0$ is the IDNLS equation (4.13). Notice that to define the fIDNLS equation, we used a power law dispersion relation, ingredient 1 of the method. However, this dispersion relation leads to the operator $(2 - \Lambda_+ - \Lambda_+^{-1})^{1+\epsilon}$ the meaning of which is currently unclear. To define this operator, we will need to use the 2nd ingredient: appropriate completeness relations. The third ingredient will be making use of IST to find solutions of the fIDNLS equation.

4.5 Completeness of Squared Eigenfunctions and Fractional Operators

In this section we define the fIDNLS equation in (4.18) and, in fact, any equation of the form (4.1) that is well-posed in physical space. We do this using the observation that $\gamma(\Lambda_+)$ is a multiplication operator when acting on the eigenfunctions of Λ_+ and the fact that the eigenfunctions of Λ_+ are complete. This result is known as completeness of squared eigenfunctions, and is the second ingredient in our method. The resulting representation of $\gamma(\Lambda_+)$ will be similar to that of the discrete fractional laplacian in (4.5). The eigenfunctions of Λ_+ are $\mathbf{\Psi}_n(z)$ and $\overline{\mathbf{\Psi}}_n(z)$ each with eigenvalue z^2 (note that time t is suppressed throughout this section). Therefore, the operation of $\gamma(\Lambda_+)$ on these eigenfunctions is given by

$$\gamma(\Lambda_+)\mathbf{\Psi}_n = \gamma(z^2)\mathbf{\Psi}_n, \quad \gamma(\Lambda_+)\overline{\mathbf{\Psi}}_n = \gamma(z^2)\overline{\mathbf{\Psi}}_n. \tag{4.20}$$

Because Λ_+ is not a self-adjoint operator, completeness of squared eigenfunctions involves both $\mathbf{\Psi}_n$, $\overline{\mathbf{\Psi}}_n$ and the adjoint functions $\mathbf{\Psi}_n^A$, $\overline{\mathbf{\Psi}}_n^A$ where

$$\gamma(\Lambda_+^A)\mathbf{\Psi}_n^A = \gamma(z^2)\mathbf{\Psi}_n^A, \quad \gamma(\Lambda_+^A)\overline{\mathbf{\Psi}}_n^A = \gamma(z^2)\overline{\mathbf{\Psi}}_n^A \tag{4.21}$$

and Λ_+^A is the adjoint, with respect to $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$, of Λ_+. The eigenfunctions and adjoint eigenfunctions can be written in terms of solutions to the Ablowitz-Ladik scattering problem which is a 2×2 eigenvalue problem fo the discrete vector-valued function $\mathbf{v}_n = (v_n^{(1)}, v_n^{(2)})^T$

$$\mathbf{v}_{n+1} = \begin{pmatrix} z & q_n \\ r_n & z^{-1} \end{pmatrix} \tag{4.22}$$

where q_n and r_n act as potentials and z is an eigenvalue. Through this association, one can solve the family of nonlinear evolution equations in (4.22) (see Appendix for more details).

In [7], it was shown that the arbitrary discrete function $\mathbf{H}_n = \left(H_n^{(1)}, H_n^{(2)}\right)^T$ can be written as

$$\mathbf{H}_m = \sum_{p=1}^{2} \oint_{S^{(p)}} dz f^p(z) \sum_{m=-\infty}^{\infty} \mathbf{G}_{n,m}^{(p)}(z)\, \mathbf{H}_m \tag{4.23}$$

where $S^{(1)} = S_R$ ($S^{(2)} = S_\delta$) is a circular contour evaluated counterclockwise centered at the origin of radius R (δ) such that all of the singularities of the integrand are inside (outside) of the contour and

$$\mathbf{G}_{n,m}^{(1)}(z) = \mathbf{\Psi}_n(z)\mathbf{\Psi}_m^A(z)^T/h_n, \;\; f^{(1)}(z) = \frac{i}{2\pi a^2(z)} \tag{4.24}$$

$$\mathbf{G}_{n,m}^{(2)}(z) = \overline{\mathbf{\Psi}}_n(z)\overline{\mathbf{\Psi}}_m^A(z)^T/h_n, \;\; f^{(2)}(z) = \frac{-i}{2\pi \overline{a}^2(z)} \tag{4.25}$$

with $h_n = 1 - r_n q_n$. The eigenfunctions $\mathbf{\Psi}_n(z)$,$\mathbf{\Psi}_n^A(z)$,$\overline{\mathbf{\Psi}}_n(z)$,$\overline{\mathbf{\Psi}}_n^A(z)$ (see appendix) and scattering data $a(z)$, $\overline{a}(z)$ are defined in terms of solutions to the Ablowitz-Ladik scattering problem (see Appendix). With this completeness relation, and the operation of $\gamma(\Lambda_+)$ on $\mathbf{\Psi}_n$ and $\overline{\mathbf{\Psi}}_n$ in Eq. (4.20), we have

$$\gamma(\Lambda_+)\mathbf{H}_n = \sum_{p=1}^{2} \oint_{S^{(p)}} dz f^{(p)}(z)\gamma(z^2) \sum_{m=-\infty}^{\infty} \mathbf{G}_{n,m}^{(p)}(z)\mathbf{H}_m. \tag{4.26}$$

Therefore, the nonlinear evolution equation in (4.6) can be explicitly characterized in physical space as

$$\sigma_3 \frac{d\mathbf{u}_n}{dt} = -\sum_{p=1}^{2} \oint_{S^{(p)}} dz f^{(p)}(z)\gamma(z^2) \sum_{m=-\infty}^{\infty} \mathbf{G}_{n,m}^{(p)}(z)\mathbf{u}_m. \tag{4.27}$$

In particular, if we put $\gamma(z^2) = -i(2 - z^2 - z^{-2})^{1+\epsilon}$ and $r_n = \mp q_n^*$, the fIDNLS equation is the first component of (4.27). Using the symmetries of the eigenfunctions (see appendix), this is

$$i\partial_t q_n = \sum_{p=1}^{2} \oint_{S^{(p)}} dz f^{(p)}(z)\gamma(z^2) \sum_{m=-\infty}^{\infty} g_{n,m}^{(p)}(z) \tag{4.28}$$

with

$$g_{n,m}^{(1)}(z) = -i\frac{\nu_n \nu_m}{h_n} \psi_n^{(1)}(z)\psi_{n+1}^{(1)}(z) \tag{4.29}$$
$$\cdot \left(\phi_m^{(2)}(z)\phi_{m+1}^{(2)}(z) q_m \mp \phi_m^{(1)}(z)\phi_{m+1}^{(1)}(z) q_m^*\right)$$
$$g_{n,m}^{(2)}(z) = -i\frac{\nu_n \nu_m}{h_n} \left(\psi_n^{(2)}(1/z)\psi_{n+1}^{(2)}(1/z)\right)^* \tag{4.30}$$
$$\cdot \left(\phi_m^{(1)}(1/z)\phi_{m+1}^{(1)}(1/z) q_m^* \mp \phi_m^{(2)}(1/z)\phi_{m+1}^{(2)}(1/z) q_m\right)^*$$

where t has been suppressed.

In the appendix we show how this equation can be linearized via Gel'fand-Levitan-Marchenko type summation equations. After long time the kernel of the summation equation contains only discrete spectra, i.e., the soliton solutions. Multisoliton solutions can be found by standard methods.

4.6 Solitons and Solitary Wave Solutions of the fIDNLS and fADNLS Equations

The fIDNLS equation in (4.18) is not the only fractional generalization of the IDNLS equation in (4.13). A simpler generalization is to replace the discrete laplacian $-\Delta_n$ in (4.13) with the discrete fractional laplacian $(-\Delta_n)^{1+\epsilon}$ defined in (4.5) to give the fractional averaged DNLS (fADNLS) equation

$$i\partial_t q_n + (-\Delta_n)^{1+\epsilon} q_n \pm |q_n|^2(q_{n+1} + q_{n-1}) = 0. \tag{4.31}$$

Notice that in the figure captions we refer to the fIDNLS equation as the fractional integrable equation and the fADNLS equation as the fractional averaged equation.

The fADNLS equation is not known to be integrable, but in the limit $\epsilon \to 0$, it becomes the IDNLS equation, Eq. (4.13), which is integrable; therefore, we expect Eq. (4.31) to have some similarity the fIDNLS equation. To characterize this similarity, we will compare the solitons and solitary waves predicted by these equations. The fIDNLS equation has an exact one-soliton solution, derivable by the IST. To find the solitary wave solutions to the fADNLS equation we use the same initial condition as that of the fIDNLS equation.

Even though this solitary wave initially deforms from the exact secant profile, emitting radiation in the process, its solutions have nearly constant velocity, propagate with nearly constant amplitude, and have comparable velocities to the fIDNLS equation in certain regimes. These integrable-like properties of this equation are stronger for positive ϵ than negative ϵ and stronger for smaller wave amplitudes than larger wave amplitudes. Soliton solutions to the fIDNLS equation can be derived using the IST (see appendix and [60]); they are of the form

$$\begin{aligned} q_n(t) = & \frac{\sinh(2\eta h)}{h} e^{2i\left(v_i(z_1^2)t - \xi h n\right) - i(\psi - \pi/2)} \\ & \times \operatorname{sech}\left(2\eta h(n - n_0) - 2v_r(z_1^2)t\right) \end{aligned} \tag{4.32}$$

where $v_i(z_1^2) = \frac{1}{2}\mathrm{Im}\gamma(z_1^2)$, $v_r(z_1^2) = \frac{1}{2}\mathrm{Re}\gamma(z_1^2)$, and $z_1 = e^{h(\eta - i\xi)}$. Here we choose $\gamma(z_1^2) = -i(2 - z_1^2 - z_1^{-2})^{1+\epsilon}$ in accordance with Eq. (4.18) though (4.32) holds for all sufficiently regular γ. The free parameters in (4.32) are ϵ, h, η, ξ, ψ, and n_0.

