Boundary integral equations for the scattering of electromagnetic waves by a homogeneous dielectric obstacle

P. A. Martin
Department of Mathematics, University of Manchester, Manchester M13 9PL, England, U.K.

and

Petri Ola
Department of Mathematics, University of Helsinki, 00100 Helsinki, Finland

(MS received 30 January 1992)

Synopsis
Time-harmonic electromagnetic waves are scattered by a homogeneous dielectric obstacle. The corresponding electromagnetic transmission problem is reduced to a single integral equation over $\Sigma$ for a single unknown tangential vector field, where $\Sigma$ is the interface between the obstacle and the surrounding medium. In fact, several different integral equations are derived and analysed, including two previously-known equations due to E. Marx and J. R. Mautz, and two new singular integral equations. Mautz's equation is shown to be uniquely solvable at all frequencies. A new uniquely-solvable singular integral equation is also found. The paper also includes a review of methods using pairs of coupled integral equations over $\Sigma$. It is these methods that are usually used in practice, although single integral equations seem to offer some computational advantages.

1. Introduction
It is well known that the problem of time-harmonic electromagnetic scattering by a perfectly-conducting obstacle can be reduced to a single integral equation over the boundary of the obstacle. It is also well known that the simplest of these equations suffer from irregular frequencies, at which they are not uniquely solvable. Various methods for eliminating irregular frequencies have been devised; see, for example, [18], [7, §6.17], [2, §4.6], [12], [20], [8].

The situation for dielectric obstacles is more complicated. If the obstacle is inhomogeneous, so that its material properties vary with position, integral equations can still be derived but the domain of integration is usually the whole volume occupied by the obstacle; see, for example, [19, §22], [7, §6.26], [3]. However, for homogeneous dielectrics, the problem can be reduced to integral equations over the interface between the two materials. It is equations of this type that we consider below.

The above electromagnetic transmission problem is usually reduced to a pair of coupled integral equations for a pair of unknown tangential vector fields. Such formulations have been reviewed recently by Harrington [6]. We give a complementary review in Section 6, where we use an operator notation familiar from the book [2]; for convenience, we include an appendix relating our notation to that used in [6]. Some pairs of integral equations have irregular frequencies, others do not. A new proof is given of a theorem in which all the irregular
frequencies of the so-called $E$-field formulation are identified. This proof uses some properties of a certain operator ($\tilde{\mathbf{s}}_\lambda$, defined by (5.1) below), involving products of the standard electromagnetic boundary integral operators. These properties are proved in Section 5, using the theory of pseudodifferential operators.

In Sections 7 and 8, we consider methods for solving the electromagnetic transmission problem using a single integral equation for a single unknown tangential vector field. A systematic derivation is given (in Section 7) of two different two-parameter integral equations. As a special case, we recover a known hypersingular integral equation due to Marx [14, 15]. In fact, for almost all values of the two parameters, our single integral equations are hypersingular integral equations. Exceptionally, we find two new singular integral equations, although these are shown to suffer from irregular frequencies.

In Section 8, we derive single integral equations that are shown to be uniquely solvable at all frequencies. Thus, in Section 8.1 we derive an equation previously obtained by Mautz [16]. We establish the existence of a unique solution to this hypersingular integral equation by adapting a regularisation method due to Kress [11]. Mautz's equation is fairly simple and so is worth investigating for computational work. In Section 8.2, we derive a new uniquely-solvable singular integral equation by adapting another method due to Kress [12]; this integral equation is attractive theoretically, although the kernel is rather complicated.

The paper can be viewed as the electromagnetic counterpart of [10], in which the acoustic transmission problem was studied. In [10], we found an abundance of Fredholm integral equations of the second kind; here, hypersingular integral equations are the norm, and it is this difference that makes existence results more difficult to establish.

2. Statement of the problem

Let $B_i$ denote a bounded three-dimensional domain with a smooth closed boundary, $S$, and simply-connected exterior, $B_e$. We consider the following problem:

**Transmission Problem.** Find electric fields $E_e$ and $E_i$, and magnetic fields $H_e$ and $H_i$, which satisfy Maxwell's equations

\[
\begin{align*}
\text{curl } E_e - i\mu_e \omega H_e &= 0, & P & \in B_e, \\
\text{curl } H_e + i\varepsilon_e \omega E_e &= 0, & P & \in B_e, \\
\text{curl } E_i - i\mu_i \omega H_i &= 0, & P & \in B_i, \\
\text{curl } H_i + i\varepsilon_i \omega E_i &= 0, & P & \in B_i,
\end{align*}
\]

and two transmission conditions on the interface,

\[
\mathbf{n} \times E = \mathbf{n} \times E_i \quad \text{and} \quad \mathbf{n} \times H = \mathbf{n} \times H_i, \quad p \in S,
\]

where the total fields in $B_e$ are given by

\[
\begin{align*}
E(P) &= E_e + E_{\text{inc}}, & H(P) &= H_e + H_{\text{inc}}, & P & \in B_e,
\end{align*}
\]

and $\{E_{\text{inc}}, H_{\text{inc}}\}$ is the given incident field. In addition, the scattered fields $\{E_s, H_s\}$ must satisfy a Silver-Müller radiation condition [2, §4.2],

\[
\sqrt{\mu_e} \mathbf{e}_f \times H_e + \sqrt{\varepsilon_e} E_e = \sigma(r_p^{-1}) \quad \text{as} \quad r_p \to \infty,
\]

uniformly for all directions $\mathbf{e}_f$. 

https://doi.org/10.1017/S0308210500021296 Published online by Cambridge University Press
We have suppressed a time dependence of $e^{-i\omega t}$ throughout. We assume that the electric permittivities $\varepsilon_e$ and $\varepsilon_\mu$, and the magnetic permeabilities $\mu_e$ and $\mu_\mu$ are given positive real constants.

We shall use the following notation: capital letters $P, Q$ denote points of $B_i \cup B_j$; lower-case letters $p, q$ denote points of $S$; and $n(q)$ denotes the unit normal at $q$ pointing into $B_i$. We choose the origin $O$ at some point in $B_i$; $r_p$ is the position vector of $P$ with respect to $O$, $r_p = |r_p|$ and $\hat{r}_p = r_p/r_p$.

It is known that the transmission problem has precisely one solution. These existence and uniqueness results are proved in [19, §§21 and 23].

We shall also need to consider two interior problems:

**INTERIOR MAXWELL PROBLEM.** Find a field $\{E, H\}$ which satisfies Maxwell's equations
\[
\text{curl } E - i\mu \omega H = 0 \quad \text{and} \quad \text{curl } H + i\varepsilon \omega E = 0, \quad P \in B_i
\]
and the boundary condition
\[
n \times E = 0, \quad p \in S.
\]
If this problem has a non-trivial solution, we say that $k^2 = \omega^2 \mu \varepsilon$ is an eigenvalue of the interior Maxwell problem. All such eigenvalues are known to be real [2, p. 125]. Physically, the interior Maxwell problem corresponds to a perfectly-conducting cavity resonator. It is a special case of the next problem.

**ASSOCIATED INTERIOR PROBLEM.** Find a field $\{E, H\}$ which satisfies Maxwell's equations (2.4) in $B_i$ and the boundary condition
\[
a(n \times E) + b \{n \times (n \times H)\} = 0, \quad p \in S.
\]
Here, $a$ and $b$ are constants. This problem is equivalent to the interior Maxwell problem if $a = 0$ or $b = 0$. Suppose $a \neq 0$ and set $\lambda = b/a$, whence the associated interior problem reduces to an impedance problem. From Maxwell's equations and the divergence theorem, we have
\[
\text{Re} \int_S (E \times H) \cdot n \, ds = 0,
\]
where the overbar denotes complex conjugation. Then, the boundary condition (2.5) implies that
\[
\text{Re} (\lambda) \int_S |n \times H|^2 \, ds = 0.
\]
It follows that the associated interior problem only has the trivial solution if Re $\lambda \neq 0$.