To find the localized wave solutions to the fADNLS equation, we numerically evolved the equation at discrete time steps $\{t_m\}_{m=0}^{M}$ with $t_0 = 0$ using a Fourier split-step scheme. The initial condition $q_n(t_0) = q_n(0)$ is given by (4.32) with $t = 0$. The Fourier split-step scheme propagates the approximation from t_m to t_{m+1} by separately evaluating the linear and nonlinear parts of the equation; cf. Refs. [71–73]. Explicitly, we compute

$$q_n(t_{m+1}) = e^{-i\Delta t_m \mathcal{L}/2} e^{i\int_{t_m}^{t_{m+1}} d\xi \mathcal{N}} e^{-i\Delta t_m \mathcal{L}/2} q_n(t_m) \tag{4.33}$$

where $\mathcal{L}q_n = (-\Delta_n)^{1+\epsilon} q_n$ and $\mathcal{N}q_n = \pm|q_n|^2(q_{n+1} + q_{n-1})$. The particular operator splitting in equation (4.33) makes the solution method $\mathcal{O}(\Delta t^2)$ accurate [74, 75]. The linear step, $e^{-i\Delta t_m \mathcal{L}/2}$, is evaluated using discrete Fourier transforms, while the nonlinear step, $e^{i\Delta t_m \mathcal{N}}$, is evaluated by solving the associated differential equation, equation (4.31) with $(-\Delta_n)^{1+\epsilon} q_n \to 0$, using a fourth-order Runge-Kutta scheme. Throughout this manuscript, solutions to the fADNLS equation were computed with the parameters $h = 1$, $\xi = 0.5$, and $\psi = \pi/2$ and with $N = 2,000$ grid points and time discretization $\Delta t = 0.01$.

The fADNLS equation initialized with the soliton solution to fIDNLS, i.e., putting $t = 0$ into Eq. (4.32), leads to radiation emission for non-zero ϵ. Figure 4.1 shows this radiation for small ($\eta = 0.05$), medium ($\eta = 0.5$), and large ($\eta = 1$) amplitude initial conditions at simulation time $T = 300$ with $\epsilon = 0.1$. Recall that amplitude is related to the paramters η and h (h is taken to be 1) by $A = \sinh(2\eta h)/h$. The heights of the three solutions are normalized to 1 to compare the relative amount of radiation; the radiation increases with increasing amplitude, with the large amplitude solution having radiation about 2% of the height of the solution, the medium amplitude having 1%, and the small amplitude having negligible radiation.

The positions of the peaks of the fADNLS equation (solid lines) are given along with linear fits (dashed lines) in Figure 4.2 for medium amplitude initial conditions and $\epsilon = -0.25$, 0.0, +0.25. The linear fit shows that the positive ϵ solution propagates with nearly constant velocity, while the negative ϵ one has quadratic character which causes it to slow down over time. The amplitudes of these localized wave solutions have breathing patterns. Figure 4.3 shows that when we average over these oscillations, the amplitude settles down to a constant for $\epsilon = 0.25$ after deformation from the secant profile, but grows a little bit over time for $\epsilon = -0.25$. The averaged amplitude was obtained by taking the mean of the amplitude for ± 10 time units around each point. These results suggest that for ϵ positive and sufficiently small the localized wave solutions to the fADNLS equation have structure similar to integrable solitons, while those for ϵ negative are less similar.

A comparison of a small amplitude soliton solution to the fIDNLS equation and solitary wave solution to the fADNLS equation is given in Figure 4.4 for $\epsilon = 0.1$. The solitary wave spreads out, deforming from the hyperbolic secant profile of the soliton. However, the peak velocities of the two waves are nearly identical, 1.83864 for the soliton and $1.838713 \pm 1 \times 10^{-6}$ for the solitary wave. The soliton moves with exactly constant velocity, but the solitary wave does have an acceleration of $(-1.513 \pm 0.002) \times 10^{-6}$. However, this acceleration is small enough that we can still compare the velocities of these two waves. The velocity and acceleration were estimated by fitting a quadratic curve to the solitary wave peak position and error bounds were obtained by doubling the time discretization, i.e., computing the difference between the results for $\Delta t = 0.01$ and $\Delta t = 0.02$. For larger values of ϵ and for larger amplitude waves the agreement between these two equations diverges.

The peak velocity for the one soliton solution to the fIDNLS equation is given by

$$c_p(\eta, \xi, h) = \frac{v_r}{\eta h}, \; v_r = -2\mathrm{Im}\big(\sinh^{1+\epsilon}(h[\eta - i\xi]/2)\big) \tag{4.34}$$

which is determined analytically from the form of the soliton in equation (4.32). The peak velocity of the fIDNLS soliton is related to its amplitude in a much more complicated manner than for the fKdV and fNLS equations which have power law relationships between their amplitude and velocity, i.e., anomalous dispersion. Figure 4.5 shows this velocity as a function of ϵ for $h = 1$; $\xi = 0.5$; and small, medium, and large amplitudes.

4.7 Conclusion

In this paper, the fractional integrable discrete nonlinear Schrödinger equation was obtained and it's properties were investigated. We did this by applying three principal mathematical constituents which were introduced in our earlier work [1], [2]: the inverse scattering transform, power law dispersion relations, and completeness relations, to the Ablowitz-Ladik scattering problem. We linearized the equation via Gel'fand-Levitan-Marchenko type summation equations. After long time the kernel of the summation equation contains only discrete spectra; we then obtained an explicit one-soliton solution to this equation, showing that it's velocity depends on the fractional parameter ϵ in a more complicated way than its continuous counterpart in the fractional nonlinear Schrödinger equation. Multi-soliton solutions can be obtained by standard methods; but they are outside the scope of this paper. Using a Fourier split step method, we compared the predictions of the integrable discretization to the fractional averaged nonlinear Schrödinger equation, a related non-integrable equation. We demonstrated that for small amplitude initial data, the two equations predicted nearly identical velocities and similar structure, while for large amplitudes they exhibited qualitatively similar characteristics.

4.8 Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

4.9 Acknowledgements

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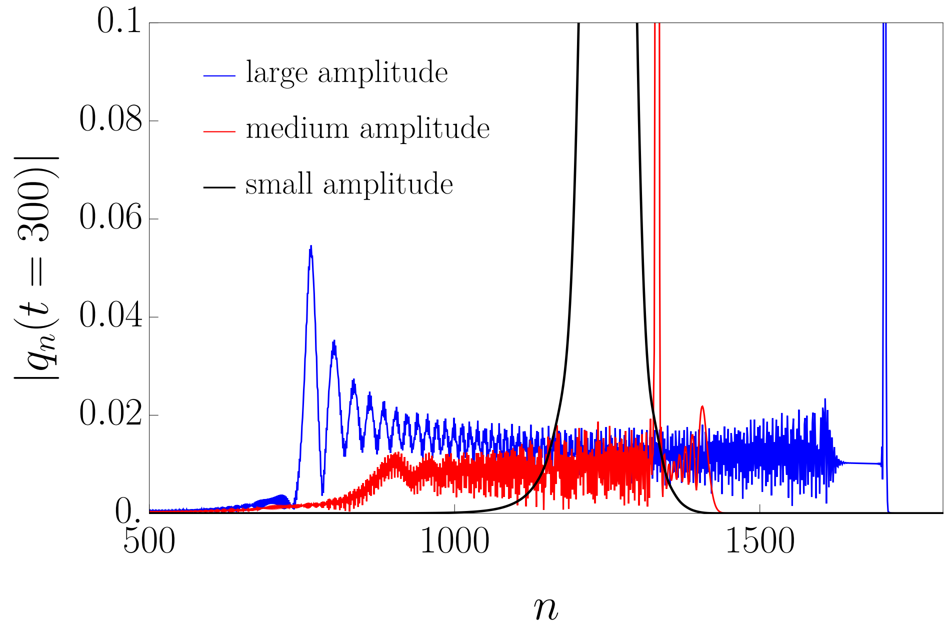

Figure 4.1 *Radiation emission for small, medium, and large initial data.* Solitary wave solutions to the fractional averaged equation emits more radiation for larger amplitude initial conditions and larger fractional order ϵ ($\epsilon = 0.1$ is shown here). The initial amplitudes corresponding to the small, medium, and large solutions are $A = 0.100$, 1.175, and 3.627, respectively; however, each solitary wave solution is normalized to peak height 1.

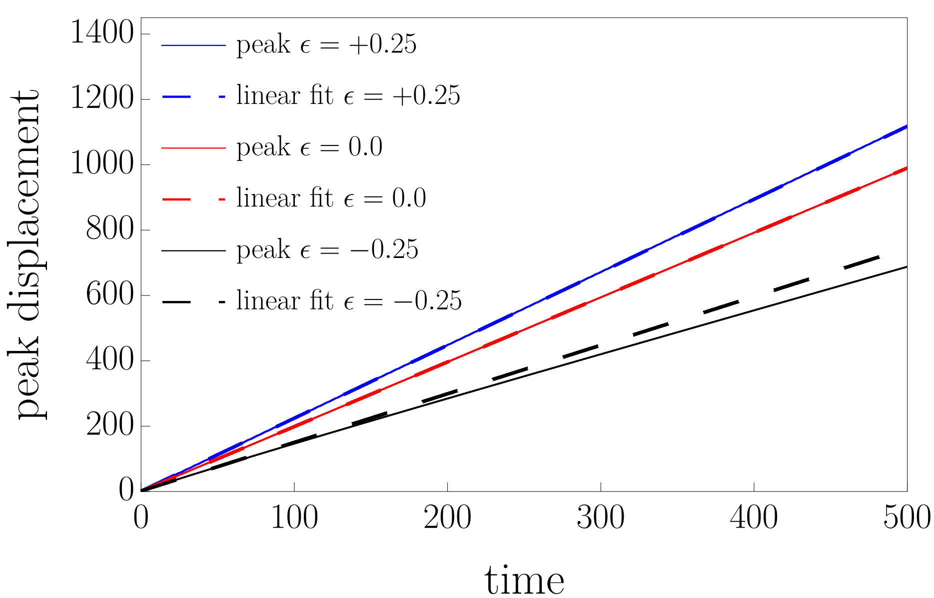

Figure 4.2 *Linearity of solitary wave peak displacement.* Medium amplitude solitary wave solutions of the fractional averaged equation have a nearly linear relationship between displacement and time for positive and zero ϵ. For negative ϵ, the solitary wave slows down over time.

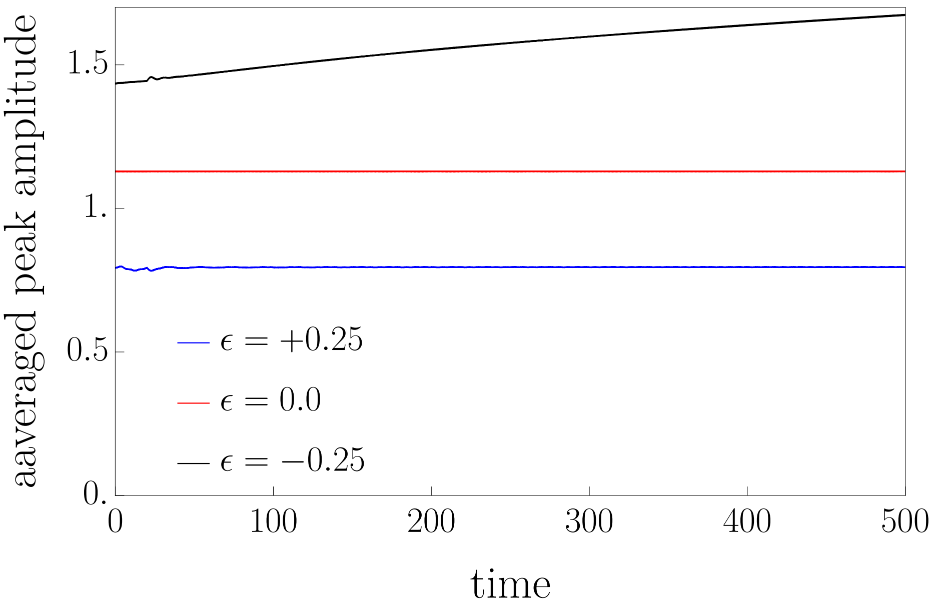

Figure 4.3 *Time-averaged solitary wave peak amplitude.* The time-averaged amplitude of medium solitary wave solutions of the fractional averaged equation are nearly constant for positive and zero ϵ and grow slightly for negative ϵ. The results in this plot and Figure 4.2 suggest that the solitary waves for positive ϵ are closer to solitons than for negative ϵ.

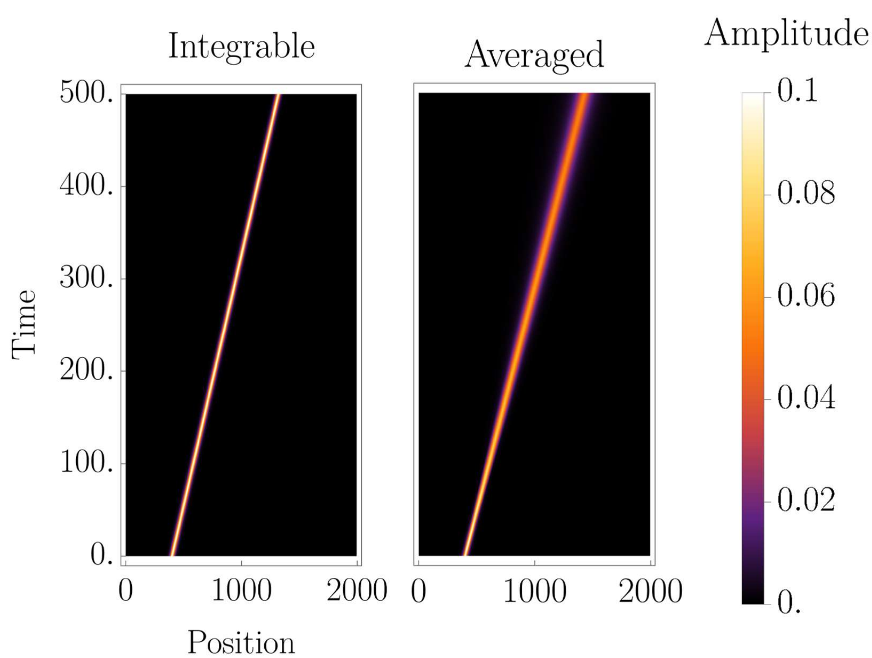

Figure 4.4 *Integrable and averaged dynamics for small initial conditions.* The soliton solution to the fractional integrable equation propagates at a constant velocity without dissipating for $\epsilon = 0.1$. Even though the profile of the solitary wave solution to the fractional averaged equation deforms from the initial solitonic profile, its peak propagates at a nearly identical velocity to the soliton; the soliton has velocity 1.83864 and the solitary wave $1.83871 \pm 1 \times 10^{-6}$.

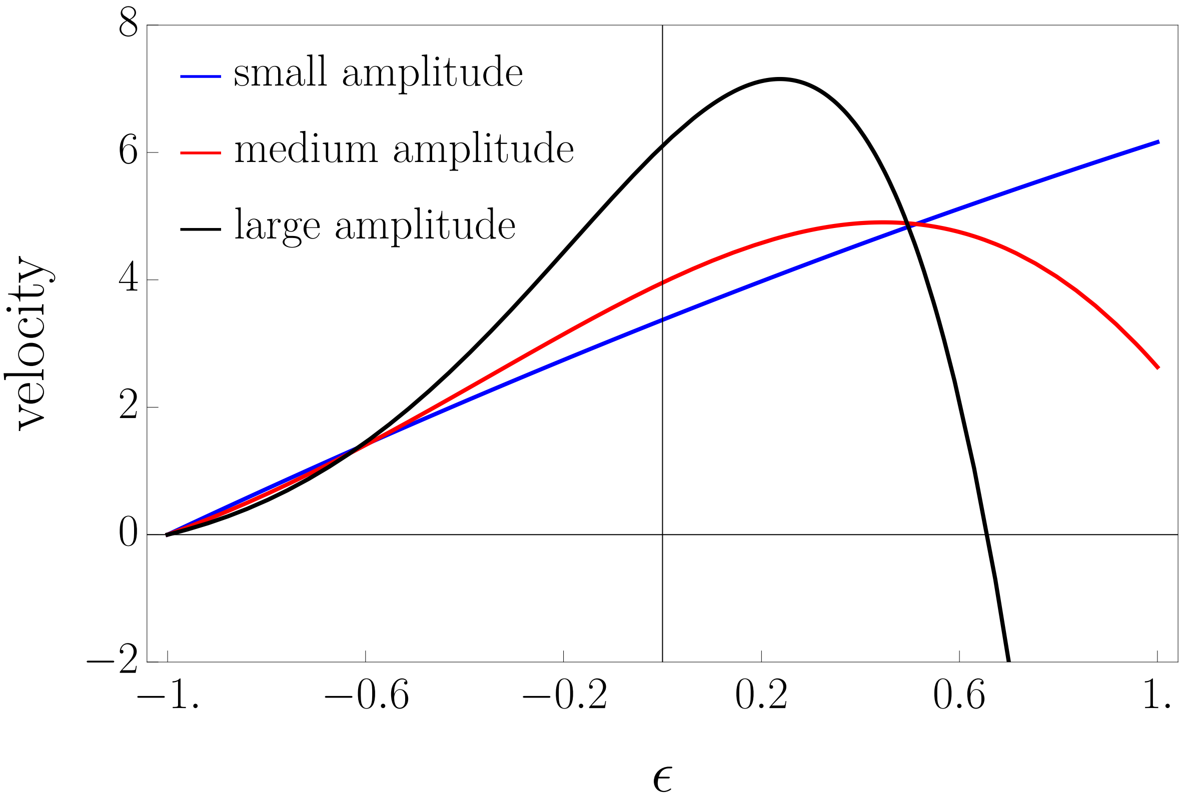

Figure 4.5 *Fractional soliton velocity.* Velocity of the one-soliton solution to the fractional integrable equation exhibits super-dispersive transport for small amplitudes ($A = 0.100$). However, for medium ($A = 1.175$) and large ($A = 3.627$) amplitudes, the velocity has a turning point where increasing ϵ decreases the velocity. This is a fundamentally discrete phenomenon not shared by known continuous fractional integrable equations; cf. [1, 2].