### 3. Potential theory

Introduce two free-space fundamental solutions, $G_\alpha$, defined by
\[
G_\alpha(P, Q) = \exp(i k_\alpha R)/(2\pi R),
\]
where $R = |r_p - r_Q|$ is the distance between $P$ and $Q$,
\[
k_\alpha = \omega \sqrt{\varepsilon_\alpha \mu_\alpha}.
\]
and \( \alpha = e \) or \( i \). Next, define a single-layer potential by

\[
(S_\alpha v)(P) = \int_S v(q) G_\alpha(P, q) \, ds_q, \quad P \in B_e \cup B_i.
\] (3.1)

In electromagnetic theory, we usually apply \( S_\alpha \) to a vector-valued function of position, \( \mathbf{a}(q) \), say; we define

\[
(C_\alpha \mathbf{a})(P) = \text{curl} \{ S_\alpha \mathbf{a} \} \quad \text{and} \quad (F_\alpha \mathbf{a})(P) = \text{curl} \{ C_\alpha \mathbf{a} \}.
\]

We are interested in the tangential components of these vector fields evaluated on \( S \) when \( \mathbf{a}(q) \) itself is a tangential density (so that \( \mathbf{a}(q) \cdot \mathbf{n}(q) = 0 \) for all \( q \in S \)). For continuous tangential densities, we have

\[
\mathbf{n} \times C_\alpha \mathbf{a} = \pm \mathbf{a} + M_\alpha \mathbf{a},
\]

where the upper (lower) sign corresponds to \( P \to p \in S \) from \( B_e \) (\( B_i \)) and \( M_\alpha \) is a boundary integral operator defined by

\[
(M_\alpha \mathbf{a})(p) = \mathbf{n}(p) \times \text{curl} \{ S_\alpha \mathbf{a} \}, \quad p \in S.
\]

For sufficiently smooth tangential densities \( \mathbf{a} \) (we shall be more precise later), we also have

\[
\mathbf{n} \times F_\alpha \mathbf{a} = P_\alpha \mathbf{a}
\]
on \( S \), where

\[
(P_\alpha \mathbf{a})(p) = \mathbf{n}(p) \times \text{curl} \text{curl} \{ S_\alpha \mathbf{a} \}, \quad p \in S.
\]

Note that \( M_\alpha \) and \( P_\alpha \) are related to the operators \( M_\alpha \) and \( N_\alpha \) in [2, §2.7] by

\[
M_\alpha \mathbf{a} = 2M_\alpha \mathbf{a} \quad \text{and} \quad N_\alpha \mathbf{a} = 2P_\alpha \{ \mathbf{n} \times \mathbf{a} \}.
\] (3.2)

We shall make extensive use of the Stratton-Chu representation. Applied in \( B_e \) to \( \{ \mathbf{E}_e, \mathbf{H}_e \} \), it gives

\[
C_e \{ \mathbf{n} \times \mathbf{E}_e \} + \frac{i}{\omega \varepsilon_e} F_e \{ \mathbf{n} \times \mathbf{H}_e \} = \begin{cases} 2\mathbf{E}_e(P), & P \in B_e, \\ 0, & P \in B_i. \end{cases}
\] (3.3)

and

\[
C_e \{ \mathbf{n} \times \mathbf{H}_e \} - \frac{i}{\omega \mu_e} F_e \{ \mathbf{n} \times \mathbf{E}_e \} = \begin{cases} 2\mathbf{H}_e(P), & P \in B_e, \\ 0, & P \in B_i. \end{cases}
\] (3.4)

An application in \( B_i \) to \( \{ \mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}} \} \) (with the exterior material) yields similar formulae which, when added to (3.3) and (3.4), give

\[
C_e \{ \mathbf{n} \times \mathbf{E} \} + \frac{i}{\omega \varepsilon_e} F_e \{ \mathbf{n} \times \mathbf{H} \} = \begin{cases} 2\mathbf{E}_e(P), & P \in B_e, \\ -2\mathbf{E}_{\text{inc}}(P), & P \in B_i. \end{cases}
\] (3.5)

and

\[
C_e \{ \mathbf{n} \times \mathbf{H} \} - \frac{i}{\omega \mu_e} F_e \{ \mathbf{n} \times \mathbf{E} \} = \begin{cases} 2\mathbf{H}_e(P), & P \in B_e, \\ -2\mathbf{H}_{\text{inc}}(P), & P \in B_i. \end{cases}
\] (3.6)
Finally, an application in $B_i$ to $\{E_i, H_i\}$ gives

$$C_i\{n \times E_i\} + \frac{i}{\omega E_i} F_i\{n \times H_i\} = \begin{cases} 0, & P \in B_e, \\ -2E_i(P), & P \in B_i. \end{cases}$$  (3.7)

and

$$C_i\{n \times H_i\} - \frac{i}{\omega H_i} F_i\{n \times E_i\} = \begin{cases} 0, & P \in B_e, \\ -2H_i(P), & P \in B_i. \end{cases}$$  (3.8)

Computing the tangential components of (3.5), (3.6), (3.7) and (3.8) on $S$, we obtain

$$(I - M_e)\{n \times E\} - \frac{i}{\omega E} P_e\{n \times H\} = 2n \times E_{inc},$$  (3.9)

$$(I - M_e)\{n \times H\} + \frac{i}{\omega H} P_e\{n \times E\} = 2n \times H_{inc},$$  (3.10)

$$(I + M_i)\{n \times E_i\} + \frac{i}{\omega E_i} P_i\{n \times H_i\} = 0,$$  (3.11)

$$(I + M_i)\{n \times H_i\} - \frac{i}{\omega H_i} P_i\{n \times E_i\} = 0.$$  (3.12)

4. Properties of boundary integral operators

In this section, we begin by defining appropriate function spaces. Properties of $M$ and $P$, considered as operators acting on these spaces, are given; usually, we omit the subscript $\alpha$ in this section. Next, we introduce the adjoints of $M$ and $P$. Finally, we obtain some identities satisfied by products of $M$ and $P$.

4.1. Function spaces

We seek classical solutions of the transmission problem, that is

$$\{E_e, H_e\} \in C^1(B_e) \cap C^{0,\beta}(B_e) \quad \text{and} \quad \{E_i, H_i\} \in C^1(B_i) \cap C^{0,\beta}(B_i),$$

where $C^{0,\beta}(\Omega)$ is the usual Banach space of Hölder-continuous functions on $\Omega$ (with $0 < \beta < 1$) and $\overline{\Omega}$ is the closure of $\Omega$. For functions defined on the interface $S$, we use the space $T^{0,\beta}(S)$, where

$$T^{0,\beta}(S) = \{a(q): a \in C^{0,\beta}(S) \text{ and } a \cdot n = 0\}$$

contains all Hölder-continuous tangential densities $a(q)$; the spaces $T^{m,\beta}(S)$ are defined similarly. Kress [12] has pointed out that the natural space to use is $T^{d,\beta}(S)$, where

$$T^{d,\beta}(S) = \{a(q): a \in T^{d,\beta}(S) \text{ and } \text{Div } a \in C^{0,\beta}(S)\}$$

contains all $a$ in $T^{d,\beta}(S)$ with a Hölder-continuous surface divergence, $\text{Div } a$. $T^{d,\beta}(S)$ is a normed space, with norm

$$\|a\|_{T^{d,\beta}(S)} = \max \{ \|a\|_{T^{d,\beta}(S)}, \|\text{Div } a\|_{C^{0,\beta}(S)} \}.$$
From [2], [12] and [9], we have the following properties:

\[ M : T^{0,\beta}(S) \rightarrow T^{0,\beta}(S) \quad \text{and} \quad M : T_{d}^{0,\beta}(S) \rightarrow T_{d}^{0,\beta}(S) \]

are compact;

\[ P : T^{1,\beta}(S) \rightarrow T^{0,\beta}(S) \quad \text{and} \quad P : T_{d}^{0,\beta}(S) \rightarrow T_{d}^{0,\beta}(S) \]

are bounded; and

\[ P_{e} - P_{i} : T^{0,\beta}(S) \rightarrow T^{1,\beta}(S) \quad (4.1) \]

is bounded, whence

\[ P_{e} - P_{i} : T^{0,\beta}(S) \rightarrow T^{0,\beta}(S) \]

is compact. Note that, according to (4.1), \((P_{e} - P_{i})a\) is smoother than \(a\). However, \(\text{Div}\ \{P_{e} - P_{i}\}a\) is not smoother than \(\text{Div}\ a\), and so \((P_{e} - P_{i})\) is not compact from \(T^{0,\beta}(S)\) into itself.

4.2. Adjoints

We define the adjoint of \(M\), namely \(M'\), so that

\[ \langle Ma, b \rangle = \langle a, M'b \rangle \]

for every \(a, b \in T^{0,\beta}(S)\), where \(\langle a, b \rangle = \int_{S} a \cdot b \, ds\). From [2, §2.7] and (3.2), we have

\[ M'a = n \times M\{n \times a\}. \]

From (3.2), we have

\[ N\{n \times a\} = 2P\{n \times (n \times a)\} = -2Pa, \quad a \in T^{0,\beta}(S). \]

Since \(N\) is known to be self adjoint, we have

\[ \langle Pa, b \rangle = -\langle n \times a, P\{n \times b\} \rangle \]

whence

\[ P'a = n \times P\{n \times a\}. \]

It follows that, for \(a \in T^{0,\beta}(S)\), we have

\[ n \times M'a = -M\{n \times a\} \quad \text{and} \quad n \times M_{c}'\{M_{c}a\} = M_{c}M_{i}\{n \times a\}; \quad (4.2) \]

the same relations also hold for \(P\).