CHAPTER 5

CONCLUSION AND OUTLOOK

In this thesis, we presented the first known fractional integrable nonlinear evolution equations with non-local fractional operators and smooth solutions. This work connects the fields of nonlinear dynamics and fractional calculus in a novel manner. Here, we review the new class of equations introduced in this thesis and suggest potential future research topics in this area.

5.1 Conclusion

In chapter 2 we introduce our method of finding these fractional equations and exemplify it by deriving the fractional Korteweg-deVries and fractional Nonlinear Schrödinger equations for decaying initial data. The method can be applied to any system that admits three key mathematical ingredients: power law dispersion relations, completeness relations, and the inverse scattering transform. Using a spectral representation of the fractional nonlinear operators in each of these equations, we gave the explicit forms of both of these equations. By deriving the one soliton solution to each of these equations using the inverse scattering transform, we demonstrated that they predict anomalous dispersion, a simple physical property where amplitude is related to velocity by a power law.

We then applied this method to derive the fractional modified Korteweg-deVries, fractional sine-Gordon, and fractional sinh-Gordon equations in chapter 3. Using the three key mathematical components, we gave an explicit form of each of these equations. We also presented a detailed review of inverse scattering for these equations, describing how a solution to each of these fractional problems can be found by direct scattering, time evolution, and inverse scattering given decaying initial data. We present the one-soliton solution for the fractional modified Korteweg-deVries and sine-Gordon equations and demonstrate that they satisfy their respective equations explicitly. These soliton solutions predict anomalous dispersion like the fractional Korteweg-deVries and nonlinear Schödinger equations.

Finally, in chapter 4, we showed that our method can also be applied to discrete problems by deriving the fractional integrable discrete nonlinear Schrödinger equation. We demonstrate that this discrete equation has more complicated physical predictions than its continuous counter part. Specifically, we showed that the velocity of the one soliton solution to this equation can exhibit a turning point where it switches from increasing to decreasing as the fractional parameter increases. We then compared the soliton solution of this equation to the solitary wave solution to the closely related but (likely) nonintegrable fractional averaged discrete nonlinear Schrödinger equation, demonstrating that the two equations predict similar behavior especially for small amplitude waves and positive fractional parameter. The averaged equation is numerically approximated using a Fourier split step method in which the discrete fractional laplacian in the equation is computed using discrete Fourier transforms. This comparison showed that fractional integrable nonlinear equations have characteristics that are shared by nonintegrable fractional nonlinear equations.

5.2 Outlook and Future Research

The method we presented in this manuscript can be applied to any integrable system with the three key mathematical components: power law dispersion, completeness relations, and inverse scattering transform. Therefore, a natural way of extending this research is to find more fractional equations that are integrable. If the three components are already known, then the task is simple. However, for many integrable systems completeness of squared eigenfunctions is not known; therefore, finding completeness relations and using them to define fractional operators is an important extension of this work.

Currently, our theory can only be applied to find bright solitonic solutions of fractional integrable nonlinear Schrödinger equations because the method requires decaying solutions. Many of these integrable equations, e.g., the Korteweg-deVries and nonlinear Schrödinger equations, have dark soliton solutions which do not decay at infinity. Therefore, we can expect the fractional generalization of these two equations to have dark soliton solutions. Because of the nonlocality of the nonlinear fractional operators in these equations, the non-decaying boundary conditions fundamentally alter the operator in a way we have not found. Finding the way in which these fractional equations change when the solution has non-zero boundary conditions will lead to these dark solitons.

Another important aspect of this work is comparing integrable fractional equations and (likely) nonintegrable fractional equations as we did in chapter 4. Expanding on our preliminary exploration of this comparison will elucidate when the simple physical predictions of integrable equation are shared by closely related nonintegrable equations. This is valuable because we can solve for the dynamics of these fractional integrable equations exactly, while we often can only describe fractional nonintegrable equations numerically. However, nonintegrable equations in physical space are, in general, simpler than their integrable counterparts.

It is surprising how often integrable evolution equations involving integer operators appear in application, so we can anticipate these fractional integrable equations to have physical applications. These equations will most likely be an effective description for a real physical system, i.e., averaging over more detailed dynamics to give an approximation of a physical systems. As we presented in this thesis, the generalization of well known integrable equations to be of fractional order gives equations which are complicated in physical space but simple in scattering space. In scattering space, the generalization is simply a change in the dispersion relation (to be a power law). Therefore, if a physical system is well described in scattering space — just as optical systems, for example, are well described in Fourier space — then that system could potentially be tuned to be described by a fractional integrable equation.

Another way in which these equation could potentially be realized in physical systems is through stochastic processes. These are an effective way to relate linear fractional evolution equations to physical systems; for example, the Riesz fractional derivative in the diffusion equation describes diffusion with power law jumps. So, if the fractional operators of these fractional integrable equations can be related to stochastic processes, this could give a direct statistical signature of a physical system described by a fractional integrable equation. This may be difficult because it seems unnatural to describe a equation like the Korteweg-deVries equation in terms of stochastic processes.

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APPENDIX A

FRACTIONAL INTEGRABLE NONLINEAR SOLITON EQUATIONS SUPPLEMENTAL MATERIALS

A.1 Scattering Theory

The inverse scattering transformation (IST) solves nonlinear wave equations by associating them with linear eigenvalue problems. The fKdV equation is associated with the time-independent Schrödinger equation, (A.1), while the fNLS equation is associated with the AKNS system, (A.10)-(A.11). Here, we define the eigenfunctions and scattering data of the linear eigenvalue problem and provide some important properties.

A.1.1 Scattering Theory of the Time-Independent Schrödinger Equation

The Schrödinger scattering problem is

$$v_{xx} + \left(k^2 + q(x,t)\right) v = 0, \quad |x| < \infty \tag{A.1}$$

where v is the eigenfunction and k^2 is the eigenvalue. This is associated to the following class of integrable nonlinear equations for $q(x,t)$

$$q_t + \gamma(L^A) q_x = 0, \quad L^A \equiv -\frac{1}{4}\partial_x^2 - q + \frac{1}{2} q_x \int_x^\infty dy \tag{A.2}$$

where $\gamma(L^A)$ is defined in Eq. (A.40). Appropriate eigenfunctions are solutions to Eq. (A.1) with asymptotic boundary conditions

$$\varphi(x,k,t) \sim e^{-ikx}, \quad \overline{\varphi}(x,k,t) \sim e^{ikx}, \quad x \to -\infty, \tag{A.3}$$

$$\psi(x,k,t) \sim e^{ikx}, \quad \overline{\psi}(x;k,t) \sim e^{-ikx}, \quad x \to \infty. \tag{A.4}$$

Because ψ and $\overline{\psi}$ are linearly independent and Eq. (A.1) is second order, we have

$$\varphi(x,k,t) = a(k,t)\overline{\psi}(x,k,t) + b(k,t)\psi(x,k,t). \tag{A.5}$$

The eigenfunctions are related via

$$\varphi(x,k,t) = \overline{\varphi}(x,-k,t), \;\; \psi(x,k,t) = \overline{\psi}(x,-k,t) \tag{A.6}$$

and the scattering data are obtained from

$$a(k,t) = \frac{W(\varphi,\psi)}{2ik}, \quad b(k,t) = \frac{W(\overline{\psi},\varphi)}{2ik} \tag{A.7}$$

where $W(u,v)$ is the Wronskian $W(u,v) = uv_x - vu_x$. The associated transmission and reflection coefficients of the Schrödinger scattering problem (A.1) are written in terms of a, b as

$$\tau(k,t) = \frac{1}{a(k,t)}, \quad \rho(k,t) = \frac{b(k,t)}{a(k,t)}. \tag{A.8}$$

Discrete eigenvalues correspond to zeros of a at k_j, $j = 1, 2, ..., J$ where J gives the number of solitons in the solution. When q is real the eigenvalues are purely imaginary; i.e. $k_j = i\kappa_j$, κ_j real, $\kappa_j > 0$. At these eigenvalues, which are simple, the eigenfunctions decay exponentially — they are bound states. At these discrete eigenvalues, the eigenfunctions are related by

$$\varphi_j(x,t) = b_j(t)\psi_j(x,t) \tag{A.9}$$

where $\varphi_j(x,t) = \varphi(x,k_j,t)$, $\psi_j(x,t) = \psi(x,k_j,t)$.

A.1.2 Scattering Theory of the AKNS System

The AKNS scattering problem is:

$$v_x^{(1)} = -ikv^{(1)} + q(x,t)v^{(2)}, \tag{A.10}$$
$$v_x^{(2)} = ikv^{(2)} + r(x,t)v^{(1)} \tag{A.11}$$

where $v^{(n)}$ represents the nth component of the vector $\underline{v} = [v^{(1)}, v^{(2)}]^T$. This is associated to the following set of integrable nonlinear equations

$$\sigma_3\partial_t\mathbf{u} + 2A_0(\mathbf{L}^A)\mathbf{u} = 0, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.12}$$

where $\mathbf{u} = [r,q]^T$ and the operator

$$\mathbf{L}^A \equiv \frac{1}{2i}\begin{pmatrix} \partial_x - 2rI_-q & 2rI_-r \\ -2qI_-q & -\partial_x + 2qI_-r \end{pmatrix} \tag{A.13}$$

with $I_- = \int_{-\infty}^x dy$. With sufficient decay and smoothness, we define eigenfunctions for the AKNS system as solutions to Eqs. (A.10)-(A.11) satisfying the boundary conditions

$$\boldsymbol{\phi}(x,k,t) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \ \overline{\boldsymbol{\phi}}(x,k,t) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, x \to -\infty, \tag{A.14}$$
$$\boldsymbol{\psi}(x,k,t) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \quad \overline{\boldsymbol{\psi}}(x,k,t) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, x \to \infty. \tag{A.15}$$

The eigenfunctions $\boldsymbol{\psi}$, $\overline{\boldsymbol{\psi}}$ are linearly independent so that

$$\boldsymbol{\phi}(x,k,t) = b(k,t)\boldsymbol{\psi}(x,k,t) + a(k,t)\overline{\boldsymbol{\psi}}(x,k,t), \tag{A.16}$$

$$\overline{\boldsymbol{\phi}}(x,k,t) = \overline{a}(k,t)\boldsymbol{\psi}(x,k,t) + \overline{b}(k,t)\overline{\boldsymbol{\psi}}(x,k,t). \tag{A.17}$$

The scattering data is obtained from

$$a(k) = W(\boldsymbol{\phi},\boldsymbol{\psi}), \quad \overline{a}(k) = W(\overline{\boldsymbol{\psi}},\overline{\boldsymbol{\phi}}), \tag{A.18}$$

$$b(k) = W(\overline{\boldsymbol{\psi}},\boldsymbol{\phi}), \quad \overline{b}(k) = W(\overline{\boldsymbol{\phi}},\boldsymbol{\psi}) \tag{A.19}$$

with the Wronskian given by $W(\mathbf{u},\mathbf{v}) = u^{(1)}v^{(2)} - u^{(2)}v^{(1)}$. The transmission and reflection coefficients, τ, $\overline{\tau}$, ρ, and $\overline{\rho}$, are defined analogously to Eq. (A.8).

The zeros of a and $\overline{a}$ at $k_j = \xi_j + i\eta_j$, $\eta_j > 0$, $j = 1, 2, ..., J$ and $\overline{k}_j = \overline{\xi}_j + i\overline{\eta}_j$, $\overline{\eta}_j < 0$, $j = 1, 2, ..., \overline{J}$ are the eigenvalues. We assume the eigenvalues are 'proper': i.e. they are simple, not on the real k axis, and $J = \overline{J}$; cf. Ref. [60]. The bound state eigenfunctions are related by

$$\boldsymbol{\phi}_j(x,t) = b_j(t)\boldsymbol{\psi}_j(x,t), \quad \overline{\boldsymbol{\phi}}_j(x,t) = \overline{b}_j(t)\overline{\boldsymbol{\psi}}_j(x,t). \tag{A.20}$$

When $r = \mp q^*$, we have the symmetry reductions

$$\overline{\boldsymbol{\psi}}(x,k,t) = \Sigma\boldsymbol{\psi}^*(x,k^*,t), \;\; \overline{\boldsymbol{\phi}}(x,k,t) = \mp\Sigma\boldsymbol{\phi}^*(x,k^*,t) \tag{A.21}$$

for the eigenfunctions where $\Sigma = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$. We also have $\overline{a}(k) = a^*(k^*)$ and $\overline{b}(k) = \mp b^*(k^*)$ for the scattering data.

From the scattering eigenfunctions $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$, we can construct the eigenfunctions of the operator $\mathbf{L}$, $\boldsymbol{\Psi}$ and $\overline{\boldsymbol{\Psi}}$, and its adjoint $\mathbf{L}^A$, $\boldsymbol{\Psi}^A$ and $\overline{\boldsymbol{\Psi}}^A$

$$\boldsymbol{\Psi}(x,k,t) = \left[(\psi^{(1)}(x,k,t))^2, (\psi^{(2)}(x,k,t))^2\right]^T, \tag{A.22}$$

$$\overline{\boldsymbol{\Psi}}(x,k,t) = \left[(\overline{\psi}^{(1)}(x,k,t))^2, (\overline{\psi}^{(2)}(x,k,t))^2\right]^T, \tag{A.23}$$

$$\boldsymbol{\Psi}^A(x,k,t) = \left[(\phi^{(2)}(x,k,t))^2, -(\phi^{(1)}(x,k,t))^2\right]^T, \tag{A.24}$$

$$\overline{\boldsymbol{\Psi}}^A(x,k,t) = \left[(\overline{\phi}^{(2)}(x,k,t))^2, -(\overline{\phi}^{(1)}(x,k,t))^2\right]^T. \tag{A.25}$$

We use these functions to define the fNLS equation.

A.2 Direct Scattering

To solve fKdV and fNLS by the inverse scattering transform, we first map the initial condition into scattering space; this is analogous to taking the Fourier transform of a linear PDE. This process involves analyzing linear integral equations for the eigenfunctions, determining their analytic properties, and then obtaining the scattering data using Wronskian relations.

A.2.1 Direct Scattering for the Time-Independent Schrödinger Equation

The eigenfunctions φ and ψ of the time-independent Schrödinger Eq. (A.1) solve linear integral equations which have uniformly convergent Neumann series for $q(x,0)$ in L_2^1 [16]. This series can be used to construct the eigenfunctions explicitly. Then, the scattering data at $t=0$, that is $a(k,0)$, $b(k,0)$, $\tau(k,0)$, and $\rho(k,0)$, can be obtained from the Wronskian relations in Eq. (A.7) along with the definitions of the transmission and reflection coefficients in Eq. (A.8).

A.2.2 Direct Scattering for the AKNS System

Similarly, the eigenfunctions $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ of the AKNS system solve linear integral equations with convergent Neumann series [16] and the initial scattering data $a(k,0)$, $\overline{a}(k,0)$, $b(k,0)$, and $\overline{b}(k,0)$ is constructed from the Wronskian relations in Eqs. (A.18) and (A.19).

A.3 Time Evolution of the Scattering Data

The scattering data evolve in time according to elementary differential equations.

A.4 Time Evolution for the Time-Independent Schrödinger Equation

Following the procedure in cf. Ref. [16] the time dependence of the scattering data is given by

$$a(k,t) = a(k,0), \quad b(k,t) = b(k,0)e^{-2ik\gamma(k^2)t}, \tag{A.26}$$

$$\rho(k,t) = \rho(k,0)e^{-2ik\gamma(k^2)t}, \ c_j(t) = c_j(0)e^{\kappa_j\gamma(-\kappa_j^2)t} \tag{A.27}$$

where $c_j^2(t) = -ib_j(t)/a_j'(t)$ with $a_j'(t) \equiv \partial_k a(k,t)|_{k=i\kappa_j}$ and γ comes from the nonlinear evolution equation in Eq. (A.2). Notice that $a(k,t)$ is a constant of motion.

A.4.1 Time Evolution for the AKNS System

For the general evolution equation associated with the AKNS system in Eq. (A.12), we find the time evolution to be [16]

$$
\begin{aligned}
a(k,t) &= a(k,0), & \overline{a}(k,t) &= \overline{a}(k,0), & \text{(A.28)}\\
b(k,t) &= b(k,0)e^{-2A_0(k)t}, & \overline{b}(k,t) &= \overline{b}(k,0)e^{2A_0(k),t} & \text{(A.29)}\\
C_j(t) &= C_j(0)e^{-2A_0(k_j)t}, & \overline{C}_j(t) &= \overline{C}_j(0)e^{2A_0(\overline{k}_j)t} & \text{(A.30)}
\end{aligned}
$$

where $C_j(t) = b_j(t)/a'_j(t)$ and $\overline{C}_j(t) = \overline{b}_j(t)/\overline{a}'_j(t)$ with $a'_j(t) \equiv \partial_k a(k,t)|_{k=i\kappa_j}$ and $\overline{a}'_j(t) \equiv \partial_k \overline{a}(k,t)|_{k=i\overline{\kappa}_j}$. Recall that $A_0(k)$ is related to the nonlinear evolution equation in Eq. (A.12).

A.5 Inverse Scattering

Inverse scattering is analogous to the inverse Fourier transform, except evaluating an integral on the real line in the case of Fourier transforms is now replaced by solving a linear integral equation in the case of the IST.

A.5.1 Inverse Scattering for the Time-Independent Schrödinger Equation

The inverse scattering and solution $q(x,t)$ of fKdV can be constructed by solving the following Gel'fand-Levitan-Marchenko (GLM) integral equation for $K(x,y;t)$:

$$
\begin{aligned}
K(x,y;t) + \tilde{F}(x+y;t) + \int_x^\infty ds K(x,s;t)\tilde{F}(s+y;t) &= 0, \qquad \text{(A.31)}\\
\tilde{F}(x;t) = \sum_{j=1}^{J} c_j^2(t)e^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\rho(k,t)e^{ikx}.
\end{aligned}
$$

Here $y > x$ and recall that J is the number of zeros of a or, equivalently, the number of solitons in the solution q. Here, the time dependence of $\rho(k,t)$ and $c_j(t)$ are given in Eq. (A.27). Then the solution of the fKdV is obtained from

$$
q(x,t) = 2\frac{\partial}{\partial x}K(x,x;t). \qquad \text{(A.32)}
$$

A.5.2 Inverse Scattering for the AKNS System

The solution of fNLS and the general fractional q, r system can be constructed by solving the following GLM-type integral equations

$$
\begin{aligned}
\mathbf{K}(x,y;t) + \begin{pmatrix}1\\0\end{pmatrix}\overline{F}(x+y;t) \qquad \text{(A.33)}\\
+ \int_x^\infty ds\,\overline{\mathbf{K}}(x,s;t)\overline{F}(s+y;t) = 0,
\end{aligned}
$$

$$\overline{\mathbf{K}}(x,y;t) + \binom{0}{1} F(x+y;t) \\ + \int_x^\infty ds\, \mathbf{K}(x,s;t)F(s+y;t) = 0 \tag{A.34}$$

where

$$F(x;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\rho(k,t)e^{ikx} - i\sum_{j=1}^{J} C_j(t)e^{ik_jx}, \tag{A.35}$$

$$\overline{F}(x;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\overline{\rho}(k,t)e^{-ikx} + i\sum_{j=1}^{\overline{J}} \overline{C}_j(t)e^{-i\overline{k}_jx}. \tag{A.36}$$

The time dependence of $\rho(k,t) = b(k,t)/a(k)$, $\overline{\rho}(k,t) = \overline{b}(k,t)/\overline{a}(k)$, $C_j(t)$, and $\overline{C}_j(t)$ are given in Eqs. (A.28)-(A.30).