4.3. Operator products

For Maxwell’s equations in \(B_{i}\), the Stratton–Chu representation gives (3.11) and (3.12), namely

\[ (I + M)\{n \times E\} + \frac{i}{\omega \varepsilon} P\{n \times H\} = 0, \quad (4.3) \]

and

\[ (I + M)\{n \times H\} - \frac{i}{\omega \mu} P\{n \times E\} = 0. \quad (4.4) \]
Let us represent the field \( \{E, H\} \) as

\[
E(P) = (C_m)(P), \quad H(P) = -\frac{i}{\omega \mu} (F_m)(P), \quad P \in B,
\]

where \( m \in T^0_\beta(S) \); \( \{E, H\} \) satisfies Maxwell's equations for any \( m \). On \( S \), we have

\[
n \times E = (-I + M)m \quad \text{and} \quad n \times H = -\frac{i}{\omega \mu} Pm.
\]

Substituting these boundary values into (4.3) and (4.4), we obtain the formulae

\[
P^2 = k^2(I - M^2)
\]

and

\[
MP + PM = 0,
\]

where \( k^2 = \omega^2 \mu \varepsilon \). The formula (4.5) is equivalent to [2, Eq. (4.56)]; it will be used later. The same formulae hold for the corresponding adjoint operators. Analogous formulae for the boundary integral operators of acoustics are well known [21, 9].

5. Pseudodifferential operators

In the sequel, we shall make use of the operator

\[
\mathcal{A}_\lambda = (I + M_1)(I + M_2) + (\lambda/k_1^2)P_lP_e,
\]

where \( \lambda \geq 0 \) is a real parameter. From (4.5), we have

\[
P_lP_e = P_l(P_e - P_l) + k_1^2(I - M_1^2),
\]

whence, using some results from Section 4.1, we see that

\[
\mathcal{A}_\lambda : T^{0,\beta}(S) \to T^{0,\beta}(S)
\]

is a bounded operator.

The continuity of \( \mathcal{A}_\lambda \) can also be shown by interpreting it as a pseudodifferential operator [22, 24]. Thus, at any \( p \in S \), introduce a local system of orthonormal coordinates \((x, y, z)\) so that \( \mathbf{u}(p) = (0, 0, 1) \). Then, the principal symbol of \( \mathcal{A}_\lambda \) for the smooth bounded surface \( S \), \( \sigma(\mathcal{A}_\lambda) \), is the same as when \( S \) is replaced by the tangent plane at \( p \); the latter is readily calculated by taking Fourier transforms in the \((x, y)\)-plane. Let

\[
\mathcal{F} w = \hat{w}(\xi_1, \xi_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x, y) \exp(i\xi \cdot \mathbf{x}) \, dx \, dy,
\]

where \( \mathbf{x} = (x, y) \) and \( \xi = (\xi_1, \xi_2) \). Also, for any \( \mathbf{u} = (u, v, 0) \), we have

\[
n \times \nabla \times \nabla \mathbf{u} = \left(-\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}, \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial z^2}, 0\right).
\]

Since

\[
\mathcal{F} \{e^{ik|\mathbf{x}|} / (2\pi |\mathbf{x}|)\} = (\xi^2 - k^2)^{-\frac{1}{2}},
\]

(5.3)
where $\xi^2 = |\xi|^2$, we find that

$$
\sigma(P) = \frac{1}{|\xi|} \begin{pmatrix} \xi_1 \xi_2 \\ -\xi_1 \xi_2 \end{pmatrix}, \quad (5.4)
$$

in agreement with [23, Lemma 2.7]. Similarly,

$$
\sigma(P_e - P) = \frac{k_e^2 - k_i^2}{2|\xi|^3} \begin{pmatrix} \xi_1 \xi_2 \xi_1 \xi_2 & \xi_1^2 - 2\xi_2^2 \\ -\xi_1^2 + 2\xi_2^2 & -\xi_1 \xi_2 \end{pmatrix},
$$

whence

$$
\sigma(P_a(P_e - P)) = \frac{k_e^2 - k_i^2}{|\xi|^3} \begin{pmatrix} \xi_1^2 & -\xi_1 \xi_2 \\ -\xi_1 \xi_2 & \xi_1^2 \end{pmatrix}.
$$

Then, from (5.2), we obtain

$$
\sigma(P_t P_e) = \frac{1}{|\xi|^2} \begin{pmatrix} k_i^2 \xi_1^2 + k_e^2 \xi_2^2 \\ (k_i^2 - k_e^2) \xi_1 \xi_2 \xi_1 \xi_2 & k_i^2 \xi_1^2 + k_e^2 \xi_2^2 \end{pmatrix}.
$$

This shows that $P_t P_e$ is a pseudodifferential operator of order zero. In general, if $A$ is a classical pseudodifferential operator, of integer order $m$, defined on a compact $C^\infty$-manifold $S$, we have [24, chap. 11]

$$
A : C^{k,\beta}(S) \rightarrow C^{k-m,\beta}(S), \quad 0 < \beta < 1,
$$

for every integer $k \geq m$.

Note that (5.4) shows that $P_a$ is a pseudodifferential operator of order 1, whence $P_t P_e$ is expected to be of order 2; since $\sigma(P_t) \sigma(P_e) = 0$, we deduce merely that the order of $P_t P_e$ is less than or equal to 1.

From (5.1), we have

$$
\sigma(A) = (1 + \lambda) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\lambda(k_e^2 - k_i^2)}{k_i^2 \xi_2^2} \begin{pmatrix} \xi_2^2 & -\xi_1 \xi_2 \\ -\xi_1 \xi_2 & \xi_1^2 \end{pmatrix},
$$

which is independent of $\omega$. Since

$$
det \{ \sigma(A) \} = (1 + \lambda)(1 + \lambda k_e^2/k_i^2)
$$

does not vanish (for $\lambda \geq 0$), we deduce that $A$ is an elliptic pseudodifferential operator of order zero. In fact, $A$ is a Fredholm operator with index zero (this means that the Fredholm alternative holds). We show this using a homotopy argument. Thus, consider the family of Fredholm operators

$$
A_t : T^{0,\beta}(S) \rightarrow T^{0,\beta}(S),
$$

parametrised by $t$, with $0 \leq t \leq 1$ and a fixed $\lambda > 0$. From [22, Proposition 8.1], we know that index $(A_0)$ is locally constant as $t$ is varied. Hence

$$
\text{index } (A_t) = \text{index } (A_0) = 0,
$$

since $A_t - A$ is compact.

Since $A$ is an elliptic operator of order zero, we have the following regularity results: any solution in $T^{0,\beta}(S)$ of the inhomogeneous equation $A \mathbf{a} = \mathbf{f}$ inherits additional smoothness from $\mathbf{f}$, so that $\mathbf{f} \in T^{m,\beta}(S)$ implies that $\mathbf{a} \in T^{m,\beta}(S)$, where
$m \geq 0$ and $0 < \beta < 1$; in particular, if $a$ solves the homogeneous equation $A_3 a = 0$, then

$$a \in \bigcap_{m \geq 0} T^m \beta(S) \subset C^\omega(S).$$

### 6. Pairs of coupled integral equations

The usual method of solving the transmission problem is to reduce it to a pair of coupled boundary integral equations. There are two standard approaches, namely the direct method and the indirect method.

#### 6.1. The direct method

If we use (2.1) in the Stratton–Chu representations (3.5)–(3.8), we obtain the representations

$$2E_e(P) = \frac{i}{\omega e_e} (F_e J)(P) - (C_e M)(P), \quad (6.1)$$

$$H_e(P) = -\frac{i}{\omega \mu_e} \text{curl } E_e(P), \quad P \in B_e,$$

$$-2E_i(P) = \frac{i}{\omega e_i} (F_i J)(P) - (C_i M)(P), \quad (6.2)$$

$$H_i(P) = -\frac{i}{\omega \mu_i} \text{curl } E_i(P), \quad P \in B_i,$$

where, as is customary, we have defined $J$ and $M$ by

$$J(p) = n \times H \quad \text{and} \quad M(p) = -n \times E, \quad p \in S.$$

Similarly, using (2.1) in (3.9)–(3.12), we obtain

$$(I - M_e)J - \frac{i}{\omega \mu_e} P_e M = 2n \times H_{inc}, \quad (6.3)$$

$$(I - M_e)M + \frac{i}{\omega e_e} P_e J = -2n \times E_{inc}, \quad (6.4)$$

$$(I + M_i)J + \frac{i}{\omega \mu_i} P_i M = 0, \quad (6.5)$$

$$(I + M_i)M - \frac{i}{\omega e_i} P_i J = 0. \quad (6.6)$$

These are four boundary integral equations for the two unknowns $J(g)$ and $M(g)$. To proceed, we must choose two equations or two linear combinations of equations. Let us consider the two combinations

$$\alpha_1(6.3) + \alpha_2(6.4) + \alpha_3(6.5) \quad \text{and} \quad \beta_1(6.3) + \beta_2(6.4) + \beta_3(6.6), \quad (6.7)$$

where the $\alpha$'s and $\beta$'s are constants to be chosen. Harrington [6] describes several possible choices; see Table 1. For all these choices, we always have existence: $J$ and $M$ are just the tangential components of $H$ and $E$, respectively, and we already know that the transmission problem always has precisely one solution. However, the question of uniqueness is less obvious.
TABLE 1
Direct method: choices of constants in (6.7).