The solution of the fractional q, r system is obtained from

$$q(x,t) = -2K^{(1)}(x,x;t),\ r(x,t) = -2\overline{K}^{(2)}(x,x;t) \tag{A.37}$$

where $K^{(n)}$ and $\overline{K}^{(n)}$ for $n = 1, 2$ denote the nth component of the vectors $\mathbf{K}$ and $\overline{\mathbf{K}}$ respectively. If the symmetry $r = \mp q^*$ holds then the GLM equations have the scalar reduction

$$\overline{F}(x;t) = \mp F^*(x;t)$$

and consequently

$$\overline{\mathbf{K}}(x,y;t) = \begin{pmatrix} K^{(2)}(x,y;t) \\ \mp K^{(1)}(x,y;t) \end{pmatrix}^*. \tag{A.38}$$

In this case, the inverse problem reduces to

$$K^{(1)}(x,y;t) = \pm F^*(x+y;t) \\ \mp \int_x^\infty ds \int_x^\infty ds' K^{(1)}(x,s';t)F(s+s';t)F^*(y+s;t) \tag{A.39}$$

and an equation for $K^{(2)}$ which we do not write here. Then the solution to fNLS can be obtained from Eq. (A.37). We also note that when $r = \mp q$ with q real, then $F(x;t)$ and $K^{(1)}(x,y;t)$ are real.

A.6 Alternative Representation of the fKdV Operator

The operator $\gamma(L^A)$ acting on an arbitrary function $h(x)$ may be represented by the spectral expansion

$$\gamma(L^A)h(x) = \int_{\Gamma_\infty} dk\gamma(k^2)\frac{\tau^2(k)}{4\pi ik}\int_{-\infty}^{\infty} dyG(x,y,k)h(y) \tag{A.40}$$

in terms of the eigenfunctions of the Schrödinger scattering problem where time t is suppressed. This expression can be evaluated using contour integration to give a representation of $\gamma(L^A)$ in terms of integrals along the real line and a sum over discrete values along the imaginary axis:

$$\begin{aligned}\gamma(L^A)h(x) = &\int_{-\infty}^{\infty} dk\,\gamma(k^2)\frac{\tau^2(k)}{4\pi ik}\int_{-\infty}^{\infty} dy\, G_c(x,y,k)h(y)\\ &+\sum_{j=1}^{J}\gamma(-\kappa_j^2)\int_{-\infty}^{\infty} dy\, G_{d,j}(x,y,k)h(y).\end{aligned} \tag{A.41}$$

Here the continuous contribution G_c is defined by

$$G_c(x,y,k) = \partial_x\left(\psi^2(x,k)\varphi^2(y,k) - \varphi^2(x,k)\psi^2(y,k)\right), \tag{A.42}$$

and the discrete contribution, which comes from the poles of τ at $k_j = i\kappa_j$, $j = 1, 2, ..., J$, is given by

$$\begin{aligned}G_{d,j} = &\, i\eta_j\frac{\partial}{\partial x}\left[\psi^2(y)\psi(x)\partial_k\varphi(x) - \psi^2(y)\varphi(x)\partial_k\psi(x)\right]_{k=k_j}\\ &- i\eta_j\frac{\partial}{\partial x}\left[\psi^2(x)\psi(y)\partial_k\varphi(y) - \psi^2(x)\varphi(y)\partial_k\psi(y)\right]_{k=k_j}\end{aligned} \tag{A.43}$$

where $\eta_j = \frac{b(k_j)}{\kappa_j a'(k_j)^2}$. Note that we have suppressed k in the above expression, i.e., $\psi(x) = \psi(x,k)$ and $\varphi(x) = \varphi(x,k)$.

A.7 Explicit Form of the fNLS Equation

The set of nonlinear evolution equations which may be associated with the AKNS scattering problem, Eqs. (A.10-A.11), is given in Eq. (A.12). Further, $A_0(\mathbf{L}^A)$ may be represented as

$$A_0(\mathbf{L}^A)\mathbf{v}(x) = \sum_{n=1}^{2}\int_{\Gamma_\infty^{(n)}} dkA_0(k)f_n(k)\int_{-\infty}^{\infty} dy\mathbf{G}_n(x,y,k)\mathbf{v}(y). \tag{A.44}$$

We obtain fNLS by taking $A_0(\mathbf{L}^A) = 2i(\mathbf{L}^A)^2|2\mathbf{L}^A|^\epsilon$ and $r = \mp q^*$ in Eq. (A.12). If we split off $2i(\mathbf{L}^A)^2$ and operate on $\mathbf{u} = (r,q)^T$, we find

$$2i(\mathbf{L}^A)^2\mathbf{u} = \frac{1}{2i}\begin{pmatrix}\mp q^*_{xx} - 2(q^*)^2 q\\ q_{xx} \pm 2q^2 q^*\end{pmatrix}. \tag{A.45}$$

Then, representing $|2\mathbf{L}^A|^\epsilon$ with the spectral expansion in Eq. (A.44), we have

$$2i(\mathbf{L}^A)^2|2\mathbf{L}^A|^\epsilon\mathbf{u} = \sum_{n=1}^{2}\frac{1}{2i}\int_{\Gamma_\infty^{(n)}} dk|2k|^\epsilon f_n(k) \tag{A.46}$$

$$\times\int_{-\infty}^{\infty} dy\mathbf{G}_n(x,y,k)\begin{pmatrix}\mp q_{yy}^* - 2(q^*)^2 q\\ q_{yy} \pm 2q^2q^*\end{pmatrix}. \tag{A.47}$$

Putting Eq. (A.47) into Eq. (A.12), multiplying by $-i$, and taking the second component gives

$$iq_t = \sum_{n=1}^{2}\int_{\Gamma_\infty^{(n)}} dk|2k|^\epsilon f_n(k)\int_{-\infty}^{\infty} dyF_n(x,y,k) \tag{A.48}$$

where

$$F_1(x,y,k) = -\phi_1^2(x,k)\big[\psi_1^2(y,k)(\mp q_{yy}^* - 2(q^*)^2q) \tag{A.49}$$

$$+\psi_2^2(y,k)(q_{yy} \pm 2q^2q^*)\big] \tag{A.50}$$

and F_2 is the same, but with ψ_n replaced by $\overline{\psi}_n$ and ϕ_n replaced by $\overline{\phi}_n$ for $n = 1, 2$ where $\overline{\psi}_n$ and $\overline{\phi}_n$ are related to ψ_n and ϕ_n by equation (A.21)

A.8 Conserved Quantities

Like their integer counterparts, the fKdV and the fNLS equation also admit an infinite number of conserved quantities. In fact, the derivation of these using IST methods is independent of the exact form of γ and A_0, so their conserved quantities are the same as KdV and NLS; cf. [12]. However, the fluxes associated with these conservation laws corresponding to these conserved quantities are not the same.

A.9 Direct Calculation for the fKdV Soliton

It can be shown that the fractional soliton solution given by

$$q_K(x,t) = 2\kappa^2\text{sech}^2 w_\epsilon(x,t) \tag{A.51}$$

solves the fKdV equation where $w_\epsilon(x,t) = \kappa[(x - x_1) - (4\kappa^2)^{1+\epsilon}t]$. This one soliton corresponds to $J = 1$, i.e., one bound state solution of the time-independent Schrödinger equation, (A.1), at $k = i\kappa$ and a reflectionless potential $\rho(k,t) = 0$ for real k. To show this, we verify that the fKdV equation

$$q_t + \int_{\Gamma_\infty} dk|4k^2|^{1+\epsilon}\frac{\tau^2(k)}{2\pi ik}\int_{-\infty}^{\infty} dy\, G(x,y,k)q_y = 0 \tag{A.52}$$

is satisfied when $q = q_K$. For this case, the Schrödinger eigenfunctions — which are found by solving Eq. (A.1) with $q = q_K$ — can be written as

$$\psi(x,k,t) = v_1(x,k,t), \tag{A.53}$$
$$\varphi(x,k,t) = v_{-1}(x,k,t) \tag{A.54}$$

where

$$v_\sigma(x,k,t) = e^{i\sigma kx}\left(\frac{k+\sigma i\kappa\tanh w_\epsilon(x,t)}{k+i\kappa}\right) \tag{A.55}$$

with $\sigma = \pm 1$. Further, $\partial_x q_K$ is

$$\partial_x q_K(x,t) = -4\kappa^3 \mathrm{sech}^2 w_\epsilon(x,t)\tanh w_\epsilon(x,t). \tag{A.56}$$

To calculate $\gamma(L^A)\partial_x q_K$, we compute the integrals $I_c \equiv \int_{-\infty}^{\infty} dy\, G_c(x,y,k)\partial_y q_K(y,t)$ and $I_d \equiv \int_{-\infty}^{\infty} dy\, G_{d,1}(x,y,k)\partial_y q_K(y,t)$ according to Eqs. (A.41)-(A.43). These two integral can be written as

$$I_c = \partial_x(v_1)^2 I_{-1}^{(1)} + \partial_x(v_{-1})^2 I_1^{(1)}|_{k=i\kappa}, \tag{A.57}$$
$$\begin{aligned} I_d &= i\eta\partial_x\left(v_1\partial_k v_{-1} - v_{-1}\partial_k v_1\right) I_1^{(1)}|_{k=i\kappa} \\ &\quad - i\eta\partial_x(v_1)^2\left(I_1^{(2)} - I_{-1}^{(2)}\right)|_{k=i\kappa} \end{aligned} \tag{A.58}$$

where

$$I_\sigma^{(1)} = \int_{-\infty}^{\infty} dy\ v_\sigma^2(y,k,t) q_y(y,t), \tag{A.59}$$
$$I_\sigma^{(2)} = \int_{-\infty}^{\infty} dy\ v_\sigma(y,k,t)\partial_k v_{-\sigma}(y,k,t) q_y(y,t) \tag{A.60}$$

and $\eta = b(i\kappa,t)/\kappa(\partial_k a(k,t)|_{k=i\kappa})^2$ Using the substitution $w = w_\epsilon(y,t)$, the integral $I_\sigma^{(1)}$ can be shown to be proportional to $\int_{-\infty}^{\infty} dw\ f_\sigma(w)$ where

$$f_\sigma(w) = e^{2i\sigma kw/\kappa}\left(k + i\sigma\kappa\tanh w\right)^2 \mathrm{sech}^2 w\tanh w. \tag{A.61}$$

If we consider the rectangular contour with corners at $w = -R$ and $w = R + i\pi$ and define the bottom, top, right, and left contour by C_B, C_T, C_R, and C_L respectively, we have

$$\begin{aligned} &\left(\int_{C_B} dw + \int_{C_T} dw + \int_{C_R} dw + \int_{C_L} dw\right) f_\sigma(w) \\ &\qquad = 2\pi i \mathrm{Res}\left(f_\sigma, i\pi/2\right). \end{aligned} \tag{A.62}$$

However, the contours along C_R and C_L vanish as $R \to \infty$ and C_B may be written in terms of C_T as

$$\int_{C_T} dw\ f_\sigma(w) = -e^{-2\sigma k\pi/\kappa} \int_{C_B} dw\ f(w). \tag{A.63}$$

Because as $R \to \infty$, $\int_{C_B} dw f_\sigma(w) \to \int_{-\infty}^{\infty} dw f_\sigma(w)$, equation (A.62) becomes

$$\left(1 - e^{-2\sigma k\pi/\kappa}\right) \int_{-\infty}^{+\infty} dw f_\sigma(w) = 2\pi i \mathrm{Res}\left(f_\sigma, i\pi/2\right). \tag{A.64}$$

But as the residue of f_σ vanishes when $\sigma = \pm 1$, $\int_{-\infty}^{+\infty} dw f_\sigma(w) = 0$ and, thus, $I_\sigma^{(1)} = 0$. Therefore, the continuous contribution to $\gamma(L^A)q_x$, I_c, vanishes. We also see that the first half of the discrete part, I_d given by Eq. (A.58), is zero. With these reductions, we can express $\gamma(L^A)\partial_x q_K$ as

$$\gamma(L^A)\partial_x q_K = -i\eta\gamma(-\kappa^2)\partial_x(v_1)^2 \left(I_1^{(2)} - I_{-1}^{(2)}\right)|_{k=i\kappa}. \tag{A.65}$$

By explicitly evaluating η and $\partial_x(v_1)^2$, we find that

$$\eta\partial_x(v_1)^2|_{k=i\kappa} = \frac{\partial_x q_K}{2\kappa}. \tag{A.66}$$

Then, the integral $I_\sigma^{(2)}$ can be evaluated using the fundamental theorem of calculus to give

$$\left(I_1^{(2)} - I_{-1}^{(2)}\right)|_{k=i\kappa} = 2i\kappa. \tag{A.67}$$

So, $\gamma(L^A)\partial_x q_K$ follows as

$$\gamma(L^A)\partial_x q_K = -(2\kappa)^{2+2\epsilon}\partial_x q_K(x,t) \tag{A.68}$$

which is exactly $-\partial_t q_K(x,t)$. Thus, the soliton given in Eq. (A.51) solves the fKdV equation.

APPENDIX B

FRACTIONAL INTEGRABLE AND RELATED DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS

B.1 Scattering Theory for the Ablowitz-Ladik System

Here, we define eigenfunctions, scattering data, etc. that are used to define the fractional integrable discrete nonlinear Schrödinger (fIDNLS) equation and solve it by the IST. The Ablowitz-Ladik scattering problem

$$\mathbf{v}_{n+1} = \begin{pmatrix} z & q_n \\ r_n & z^{-1} \end{pmatrix} \mathbf{v}_n \tag{B.1}$$

is associated to the following family of nonlinear evolution equations

$$\sigma_3 \frac{d\mathbf{u}_n}{dt} + \gamma(\Lambda_+)\mathbf{u}_n = 0, \quad \mathbf{u}_n = (q_n, -r_n)^T \tag{B.2}$$

where T represents transpose, $\sigma_3 = \text{diag}(1, -1)$, $h_n = 1 - r_n q_n$, $\sum_n^+ = \sum_{k=n}^{\infty}$, and the operator Λ_+ is defined in the main manuscript. Eigenfunctions of the Ablowitz-Ladik scattering system are solutions to equation (B.1) subject to the boundary conditions

$$\boldsymbol{\phi}_n(z,t) \sim \begin{pmatrix} z^n \\ 0 \end{pmatrix}, \quad \overline{\boldsymbol{\phi}}_n(z,t) \sim \begin{pmatrix} 0 \\ z^{-n} \end{pmatrix}, \ n \to -\infty, \tag{B.3}$$

$$\boldsymbol{\psi}_n(z,t) \sim \begin{pmatrix} 0 \\ z^{-n} \end{pmatrix}, \ \overline{\boldsymbol{\psi}}_n(z,t) \sim \begin{pmatrix} z^n \\ 0 \end{pmatrix}, \quad n \to +\infty. \tag{B.4}$$

Because the "right" eigenfunctions $\boldsymbol{\psi}_n$ and $\overline{\boldsymbol{\psi}}_n$ are linearly independent, we can write the "left" eigenfunctions as

$$\boldsymbol{\phi}_n(z,t) = a(z,t)\overline{\boldsymbol{\psi}}_n(z,t) + b(z,t)\boldsymbol{\psi}_n(z,t), \tag{B.5}$$

$$\overline{\boldsymbol{\phi}}_n(z,t) = \overline{a}(z,t)\boldsymbol{\psi}_n(z,t) + \overline{b}(z,t)\overline{\boldsymbol{\psi}}_n(z,t). \tag{B.6}$$

These relations define the scattering data a, b, $\overline{a}$, and $\overline{b}$. We can write the scattering data explicitly in terms of the eigenfunctions as

$$a(z,t) = \nu_n W\big(\boldsymbol{\phi}_n, \boldsymbol{\psi}_n\big), \quad \overline{a}(z,t) = \nu_n W\big(\overline{\boldsymbol{\psi}}_n, \overline{\boldsymbol{\phi}}_n\big), \tag{B.7}$$

$$b(z,t) = \nu_n W\big(\overline{\boldsymbol{\psi}}_n, \boldsymbol{\phi}_n\big), \quad \overline{b}(z,t) = \nu_n W\big(\overline{\boldsymbol{\phi}}_n, \boldsymbol{\psi}_n\big). \tag{B.8}$$

with the Wronskian $W(\mathbf{u}_n, \mathbf{v}_n) \equiv u_n^{(1)} v_n^{(2)} - u_n^{(2)} v_n^{(1)}$ and $\nu_n \equiv \prod_{k=n}^{\infty} h_k$, $h_k = 1 - r_k q_k$. The transmission and reflection coefficients, $\tau(z,t)$, $\overline{\tau}(z,t)$ and $\rho(z,t)$, $\overline{\rho}(z,t)$, respectively, are defined by

$$\tau(z,t) = \frac{1}{a(z,t)}, \quad \rho(z,t) = \frac{b(z,t)}{a(z,t)}, \tag{B.9}$$

$$\overline{\tau}(z,t) = \frac{1}{\overline{a}(z,t)}, \quad \overline{\rho}(z,t) = \frac{\overline{b}(z,t)}{\overline{a}(z,t)}. \tag{B.10}$$

Often, the functions τ, $\overline{\tau}$, ρ, and $\overline{\rho}$ are equivalently referred to as the scattering data. The eigenfunctions

$$\boldsymbol{\phi}_n(z,t) z^{-n}, \quad \boldsymbol{\psi}_n(z,t) z^{n} \tag{B.11}$$

are analytic and bounded for $|z| > 1$ and continuous for $|z| \geq 1$ and

$$\overline{\boldsymbol{\phi}}_n(z,t) z^{n}, \quad \overline{\boldsymbol{\psi}}_n(z,t) z^{-n} \tag{B.12}$$

are analytic and bounded for $|z| < 1$ and continuous for $|z| \leq 1$. Hence a and $\overline{a}$ are analytic inside and outside the unit circle, respectively.