<table>
<thead>
<tr>
<th>Formulation</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$-field</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$H$-field</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Combined field</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Mautz–Harrington</td>
<td>1</td>
<td>0</td>
<td>$-\beta$</td>
<td>0</td>
<td>1</td>
<td>$-\alpha$</td>
</tr>
<tr>
<td>Müller</td>
<td>$\mu_e$</td>
<td>$\mu_i$</td>
<td>0</td>
<td>$\varepsilon_e$</td>
<td>$\varepsilon_i$</td>
<td></td>
</tr>
</tbody>
</table>

Müller’s system of equations is uniquely solvable [19]; see also [7, §6.27]. The combined-field formulation (also known as the PMCHW formulation) also gives a uniquely-solvable system of equations [17]. It turns out that we obtain the same system if we view the representations (6.1) and (6.2) as given, and then impose the transmission conditions, that is, we use an indirect method with the Stratton–Chu representations; this approach has been used in [4].

The Mautz–Harrington system is uniquely solvable, provided the constants $\alpha$ and $\beta$ are such that $\alpha \beta$ is real and positive [17, 6]. This system includes the Müller and combined-field systems as special cases.

Let us now consider the $E$-field formulation, namely the pair (6.4) and (6.6). All the irregular frequencies of this system are identified in the next theorem.

**Theorem 6.1.** The $E$-field system of integral equations, (6.4) and (6.6), is uniquely solvable if, and only if, $k_e^2$ is not an eigenvalue of the interior Maxwell problem.

*Proof.* Suppose that $J_0$ and $M_0$ solve the homogeneous forms of (6.4) and (6.6), namely

\[
(I - M_e)M_0 + \frac{i}{\omega \varepsilon_e} P_e J_0 = 0, \tag{6.8}
\]

\[
(I + M_i)M_0 - \frac{i}{\omega \varepsilon_i} P_i J_0 = 0. \tag{6.9}
\]

Assume that $J_0$ and $M_0$ are not both identically zero. We show that this can only occur if $k_e^2$ is an eigenvalue of the interior Maxwell problem. Define fields $\{\mathbf{E}_0^e, \mathbf{H}_0^e\}$ in $B_\alpha$ ($\alpha = e, i$) by (6.1) and (6.2) with $J$ and $M$ replaced by $J_0$ and $M_0$, respectively. On $S$, we find that

\[
n \times \mathbf{E}_0^e = n \times \mathbf{E}_e = -M_0, \]

\[
2n \times \mathbf{H}_0^e = (I + M_e)J_0 + \frac{i}{\omega \mu_e} P_e M_0, \tag{6.10}
\]

\[
2n \times \mathbf{H}_0^i = (I - M_i)J_0 - \frac{i}{\omega \mu_i} P_i M_0.
\]
Now, construct the following fields:

\[ 2\mathbf{E}_e(P) = -\frac{i}{\omega \varepsilon_e} F J_0 + C_e M_0, \quad 2\mathbf{H}_e(P) = -C_e J_0 - \frac{i}{\omega \mu_e} F M_0, \quad P \in B_e, \]

\[ 2\mathbf{E}_i(P) = \frac{i}{\omega \varepsilon_i} F J_0 - C_e M_0, \quad 2\mathbf{H}_i(P) = C_e J_0 + \frac{i}{\omega \mu_e} F M_0, \quad P \in B_i. \]

By construction, \( \{\mathbf{E}_e, \mathbf{H}_e\} \) satisfies Maxwell's equations (for the interior material) in \( B_e \) and the Silver-Müller radiation conditions. Also,

\[ \mathbf{n} \times \mathbf{E}_e = 0, \quad p \in S, \]

by (6.9). The uniqueness theorem for the exterior Maxwell problem [2, Thm. 4.18] then implies that \( \{\mathbf{E}_e, \mathbf{H}_e\} \) vanishes identically in \( B_e \). In particular,

\[ 0 = \mathbf{n} \times \mathbf{H}_e = \mathbf{n} \times \mathbf{H}_e^0 - \mathbf{J}_0. \]

Similarly, \( \{\mathbf{E}_i, \mathbf{H}_i\} \) satisfies Maxwell's equations (for the exterior material) in \( B_i \) and

\[ \mathbf{n} \times \mathbf{E}_i = 0, \quad p \in S, \]

by (6.8). So, if \( k_e^2 \) is not an eigenvalue of the interior Maxwell problem, we deduce that \( \{\mathbf{E}_i, \mathbf{H}_i\} \) vanishes identically in \( B_i \), whence

\[ 0 = \mathbf{n} \times \mathbf{H}_i = \mathbf{n} \times \mathbf{H}_i^0 - \mathbf{J}_0. \]

Thus

\[ \mathbf{n} \times \mathbf{H}_e^0 = \mathbf{n} \times \mathbf{H}_i^0 = \mathbf{J}_0, \quad (6.11) \]

the fields \( \{\mathbf{E}_e^0, \mathbf{H}_e^0\} \) satisfy the homogeneous transmission problem and hence must vanish, whence (6.11) and (6.10) imply that \( \mathbf{J}_0 = \mathbf{M}_0 = 0 \), which is contrary to our assumptions.

We have just shown that non-uniqueness for the \( E \)-field formulation implies that \( k_e^2 \) is an eigenvalue of the interior Maxwell problem. We now prove the converse. At such a value of \( k_e^2 \), we know that there is a non-trivial tangential density \( a(\varphi) \) satisfying

\[ (I + M_e)a = 0 \quad \text{and} \quad P_e a = 0; \]

this follows by using the Stratton–Chu representation in \( B_i \), with \( a = \mathbf{n} \times \mathbf{H} \). Clearly, \( a \) also satisfies the homogeneous equation

\[ \mathcal{A}_e a = \left\{ (I + M_e)(I + M_e) + \frac{1}{\omega^2 \mu_e \varepsilon_e} P_e P_e \right\} a = 0, \quad (6.12) \]

where \( \tau = \varepsilon_i / \varepsilon_e \). From Section 5, we know that \( \mathcal{A}_e \) is a Fredholm operator with index zero. Hence, there exists a non-trivial solution, \( b \in T_d^0 H(S) \), of the corresponding adjoint homogeneous equation, namely

\[ \mathcal{A}_e b = \left\{ (I + M_e')(I + M_e') + \frac{1}{\omega^2 \mu_e \varepsilon_e} P_e' P_e' \right\} b = 0. \]
Using (4.2), this equation can be written as

$$\left\{ (I - M_e)(I - M_i) + \frac{1}{\omega^2 \mu_i \varepsilon_i} P_i p_i \right\} (n \times b) = 0. \quad (6.13)$$

Now, set

$$J_0 = -\frac{i}{\omega \mu_i} P_i c \quad \text{and} \quad M_0 = (I - M_i)c,$$

for some $c(q) \in T_{q, \varphi}(S)$. It follows from (4.5) that (6.9) is satisfied **identically**, for any such $c$. Moreover, (6.8) is also seen to be satisfied if we choose

$$c = n \times b,$$

by comparison with (6.13). Thus, we have found a non-trivial solution to the homogeneous $E$-field system, (6.8) and (6.9). This concludes the proof of Theorem 6.1.

A different proof of this theorem is given in [6]; the non-trivial solutions at irregular values of $k_2^2$ are also given in [1]. We can also give a similar argument to show that the fields $\{E_\alpha, H_\alpha\}$, given by (6.1) and (6.2), will also solve the transmission problem if $J$ and $M$ solve the $E$-field system and $k_2^2$ is not an eigenvalue; see [13] for analogous arguments for the acoustic transmission problem. The usual electromagnetic duality argument gives exactly the same result for the $H$-field formulation.

### 6.2. The indirect method

Suppose that we can write

$$E_\alpha(P) = \frac{i}{\omega \varepsilon_\alpha} (P a j_\alpha)(P) - (C_\alpha m_\alpha)(P),$$

$$H_\alpha(P) = -\frac{i}{\omega \mu_\alpha} \text{curl} E_\alpha(P), \quad P \in B_\alpha,$$

where $j_\alpha(q)$ and $m_\alpha(q)$ are tangential densities and $\alpha = e, i$. The fields $\{E_\alpha, H_\alpha\}$ and $\{E_i, H_i\}$ satisfy the appropriate form of Maxwell's equations in $B_e$ and $B_i$, respectively. The Silver–Müller radiation conditions are also satisfied. Imposing

<table>
<thead>
<tr>
<th>Table 2. Indirect method: constraints on the surface currents.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Formulation</strong></td>
</tr>
<tr>
<td>Electric current</td>
</tr>
<tr>
<td>Magnetic current</td>
</tr>
<tr>
<td>Combined current</td>
</tr>
<tr>
<td>Combined source</td>
</tr>
<tr>
<td>2nd-kind Fredholm</td>
</tr>
</tbody>
</table>

https://doi.org/10.1017/S0308210500021296 Published online by Cambridge University Press
Boundary integral equations

The transmission conditions gives

\[
(I + M_e)\mathbf{m}_e + (I - M_i)\mathbf{m}_i - \frac{i}{\omega} \left( \frac{1}{\varepsilon_e} \mathbf{P}_e \mathbf{r} - \frac{1}{\varepsilon_i} \mathbf{P}_i \mathbf{r} \right) = \mathbf{n} \times \mathbf{E}_{\text{inc}},
\]

\[
(I + M_e)\mathbf{j}_e + (I - M_i)\mathbf{j}_i + \frac{i}{\omega} \left( \frac{1}{\mu_e} \mathbf{P}_e \mathbf{m}_e - \frac{1}{\mu_i} \mathbf{P}_i \mathbf{m}_i \right) = -\mathbf{n} \times \mathbf{H}_{\text{inc}}.
\]

These are two boundary integral equations for the determination of four unknowns, namely \(\mathbf{j}_e, \mathbf{j}_i, \mathbf{m}_e, \) and \(\mathbf{m}_i;\) we need two constraints.

We know that \((P_e - P_i)\) is compact in \(T^{0,\beta}(S),\) so a good theoretical choice is

\[
\frac{1}{\varepsilon_e} \mathbf{j}_e = \frac{1}{\varepsilon_i} \mathbf{j}_i = \mathbf{j}(q) \quad \text{and} \quad \frac{1}{\mu_e} \mathbf{m}_e = \frac{1}{\mu_i} \mathbf{m}_i = \mathbf{m}(q),
\]

say. This leads to the Fredholm system of the second kind

\[
(\varepsilon_e + \varepsilon_i)\mathbf{j} + (\varepsilon_e M_e - \varepsilon_i M_i)\mathbf{j} + \frac{i}{\omega} (P_e - P_i)\mathbf{m} = -\mathbf{n} \times \mathbf{H}_{\text{inc}},
\]

\[
(\mu_e + \mu_i)\mathbf{m} + (\mu_e M_e - \mu_i M_i)\mathbf{m} - \frac{i}{\omega} (P_e - P_i)\mathbf{j} = \mathbf{n} \times \mathbf{E}_{\text{inc}},
\]

which is to be solved for \(\mathbf{j}, \mathbf{m} \in T_d^{0,\beta}(S);\) note that \(\mathbf{n} \times \mathbf{E}_{\text{inc}}, \mathbf{n} \times \mathbf{H}_{\text{inc}} \in T_d^{0,\beta}(S).\) If solvable, we can then construct the solution of the transmission problem, using

\[
\mathbf{E}_e = \frac{i}{\omega} \mathbf{F}_e \mathbf{j} - \mu_e C_e \mathbf{m} \quad \text{and} \quad \mathbf{H}_e = \varepsilon_e C_e \mathbf{j} + \frac{i}{\omega} \mathbf{F}_e \mathbf{m}, \quad P \in B_e, \quad \alpha = e, i.
\]

It turns out that if we set \(\mathbf{j} = -\mathbf{M}\) and \(\mathbf{m} = \mathbf{J},\) and interchange the materials \((\varepsilon \rightarrow i),\) we obtain M\u00f6ller's system of equations; hence the system of integral equations (6.15) is uniquely solvable.

Other choices for the surface currents \(\mathbf{j}_\alpha\) and \(\mathbf{m}_\alpha\) in the representations (6.14) are possible. Harrington [6] describes four; see Table 2, wherein \(\alpha_e\) and \(\alpha_i\) are constants. The first and second formulations both exhibit irregular frequencies, whereas the third and fourth do not [6]. We have existence and uniqueness for the fifth formulation. Similar results obtain for the combined-current formulation; if we set \(\mathbf{j} = 2\mathbf{J}\) and \(\mathbf{m} = 2\mathbf{M},\) we obtain exactly the same system of equations as with the combined-field formulation [6]. We are not aware of any existence results for the other three, although it may be possible to adapt the analysis of Kress [11] to the combined-source formulation.

7. Single integral equations

In this section, we use a hybrid of the indirect and direct methods, leading to single integral equations. Specifically, we use a representation in \(B_e\) involving a single unknown tangential density \(\mathbf{j},\) and the Stratton–Chu representation in \(B_i.\) Thus, assume that we can write

\[
\mathbf{E}_e(P) = a \frac{i}{\omega \varepsilon_e} \mathbf{F}_e \mathbf{j} - b C_e \{\mathbf{n} \times \mathbf{j}\}, \quad \mathbf{H}_e(P) = a C_e \mathbf{j} + b \frac{i}{\omega \mu_e} \mathbf{F}_e \{\mathbf{n} \times \mathbf{j}\}, \quad P \in B_e.
\]

(7.1)
where the constants $a$ and $b$ are at our disposal; the use of $\mathbf{n} \times \mathbf{j}$ rather than $\mathbf{j}$ for two of the densities is convenient, and facilitates comparison with [16]. The Stratton-Chu representations in $B_i$, (3.7) and (3.8), together with (2.1), give

$$-2\mathbf{E}_i(P) = \frac{i}{\omega \varepsilon_i} F_i\{\mathbf{n} \times \mathbf{H}\} + C_i\{\mathbf{n} \times \mathbf{E}\}, \quad \mathbf{H}_i(P) = -\frac{i}{\omega \mu_i} \text{curl} \mathbf{E}_i, \quad P \in B_i. \quad (7.2)$$

Computing the tangential components on $S$, (7.1) gives

$$\mathbf{n} \times \mathbf{E}_i = a \frac{i}{\omega \varepsilon_i} P_i \mathbf{j} - b(I + M_e)(\mathbf{n} \times \mathbf{j}) = K_e \mathbf{j}, \quad (7.3)$$

say, and

$$\mathbf{n} \times \mathbf{H}_i = a(I + M_e)\mathbf{j} + b \frac{i}{\omega \mu_i} P_i(\mathbf{n} \times \mathbf{j}) = L_e \mathbf{j}, \quad (7.4)$$

say. Similarly, (7.2) gives (6.5) and (6.6), namely

$$(I + M_e)(\mathbf{n} \times \mathbf{E}) + \frac{i}{\omega \varepsilon_i} P_i(\mathbf{n} \times \mathbf{H}) = 0, \quad (7.5)$$

$$(I + M_e)(\mathbf{n} \times \mathbf{H}) - \frac{i}{\omega \mu_i} P_i(\mathbf{n} \times \mathbf{E}) = 0. \quad (7.6)$$

If we substitute from (7.3) and (7.4) into (7.5), using (2.2), we obtain

$$\left\{(I + M_e)K_e + \frac{i}{\omega \varepsilon_i} P_i L_e\right\} \mathbf{j} = \mathbf{f} \quad (7.7)$$

where

$$\mathbf{f}(p) = -(I + M_e)(\mathbf{n} \times \mathbf{E}_{inc}) - \frac{i}{\omega \varepsilon_i} P_i(\mathbf{n} \times \mathbf{H}_{inc}). \quad (7.8)$$

Similarly, (7.6) gives

$$\left\{(I + M_e)L_e - \frac{i}{\omega \mu_i} P_i K_e\right\} \mathbf{j} = \mathbf{g} \quad (7.9)$$

where

$$\mathbf{g}(p) = -(I + M_e)(\mathbf{n} \times \mathbf{H}_{inc}) + \frac{i}{\omega \mu_i} P_i(\mathbf{n} \times \mathbf{E}_{inc}). \quad (7.10)$$

Equation (7.7) is a boundary integral equation for $\mathbf{j}(q)$. Equation (7.9) is another boundary integral equation for $\mathbf{j}(q)$. Having solved either, $\{\mathbf{E}_e, \mathbf{H}_e\}$ and $\{\mathbf{E}_i, \mathbf{H}_i\}$ are to be constructed from (7.1) and

$$-2\mathbf{E}_i(P) = \frac{i}{\omega \varepsilon_i} F_i\{\mathbf{n} \times \mathbf{H}_{inc} + L_e \mathbf{j}\} + C_i\{\mathbf{n} \times \mathbf{E}_{inc} + K_e \mathbf{j}\}, \quad (7.11)$$

$$\mathbf{H}_i(P) = -\frac{i}{\omega \mu_i} \text{curl} \mathbf{E}_i.$$
Boundary integral equations

respectively. Note that \(f\) and \(g\) are both in \(T^0_{\alpha}(S)\). We have the following two theorems concerning solvability of the transmission problem and uniqueness.

**Theorem 7.1.** If \(j(q) \in T^0_{\alpha}(S)\) solves (7.7) or (7.9), \(\{E_e, H_e\}\) and \(\{E_i, H_i\}\), given by (7.1) and (7.11), respectively, solve the transmission problem.