The Ablowitz-Ladik scattering system can have discrete eigenvalues, corresponding to bound states. These occur at the zeros of a and $\overline{a}$ — which we notate by z_j for $j = 1, 2, ..., J$ and $\overline{z}_j$ for $j = 1, 2, ..., \overline{J}$, respectively — such that $|z_j| > 1$ and $|\overline{z}_j| < 1$. We assume that these eigenvalues are proper, i.e., the zeros of a and $\overline{a}$ are simple (not on the unit circle and finite in number). At these discrete eigenvalues, the eigenfunctions are related by

$$\boldsymbol{\phi}_n(z_j,t) = b(z_j,t) \boldsymbol{\psi}_n(z_j,t), \quad \overline{\boldsymbol{\phi}}_n(\overline{z}_j,t) = \overline{b}(\overline{z}_j,t) \overline{\boldsymbol{\psi}}_n(\overline{z}_j,t). \tag{B.13}$$

We also define the norming constants by

$$c_j(t) = \frac{b(z_j,t)}{a'(z_j,t)}, \quad \overline{c}_j(t) = \frac{\overline{b}(\overline{z}_j,t)}{\overline{a}'(\overline{z}_j,t)} \tag{B.14}$$

where $a'(z_j,t) = \partial_z a(z,t)|_{z=z_j}$, etc. When $r_n = \mp q_n^*$ in (B.1), we have the symmetry reductions

$$\overline{\boldsymbol{\phi}}_n(z,t) = \mathbf{P}_{\mp} \boldsymbol{\phi}_n^*(1/z^*,t), \quad \overline{\boldsymbol{\psi}}_n(z,t) = \mp \mathbf{P}_{\mp} \boldsymbol{\psi}_n^*(1/z^*,t) \tag{B.15}$$

for the eigenfunctions and $\overline{a}(z,t) = a^*(1/z^*,t)$ and $\overline{b}(z,t) = \mp b^*(1/z^*,t)$ where

$$\mathbf{P}_{\mp} = \begin{pmatrix} 0 & \mp 1 \\ 1 & 0 \end{pmatrix}. \tag{B.16}$$

From the eigenfunctions, solutions of (B.1), we can construct the eigenfunctions of the nonlinear operator Λ_+, $\mathbf{\Psi}_n(z,t)$ and $\overline{\mathbf{\Psi}}_n(z,t)$, and its adjoint Λ_+^A, $\mathbf{\Psi}_n^A(z,t)$ and $\overline{\mathbf{\Psi}}_n^A(z,t)$ by

$$\mathbf{\Psi}_n = \nu_n \boldsymbol{\psi}_n \circ \boldsymbol{\psi}_{n+1}, \quad \mathbf{\Psi}^A = -\nu_n \mathbf{P}_-(\boldsymbol{\phi}_n \circ \boldsymbol{\phi}_{n+1}) \tag{B.17}$$

$$\overline{\mathbf{\Psi}}_n = \nu_n \overline{\boldsymbol{\psi}}_n \circ \overline{\boldsymbol{\psi}}_{n+1}, \quad \overline{\mathbf{\Psi}}^A = -\nu_n \mathbf{P}_-(\overline{\boldsymbol{\phi}}_n \circ \overline{\boldsymbol{\phi}}_{n+1}) \tag{B.18}$$

where $\mathbf{u}_n \circ \mathbf{v}_m = \left(u_n^{(1)} v_m^{(1)}, u_n^{(2)} v_m^{(2)}\right)^T$.

B.2 Solving The Nonlinear Evolution Equation Using the IST

Solving nonlinear discrete evolution equations with the IST is analogous to solving linear discrete evolution equations with the Z-transform. The IST has three distinct steps: direct scattering, time evolution, and inverse scattering which are analagous to taking the Z-transform, evolving the solution in frequency space, and taking the inverse Z-transform, respectively. In direct scattering, the initial condition is mapped into scattering space by solving the scattering problem (B.1). The time evolution of the scattering data, which represents the solution in scattering space, is evolved in time by solving a simple set of differential equations. Finally, in inverse scattering, the solution in physical space is reconstructed from the scattering data by solving a system of algebraic and summation equations. In the following, we briefly outline direct scattering, time evolution, and inverse scattering for the Ablowitz-Ladik scattering system.

B.2.1 Direct Scattering

To perform direct scattering, we use the scattering problem in (B.1) to solve for the eigenfunctions $\boldsymbol{\phi}$, $\overline{\boldsymbol{\phi}}$, $\boldsymbol{\psi}$, and $\overline{\boldsymbol{\psi}}$ at $t=0$. Existence and uniqueness of these solutions can be proven by converting equation (B.1) and the appropriate boundary conditions into linear summation equations which have uniformly convergent Neumann series [60]. These series also provide an alternative method of constructing these eigenfunctions. Then, the scattering data, a, b, $\overline{a}$, and $\overline{b}$, at $t=0$ are obtained from the Wronskian relations in equations (B.7) and (B.8).

B.2.2 Time Evolution

The scattering data evolves in time according to [7]

$$\frac{d\rho}{dt} - \gamma(z^2)\rho(z,t) = 0, \quad \frac{d\overline{\rho}}{dt} + \gamma(z^2)\overline{\rho}(z,t) = 0, \tag{B.19}$$

$$\frac{dc_j}{dt} - \gamma(z_j^2)c_j(t) = 0, \quad \frac{d\overline{c}_j}{dt} + \gamma(\overline{z}_j^2)\overline{c}_j(t) = 0 \tag{B.20}$$

for $j = 1, 2, ..., J$ and $j = 1, 2, ..., \overline{J}$, respectively. We recall that γ is the function of an operator in equation (B.2) and is related to a linear dispersion relation. Also note that z_j and $\overline{z}_j$ are independent of time. To fully characterized the spectral representation of the operator $\gamma(\Lambda_+)$, and find the solution $q_n(t)$, we need the eigenfunctions at time t in addition to the scattering data. These functions are found using inverse scattering.

B.2.3 Inverse Scattering

To reconstruct the solutions to the nonlinear evolution equation (B.2) and eigenfunctions at time t, we solve the following Gel'fand-Levitan-Marchenko type summation equations for $\boldsymbol{\kappa}(n,m,t)$ [60]

$$\begin{aligned} \boldsymbol{\kappa}(n,m,t) &+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overline{F}(m+n,t) \\ &+ \sum_{j=n+1}^{\infty} \overline{\boldsymbol{\kappa}}(n,j,t)\overline{F}(m+j,t) = 0, \end{aligned} \tag{B.21}$$

$$\begin{aligned} \overline{\boldsymbol{\kappa}}(n,m,t) &+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(m+n,t) \\ &+ \sum_{j=n+1}^{\infty} \boldsymbol{\kappa}(n,j,t)F(m+j,t) = 0 \end{aligned} \tag{B.22}$$

where

$$F(n,t) = \sum_{j=1}^{J} z_j^{-n-1} c_j(t) + \frac{1}{2\pi i} \oint_{S_1} z^{-n-1} \rho(z,t)dz, \tag{B.23}$$

$$\overline{F}(n,t) = \sum_{j=1}^{\overline{J}} \overline{z}_j^{-n-1} \overline{c}_j(t) + \frac{1}{2\pi i} \oint_{S_1} z^{n-1} \overline{\rho}(z,t)dz. \tag{B.24}$$

Then, the potentials can be obtained from

$$q_n(t) = -\kappa^{(1)}(n,n+1,t), \quad r_n(t) = -\overline{\kappa}^{(2)}(n,n+1,t), \tag{B.25}$$

and the right eigenfunctions from

$$\boldsymbol{\psi}_n(z,t) = \sum_{j=n}^{\infty} z^{-j} \mathbf{K}(n,j,t), \tag{B.26}$$

$$\overline{\boldsymbol{\psi}}_n(z,t) = \sum_{j=n}^{\infty} z^{j} \overline{\mathbf{K}}(n,j,t) \tag{B.27}$$

where

$$\mathbf{K}(n,m,t) = \nu_n \boldsymbol{\kappa}(n,m,t), \tag{B.28}$$

$$\overline{\mathbf{K}}(n,m,t) = \nu_n \overline{\boldsymbol{\kappa}}(n,m,t). \tag{B.29}$$

The left eigenfunctions $\boldsymbol{\phi}_n(z,t)$ and $\overline{\boldsymbol{\phi}}_n(z,t)$ can be constructed using the relations in equations (B.5) and (B.6). If $r_n(t) = \mp q_n^*(t)$, then equations (B.21) and (B.22) reduce to

$$\kappa^{(1)}(n,m,t) - \overline{F}(n+m,t) \tag{B.30}$$

$$\pm \sum_{n''=n+1}^{\infty} \sum_{n'=n+1}^{\infty} \kappa^{(1)}(n,n'',t)\overline{F}^*(n''+n',t)\overline{F}(n'+m,t) = 0 \tag{B.31}$$

$$\begin{aligned}\kappa^{(1)}(n,m,t) - \overline{F}(n+m,t) \pm \sum_{n''=n+1}^{\infty} \sum_{n'=n+1}^{\infty} \kappa^{(1)}(n,n'',t) \\ \cdot \overline{F}^*(n''+n',t)\overline{F}(n'+m,t) = 0\end{aligned} \tag{B.32}$$

We note that under $r_n(t) = \mp q_n^*(t)$ there are induced symmetries: $\overline{\rho}(z) = \mp\rho^*(1/z^*)$ and for $r_n(t) = -q_n^*(t)$ there can be discrete states with $\overline{z} = 1/z^*$, $\overline{c}_j = (z^*)^{-2}c_j^*$.

APPENDIX C

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