**Proof.** This is similar to the proof of Theorem 5.1 in [10]. We have to check that (2.1) are satisfied. We have

\[
2(n \times E_e + n \times E_{\text{inc}} - n \times E_i) = \left( (I + M_i)K_e + \frac{i}{\omega \varepsilon_i} P_i L_e \right) j - f
\]

and

\[
2(n \times H_e + n \times H_{\text{inc}} - n \times H_i) = \left( (I + M_i)L_e - \frac{i}{\omega \mu_i} P_i K_e \right) j - g.
\]

Thus, if \(j\) solves (7.7), then (2.1)_1 is satisfied, whereas if \(j\) solves (7.9), then (2.1)_2 is satisfied. Next, construct the radiating fields

\[
\hat{E}_e(P) = \frac{i}{\omega \varepsilon_i} F_i \{n \times H_{\text{inc}} + L_e j\} + C_i \{n \times E_{\text{inc}} + K_e j\}, \quad \hat{H}_e(P) = -\frac{i}{\omega \mu_i} \text{curl} \ \hat{E}_e,
\]

for \(P \in B_e\). On \(S\), we find that \(n \times \hat{E}_e = 0\) if \(j\) solves (7.7), or \(n \times \hat{H}_e = 0\) if \(j\) solves (7.9). In either case, the exterior uniqueness theorem implies that \(\{E_e, \hat{H}_e\}\) vanishes identically. Then, in the first case, \(n \times \hat{E}_e = 0\) implies that (2.1)_2 is satisfied, whereas in the second case, \(n \times \hat{H}_e = 0\) implies that (2.1)_1 is satisfied.

The next theorem is concerned with non-trivial solutions of the homogeneous forms of (7.7) and (7.9), namely

\[
\left\{ (I + M_i)K_e + \frac{i}{\omega \varepsilon_i} P_i L_e \right\} j_0 = 0 \quad (7.12)
\]

and

\[
\left\{ (I + M_i)L_e - \frac{i}{\omega \mu_i} P_i K_e \right\} j_0 = 0. \quad (7.13)
\]

We show that uniqueness depends on the eigenvalues of the associated interior problem (Section 2).

**Theorem 7.2.** The homogeneous equations (7.12) and (7.13) have a non-trivial solution if, and only if, \(k_r^2\) is an eigenvalue of the associated interior problem.

**Proof.** Suppose that \(j_0 \neq 0\) solves (7.12) or (7.13). Define fields \(\{E_0^\alpha, H_0^\alpha\}\) in \(B_\alpha\) \((\alpha = e, i)\) using (7.1) and (7.11), with \(j\) replaced by \(j_0\). \(n \times E_{\text{inc}} = 0\) and \(n \times H_{\text{inc}} = 0\). These fields solve the homogeneous transmission problem, and so vanish identically. Now, construct the following fields:

\[
\hat{E}_i(P) = a \frac{i}{\omega \varepsilon_i} F_e j_0 - b C_e \{n \times j_0\}, \quad \hat{H}_i(P) = a C_e j_0 + b \frac{i}{\omega \mu_e} F_e \{n \times j_0\}, \quad P \in B_i.
\]
On $S$, we find that
\[ n \times \vec{E}_i = n \times E_i^0 + 2b(n \times j_0) = 2b(n \times j_0), \quad n \times \vec{H}_i = n \times H_i^0 - 2aj_0 = -2aj_0; \]
hence,
\[ a(n \times \vec{E}_i) + b(n \times (n \times \vec{H}_i)) = 0, \quad p \in S. \tag{7.14} \]
Thus, either $\{\vec{E}_i, \vec{H}_i\}$ is an eigenfunction of the associated interior problem, or it vanishes identically; we can eliminate the second possibility since it implies that $j_0 = 0$, contrary to hypothesis.

Conversely, suppose that $k_x^2$ is an eigenvalue of the associated interior problem. Then the Stratton–Chu representations give
\[ (I + M_e)(n \times \vec{E}_i) + \frac{i}{\omega \mu_e} P_e(n \times \vec{H}_i) = 0, \]
\[ (I + M_e)(n \times \vec{H}_i) - \frac{i}{\omega \mu_e} P_e(n \times \vec{E}_i) = 0. \]
Using the boundary condition (7.14), these give $K_e\{n \times \vec{H}_i\} = 0$ and $L_e\{n \times \vec{E}_i\} = 0$, respectively. Hence, $n \times \vec{H}_i$ is a non-trivial solution of both (7.12) and (7.13), if $a \neq 0$; if $a = 0$, $n \times \vec{E}_i$ is a non-trivial solution.

Let us now examine the structure of (7.7) and (7.9) in more detail, with the intention of establishing existence. Set
\[ \rho = \mu_e/\mu_e \quad \text{and} \quad \tau = \epsilon_e/\epsilon_e. \]
Then (7.7) and (7.9) can be written as
\[ a\frac{i}{\omega \mu_e} \mathcal{B}_i j - b \mathcal{A}_\rho (n \times j) = f \tag{7.15} \]
and
\[ a \mathcal{A}_\rho j + b \frac{i}{\omega \mu_i} \mathcal{B}_\rho (n \times j) = g, \tag{7.16} \]
where $\mathcal{A}_\rho$ is defined by (5.1) and
\[ \mathcal{B}_\rho = P_e(I + M_e) + \lambda(I + M_i)P_e. \tag{7.17} \]
$\mathcal{B}_\rho$ is a pseudodifferential operator of order +1; the determinant of its principal symbol is identically zero.

Consider (7.15). If we want only Fredholm operators, we must take $a = 0$; without loss of generality, we can set $b = -1$. This gives
\[ \mathcal{A}_\rho (n \times j) = f \tag{7.18} \]
while (7.16) becomes
\[ \mathcal{B}_\rho (n \times j) = i\omega \mu_i g. \tag{7.19} \]
Having solved either,
\[ E_e(P) = C_e\{n \times j\}, \quad H_e(P) = -\frac{i}{\omega \mu_e} F_e\{n \times j\}, \quad P \in B_e, \tag{7.20} \]
and \( \{E_i, H_i\} \) is given by (7.11), wherein \( K_i j = (I + M_e)(n \times j) \) and \( L_i j = -i\frac{1}{\omega \mu_e} P_e (n \times j) \).

Similar considerations for (7.16) lead to \( a = 1 \) and \( b = 0 \). This gives

\[ \mathcal{A}_i j = g \]

(7.21)

and

\[ \mathcal{B}_i j = -i\omega e_1 \]

(7.22)

having solved either,

\[ E_e(P) = \frac{i}{\omega e_1} F_e j, \quad H_e(P) = C_d j, \quad P \in B_e, \]

(7.23)

and \( \{E_i, H_i\} \) is given by (7.11), where now \( K_e = i \frac{1}{\omega e_1} P_e \) and \( L_e = (I + M_e) \).

Equation (7.22) is a hypersingular integral equation. It was derived previously by Marx [14, 15], using a different method. Glisson [5] rederived Marx's equation and noted that irregular frequencies would occur when \( k_e^2 \) was an eigenvalue of the interior Maxwell problem; this is consistent with Theorem 7.2. He also suggested using the representation (7.20), but did not derive any associated integral equations. The two singular integral equations (7.18) and (7.21) are preferable theoretically and are new. We have the following results.

**Theorem 7.3.** Assume that \( k_e^2 \) is not an eigenvalue of the interior Maxwell problem. Then (7.18) is uniquely solvable in \( T_0^{0,5}(S) \) for any \( f \in T_0^{0,5}(S) \) and (7.21) is uniquely solvable in \( T_0^{0,5}(S) \) for any \( g \in T_0^{0,5}(S) \). Moreover, if \( f \) is given by (7.8) (or \( g \) by (7.10)), the solution will be in \( T_0^{0,5}(S) \).

**Proof.** The first part follows from the Fredholm alternative, which gives existence from uniqueness; we have the latter from Theorem 7.2. For the second part, consider (7.18). Let \( \{E_e, H_e\}, \{E_v, H_v\} \) solve the transmission problem. We know that such a solution exists (Section 6), and that

\[ n \times E_e \in T_0^{0,5}(S), \]

using Maxwell's equations and the formula

\[ \text{Div} (n \times E_e) = -n \cdot \text{curl} E_e. \]

Next, consider the Fredholm integral equation of the second kind

\[ (I + M_e)(n \times j) = n \times E_e. \]

Since \( k_e^2 \) is not an eigenvalue of the interior Maxwell problem, this equation has a unique solution \( j \in T_0^{0,5}(S) \) [12]. It remains to show that \( j \) solves (7.18). To this end, construct the fields \( \{E_e, H_e\}, \{E_v, H_v\} \), using (7.20) and (7.11). We find that \( \{E_e, H_e\} \) solves the exterior Maxwell problem, with

\[ n \times E_e = n \times E_v \text{ on } S, \]

whence \( \{E_e, H_e\} = \{E_v, H_v\} \), by uniqueness for the exterior problem. The Stratton-Chu formulae, (3.7) and (3.8), then show that \( \{E_i, H_i\} = \{E_v, H_v\} \).
Finally, the interface condition
\[ 0 = 2(\mathbf{E}_e + \mathbf{E}_{inc} - \mathbf{E}_i) = 2(\mathbf{E}_e + \mathbf{E}_{inc} - \mathbf{E}_i) = \sigma_{\rho}(n \times j) - f \]
gives the desired result. A similar argument succeeds for (7.21).

**Theorem 7.4.** Assume that \( k^2 \) is not an eigenvalue of the interior Maxwell problem. Then Marx's equation, (7.22), and (7.19) are both uniquely solvable.

**Proof.** We cannot prove existence for every \( f \) and \( g \) in \( T^0_{\beta}(S) \), but only for those \( f \) and \( g \) defined by (7.8) and (7.10), respectively. Thus, given an incident field \( \{E_{inc}, H_{inc}\} \), Theorem 7.3 says that we can find \( j \) uniquely by solving (7.21). We then construct the fields \( \{E_\alpha, H_\alpha\} \), using (7.23) and (7.11). By Theorem 7.1, these fields will solve the transmission problem. In particular, the transmission condition (2.1) implies that \( j \) satisfies (7.22). Uniqueness follows from Theorem 7.2. A similar argument succeeds for (7.19).

We would like to obtain a single integral equation that does not suffer from irregular frequencies. By Theorem 7.2 and the remarks at the end of Section 2, we see that we can secure uniqueness by making different choices for \( a \) and \( b \). However, if \( a \) and \( b \) are both non-zero, we always obtain integral equations involving non-Fredholm operators, and so the question of existence remains. We discuss this in the next section.

### 8. Single integral equations without irregular frequencies

In this section, we describe two methods of obtaining single integral equations that are uniquely solvable at all frequencies.

#### 8.1. Mautz's equation

Mautz [16] has suggested choosing \( a = 1 \) and \( b = \alpha \), where \( \alpha \) is a non-zero constant. This leads to the representations

\[ E_e(P) = \frac{i}{\omega E_e} F_e j - \alpha C_e (n \times j), \quad H_e(P) = C_e j + \alpha \frac{i}{\omega \mu_e} F_e \{n \times j\}, \quad P \in B_e, \quad (8.1) \]

and to two integral equations,

\[ L_1 j = \frac{i}{\omega E_1} \mathcal{B}_1 j - \alpha \mathcal{A}_\rho (n \times j) = f \quad (8.2) \]

and

\[ L_2 j = \mathcal{A}_\rho j + \alpha \frac{i}{\omega \mu_1} \mathcal{B}_\rho (n \times j) = g. \quad (8.3) \]

Mautz derived (8.2) and noted that his equation has at most one solution if \( \text{Re} \ \alpha > 0 \). In fact, we have uniqueness for both (8.2) and (8.3) if \( \text{Re} \ \alpha \neq 0 \).

In order to prove existence at all frequencies, we modify an ingenious argument used by Kress [11], [2, §4.6] to regularise a related singular integral equation for the exterior Maxwell problem.
We start by choosing a frequency \( \omega^* \) so that \( k^*_e = \omega^* \sqrt{\mu_e \varepsilon_e} \) is such that

\[
(k^*_e)^2 \text{ is not an eigenvalue of the interior Maxwell problem} \quad (8.4)
\]

and

\[
(k^*_e)^2 \text{ is not an eigenvalue of the interior Dirichlet problem} \quad (8.5)
\]

(so that the only solution of \( (\nabla^2 + (k^*_e)^2)u = 0 \) in \( B \), with \( u = 0 \) on \( S \) is \( u = 0 \)). We use an asterisk to denote any quantity evaluated at \( \omega^* \). The aim is to show that (8.2) and (8.3) have a solution at \( \omega^* \) by actually constructing it.

By Theorem 7.1, Theorem 7.3 and (8.4), we know that we can solve the transmission problem at \( \omega^* \) by solving (7.18). Thus, we have

\[
m^* = (\mathcal{M}_\mu^*)^{-1} f^* ,
\]

where the exterior field is given by

\[
E_e^*(P) = C_e^* m^*, \quad H_e^*(P) = - \frac{i}{\omega^* \mu_e} F_e^* m^*, \quad P \in B_e ,
\]

and the interior field is given by (7.11). Now, construct the potential

\[
F(P) = (S_e^* m^*)(P), \quad P \in B_e .
\]

\( F \) satisfies the vector Helmholtz equation in \( B_e \) and, by [2, Corollary 4.14], the radiation condition

\[
\hat{r}_P \times \text{curl} F - \hat{r}_P \text{div} F + ik_e^* F = o(r_P^{-1}) \quad \text{as} \ r_P \to \infty .
\]

On \( S \), we can compute

\[
n \times F = c(q), \quad \text{say}, \quad \text{and} \quad \text{div} F + \frac{i \omega^* \mu_e}{\alpha_e^*} n \cdot F = \gamma(q), \quad \text{say},
\]

where we have written \( \alpha^* = \alpha(\omega^*) \), since \( \alpha \) can depend on \( \omega \). From the proof of Theorem 4.42 in [2], we know that \( F \) has an alternative representation, namely

\[
F(P) = C_e^* b + \frac{i \omega^* \mu_e}{\alpha_e^*} (S_e^*(n \times b) + \text{grad} (S_e^* \lambda)), \quad P \in B_e,
\]

(it is here that we use (8.5)), where the densities \( b \in T_d^{0,\beta}(S) \) and \( \lambda \in C^{0,\beta}(S) \) are given by

\[
\begin{pmatrix} b \\ \lambda \end{pmatrix} = \mathcal{M}^* \begin{pmatrix} c \\ \gamma \end{pmatrix} ,
\]

for a certain matrix of bounded operators, \( \mathcal{M}^* \) (see [2, Eq. (4.71)]). If we set

\[
j_e^* = \frac{i \omega^* \mu_e}{\alpha_e^*} n \times b ,
\]

we see that

\[
\text{curl} F = C_e^* j_e^* + \alpha^* \frac{i}{\omega^* \mu_e} F_e^*(n \times j_e^*).
\]
So, comparing with (8.1), we take

\[ \mathbf{H}^e(P) = \text{curl} \mathbf{F}, \quad \mathbf{E}^e(P) = \frac{i}{\omega \sigma_e} \text{curl} \mathbf{H}^e, \quad P \in B_e, \]

giving a representation for the exterior field in the required form. Retracing our steps shows that we can construct \( \mathbf{j}^*(q) \in T^0_{d, \beta}(S) \) from \( \mathbf{f}^*(q) \in T^0_{d, \beta}(S) \) (subject to (8.4) and (8.5)) in the form

\[ \mathbf{j}^* = \mathcal{C}^* \mathbf{f}^*, \]

where \( \mathcal{C}^* \) is a known bounded operator. But, from the derivation of (8.2), we know that \( \mathbf{j}^* \) solves

\[ \mathcal{L}_1 \mathbf{j}^* = \mathbf{f}^*. \]

We also know from Theorem 7.2 that there can be at most one \( \mathbf{j}^* \) that solves this equation, and so it follows that \( \mathcal{C}^* = (\mathcal{L}_1^*)^{-1} \).

Next, consider an arbitrary frequency \( \omega \). We have

\[ \mathcal{L}_1 \mathbf{j} = \beta \mathcal{L}_1^* \mathbf{j} + (\mathcal{L}_1 - \beta \mathcal{L}_1^*) \mathbf{j} = \mathbf{f}, \]

where \( \beta \) is a constant, whence

\[ \beta \mathbf{j} + \mathcal{C}^*(\mathcal{L}_1 - \beta \mathcal{L}_1^*) \mathbf{j} = \mathcal{C}^* \mathbf{f}. \quad (8.6) \]

This is a Fredholm integral equation of the second kind for \( \mathbf{j}(q) \) if \( \beta \neq 0 \) and \( \mathcal{L}_1 - \beta \mathcal{L}_1^* \) is compact. Now,

\[ (\mathcal{L}_1 - \beta \mathcal{L}_1^*) \mathbf{j} = (\beta \alpha^* \mathcal{A}_p^* - \alpha \mathcal{A}_n)(\mathbf{n} \times \mathbf{j}) + \frac{i}{\varepsilon_{\varepsilon}} \left( \frac{1}{\omega} \mathcal{B}_r - \frac{\beta}{\omega} \mathcal{B}_r^* \right) \mathbf{j}. \]

This will be compact from \( T^0_{d, \beta}(S) \) into itself if

\[ \beta \alpha^* = \alpha \quad \text{and} \quad \omega^* = \beta \omega; \quad (8.7) \]

thus, we require that Mautz's 'constant' \( \alpha \) be given by

\[ \alpha(\omega) = \eta/\omega, \]

where \( \eta \) is a frequency-independent constant. Since we already have uniqueness for (8.2), the Riesz theory gives the existence of \( \mathbf{j} \in T^0_{d, \beta}(S) \). It remains to show that \( \mathbf{j} \) is actually in \( T^0_{d, \beta}(S) \). Consider (8.6). We know that its right-hand side \( \mathcal{C}^* \mathbf{f} \in T^0_{d, \beta}(S) \), since \( \mathbf{f} \in T^0_{d, \beta}(S) \). Also, the choices (8.7) ensure that \( \mathcal{L}_1 - \beta \mathcal{L}_1^* \) is a pseudodifferential operator of order \(-1\); thus, it maps \( T^0_{d, \beta}(S) \rightarrow T^1_{d, \beta}(S) \subset T^0_{d, \beta}(S) \). Hence, \( \mathbf{j} \in T^0_{d, \beta}(S) \), as required, and so we obtain the following result.

**Theorem 8.1.** Mautz's single integral equation (8.2), in which \( \alpha = \eta/\omega \) and \( \eta \) is a frequency-independent constant with \( \text{Re } \eta \neq 0 \), is uniquely solvable in \( T^0_{d, \beta}(S) \) at all frequencies.

A similar argument works for the other single integral equation, (8.3); the starting point is the Fredholm equation (7.21).

**Theorem 8.2.** The single integral equation (8.3), in which \( \alpha = \eta \omega \) and \( \eta \) is a frequency-independent constant with \( \text{Re } \eta \neq 0 \), is uniquely solvable in \( T^0_{d, \beta}(S) \) at all frequencies.
8.2. A singular integral equation without irregular frequencies

Consider the hypersingular integral equation (7.16). If $\mathcal{B}_\nu$ was replaced by $\mathcal{B}_\nu V$, where $V$ is chosen so that $\mathcal{B}_\nu V$ is compact, (7.16) would be a Fredholm equation. This idea is the basis of the paper by Kress [12] on the exterior Maxwell problem. Thus, assume that we can write (cf. (7.1))

$$E_e(P) = \frac{i}{\omega e} F_e j - b C_e V j, \quad H_e(P) = C_n j + b \frac{i}{\omega \mu_e} F_e V j, \quad P \in B_e, \tag{8.8}$$

where the constant $b$ and the operator $V$ will be specified later. Proceeding as in Section 7, we use the Stratton-Chu representations in $B_i$ and then obtain (cf. (7.16))

$$\left\{ \mathcal{A}_e + b \frac{i}{\omega \mu_e} \mathcal{B}_\nu V \right\} j = g, \tag{8.9}$$

where $g$ is defined by (7.10). Having solved (8.9), $\{E_e, H_e\}$ and $\{E_i, H_i\}$ are to be constructed from (8.8) and (7.11), respectively, with

$$K_e = \frac{i}{\omega e} P_e - b (I + M_e) V \quad \text{and} \quad L_e = I + M_e + b \frac{i}{\omega \mu_e} P_e V \tag{8.10}$$

in (7.11). We have the following result.

**Theorem 8.3.** If $j(q) \in T^0_B(S)$ solves (8.9), $\{E_e, H_e\}$ and $\{E_i, H_i\}$, given by (8.8) and (7.11) (with (8.10)), respectively, solve the transmission problem.

**Proof.** Straightforward adaptation of proof of Theorem 7.1.

The next step is to examine uniqueness. It depends on the eigenvalues of the following problem.

**Interior V-Problem.** Find a field $\{E, H\}$ which satisfies Maxwell's equations (2.4) in $B_i$ and the boundary condition

$$n \times E + b V (n \times H) = 0, \quad p \in S. \tag{8.11}$$

**Theorem 8.4.** The homogeneous form of (8.9) has a non-trivial solution if, and only if $k_0^2$ is an eigenvalue of the interior V-problem.

**Proof.** Straightforward adaptation of the proof of Theorem 7.2.

We eliminate eigenvalues of the interior V-problem by making appropriate choices for $b$ and $V$. Thus, suppose that $\{E, H\}$ solves the interior V-problem. As at the end of Section 2, we deduce that

$$\text{Re} \left\{ b \int_S \tilde{H} \cdot \{V(n \times H)\} \ ds \right\} = 0,$$

where we have used (8.11). If, following Kress [12], we set

$$V a = n \times \{\tilde{S}_0 S_0 a\}, \tag{8.12}$$

where $k_0$ will be specified and

$$(\tilde{S}_0 v)(p) = \int_S v(q) \tilde{G}_0(p, q) \ ds_q,$$
we find that
\[ \text{Re} (b) \int_S |S_0(n \times H)|^2 \, ds = 0. \]
So, if \( \text{Re} b \neq 0 \), we deduce that
\[ S_0(n \times H) = 0, \quad \rho \in S. \]
But, if we choose \( k_0 \) so that \( k_0^2 \) is not an eigenvalue of the interior Dirichlet problem, \( S_0 \) is invertible. Hence, \( n \times H = 0 \) on \( S \), (8.11) gives \( n \times E = 0 \) and so \( \{ E, H \} \) vanishes identically in \( B_r \).
Now, from (5.3) and (8.12), we have
\[ \sigma(V) = \frac{1}{\xi^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]
whence (5.4) and (7.17) give
\[ \sigma(\mathcal{B}_\rho V) = \frac{1 + \rho}{|\xi|} \begin{pmatrix} \xi_2^2 & -\xi_1 \xi_2 \\ -\xi_1 \xi_2 & \xi_1^2 \end{pmatrix}. \]
Hence, \( \mathcal{B}_\rho V \) is compact from \( T^{a,\beta}(S) \) into itself. The Fredholm alternative then guarantees the unique solvability of (8.9) in \( T^{a,\beta}(S) \). It remains to show that \( j \in T^{a,\beta}_0(S) \). We proceed as in the proof of Theorem 7.3. Thus, let \( \{ \tilde{E}_a, \tilde{H}_a \} \) solve the transmission problem, whence \( n \times \tilde{H}_a \in T^{a,\beta}_0(S) \). Then
\[ \left\{ I + M + b \frac{i}{\omega \mu_e} \mathcal{P}_V \right\} j = n \times \tilde{H}_a \]
is uniquely solvable for \( j \in T^{a,\beta}(S) \) [12, Eqn. (10')]]. The result follows as before, using uniqueness for the exterior problem, and so we obtain the following.

**Theorem 8.5.** _The single integral equation (8.9), wherein \( V \) is given by (8.12), is a Fredholm equation. It is uniquely solvable in \( T^{a,\beta}(S) \) for any \( g \in T^{a,\beta}(S) \) if the two constants \( b \) and \( k_0 \) are chosen so that \( \text{Re} b \neq 0 \) and \( k_0 \) is not an eigenvalue of the interior Dirichlet problem. Moreover, if \( g \) is given by (7.10), the solution will be in \( T^{a,\beta}_0(S) \)._

9. **Discussion**

We have seen that there are various methods for reducing the electromagnetic transmission problem for a homogeneous dielectric obstacle to boundary integral equations: one can use a pair of coupled integral equations for a pair of unknowns (Sections 5 and 6) or a single integral equation for a single unknown (Sections 7 and 8). Clearly, we have not exhausted all the possibilities. Thus, we derived single integral equations by using an assumed representation (or _ansatz_ ) in \( B_e \) and the Stratton–Chu representation in \( B_r \). We could reverse this procedure, leading to different integral equations; a systematic investigation of this approach for the acoustic transmission problem is given in [10, §6]. We could also derive single integral equations without irregular frequencies by modifying the Green's function, as done in [20] and [8] for the exterior Maxwell problem.
Appendix. Notational comparisons

For a time dependence of $e^{-i\omega t}$, Harrington's notation [6] for the basic potentials is as follows:

$$
\vec{E}(J, 0) = i\omega \vec{A}(J) - \text{grad } \phi(J),
$$
$$
\vec{H}(J, 0) = \frac{1}{\mu} \text{curl } \vec{A}(J),
$$
$$
\vec{E}(0, M) = -\frac{1}{\varepsilon} \vec{F}(M),
$$
$$
\vec{H}(0, M) = i\omega \vec{F}(M) - \text{grad } \psi(M),
$$

where

$$
\vec{A}(J) = \mu S\vec{J}, \quad \phi(J) = \frac{1}{\varepsilon} S\vec{q}, \quad q = \frac{1}{i\omega} \text{Div } \vec{J},
$$
$$
\vec{F}(M) = \varepsilon S\vec{M}, \quad \psi(M) = \frac{1}{\mu} S\vec{m}, \quad m = \frac{1}{i\omega} \text{Div } \vec{M},
$$

and $S$ is defined by (3.1). By [2, Theorem 2.29] we have

$$
\phi(J) = -\frac{i}{\omega\varepsilon} \text{div } \{SJ\} \quad \text{and} \quad \psi(M) = -\frac{i}{\omega\mu} \text{div } \{SM\}.
$$

Then, we can simplify $\vec{E}(J, 0)$ and $\vec{H}(0, M)$ using $(\nabla^2 + k^2)G = 0$ and the vector identity $\text{grad } \text{div } = \text{curl } \text{curl } + \nabla^2$. The results are

$$
\vec{E}(J, 0) = \frac{i}{\omega\varepsilon} \text{curl } \{\vec{H}(J, 0)\}, \quad \vec{H}(J, 0) = \text{curl } \{SJ\},
$$
$$
\vec{E}(0, M) = -\text{curl } \{SM\}, \quad \vec{H}(0, M) = -\frac{i}{\omega\mu} \text{curl } \{\vec{E}(0, M)\}.
$$

Note that $\vec{E}(0, M) = -\vec{H}(M, 0)$ and $\vec{H}(0, M) = \frac{\varepsilon}{\mu} \vec{E}(M, 0)$.

Acknowledgments

The work of the first author was motivated by some discussions with Peter Kirby and Guy Morgan of AEA Technology, Culham. The second author acknowledges financial support from the Academy of Finland.

References


https://doi.org/10.1017/S0308210500021296 Published online by Cambridge University Press


