Acoustic scattering by inhomogeneous spheres

P. A. Martin

Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, Colorado 80401-1887

(Received 6 March 2001; revised 19 December 2001; accepted 20 February 2002)

Acoustic scattering problems are considered when the material parameters (density \(\rho\) and speed of sound \(c\)) are spherically symmetric functions of position. Explicit separated solutions are derived (i) when \(\rho(r) = \exp(\beta r)\) and \(c^{-2}\) is a linear function of \(r^{-1}\), and (ii) when \(\rho(r) = \exp(-\beta r^2)\) and \(c^{-2}\) is a linear function of \(r^2\). In both cases, the radial parts of the solutions are given in terms of Coulomb wave functions or Whittaker functions; these are well-studied special functions, closely related to confluent hypergeometric functions. Two problems are discussed in detail: scattering by an inhomogeneous sphere embedded in a homogeneous fluid, and scattering by a homogeneous sphere with a concentric inhomogeneous coating. © 2002 Acoustical Society of America.

PACS numbers: 43.20.Fn [ANN]

I. INTRODUCTION

Sound propagation in inhomogeneous media continues to be of interest. Much is known about layered, stratified media, where the properties of the media (density \(\rho\) and speed of sound \(c\)) depend on one coordinate, \(z\), say, where \(x\), \(y\), and \(z\) are Cartesian coordinates (Brekhovskikh, 1960). Such situations have obvious application to underwater acoustics.

For time-harmonic motions of frequency \(\omega\) the governing equation is

\[
\rho \text{ div}(\rho^{-1} \text{ grad } p) + k^2 p = 0, \tag{1}
\]

where \(k = \omega/c\) and \(p\) is the acoustic pressure. If the density is constant, Eq. (1) reduces to

\[
\nabla^2 p + \kappa^2 n(r)p = 0, \tag{2}
\]

where \(\nabla^2 = \text{div grad}\), \(n(r) = [c_0/c(r)]^2\) is the (square of the) refractive index at position \(r\), \(\kappa = \omega/c_0\), and \(c_0\) is a constant sound speed. We assume that \(n(r) \rightarrow 1\) as \(r \rightarrow \infty\) in all directions; this excludes layered media, of course.

There is a considerable literature on Eq. (2), especially in the context of quantum mechanics; some of this will be mentioned below. In addition, several point-source solutions (Green’s functions) are known for various functional forms of \(n(r)\); see Li et al. (1990) for a review.

For scattering problems in acoustics, there are essentially three cases, depending on properties of \(n\). First, suppose that \([1 - n(r)]\) has compact support, so that \(n(r) = 1\) for \(r = |r| > a\), say. Suppose further that \(n(r)\) is smooth for all \(r\) in three-dimensional space. Then, one can reduce the scattering problem to an integral equation over \(D\), the finite region in which \(n(r) \neq 1\). One such is the Lippmann–Schwinger equation (see, for example, Ahner, 1977; Colton and Kress, 1992, Sec. 8.2; and Newton, 1982, Sec. 10.3). Asymptotic approximations are available for \(k_0 a \ll 1\) (Ahner, 1977; Kriegsmann et al., 1983); Kriegsmann et al. (1983) also discuss Eq. (1) briefly.

Second, we could have situations in which \([1 - n(r)]\) does not have compact support, but is such that \(n(r) \rightarrow 1\) as \(r \rightarrow \infty\). The corresponding scattering problems are uncommon in acoustics.

Third, we could have \(n(r) = 1\) outside \(D\) with \(n\) discontinuous across \(\partial D\), the boundary of \(D\). The corresponding scattering problem will require transmission conditions across the interface \(\partial D\). If the material in \(D\) is actually homogeneous, so that \(n(r) = n_1\), a constant, for all \(r \in D\), boundary integral equations over \(\partial D\) can be used; see Kleinman and Martin (1988) for a review.

In this paper, we are mainly concerned with this third class of problem: acoustic scattering by a bounded inhomogeneity embedded in an unbounded homogeneous medium. We begin (Sec. II) with a derivation of the partial differential equation (1): this equation governs the acoustic pressure in an otherwise stationary but inhomogeneous compressible fluid. We give this derivation because some textbook discussions are flawed. We then suppose that the inhomogeneity is spherically symmetric, so that \(\rho\) and \(c\) are assumed to be given functions of the spherical polar coordinate \(r\) (only). Such problems have been studied by several authors. For problems of the first type [smooth \(n\) with \(n(r) = 1\) for \(r > a\)], Ahner (1977) has given low-frequency expansions. For the second type [smooth \(n\), with \(n(r) \rightarrow 1\) rapidly as \(r \rightarrow \infty\)], Colton (1978) has used so-called “transformation operators,” which map solutions of the Helmholtz equation

\[
(\nabla^2 + \kappa^2)u = 0, \tag{3}
\]

into solutions of Eq. (2); see also Colton and Kress (1978). Colton and Kress (1979) have extended this approach to consider related transmission problems; see also Sleeman (1980). Frisk and DeSanto (1970) have exploited the notion of a Jost function from quantum mechanics so as to obtain approximate solutions of Eq. (2).

We consider acoustic scattering by an inhomogeneous sphere of a radius \(a\). The medium in \(r > a\) is homogeneous, with density \(\rho_0\) and sound speed \(c_0\). For time-harmonic motions, the acoustic pressure \(p_0\) is governed by Eq. (3). Inside
the sphere, \( r < a \), the governing equation is Eq. (1). Across \( r = a \), we impose continuity of pressure and normal velocity, the latter condition being equivalent to continuity of \( \rho^{-1}(\partial P/\partial r) \). This gives a transmission problem. If the interior is homogeneous, with \( \rho = \rho_1 \) and \( c = c_1 \) constants, the problem can be solved exactly, by separation of variables. We will show that this method can be extended to certain functional forms for \( \rho(r) \) and \( c(r) \). We also consider scattering by a sphere with a homogeneous core and an inhomogeneous coating.

There are many papers on analogous electromagnetic scattering problems. For an early treatment, see the paper by Wyatt (1962). One popular technique is to replace the inhomogeneous sphere by many concentric layers, and then to use a simple approximation to the refractive index within each layer; see, for example, Perelman (1996). Kai and Mascal (1994) have reported the results of computations with as many as 10,000 layers.

In this paper, we consider the following specific functional forms

\begin{enumerate}[i)
  \item \( \rho(r) = \rho_1 e^{\beta r} \) and \( [k(r)]^2 = k_1^2 + \alpha r^{-1} \),
  \item \( \rho(r) = \rho_1 e^{-\beta r} \) and \( [k(r)]^2 = k_1^2 + \alpha r^2 \).
\end{enumerate}

Here, \( \rho_1, \beta, k_1, \) and \( \alpha \) are adjustable parameters. For both (i) and (ii), explicit solutions of Eq. (1) are derived. The radial parts of these solutions are given in terms of known special functions, namely Coulomb wave functions and Whittaker functions. These solutions then permit the explicit solution of various scattering problems for inhomogeneous spheres. Such solutions can serve as benchmarks for numerical methods, but they also have intrinsic interest.

II. GOVERNING EQUATIONS

In the linear theory of acoustics for an inhomogeneous medium, the basic equation is (Morse and Ingard, 1986, p. 408)

\[
\rho \text{ div}(\rho^{-1} \text{ grad } P) = c^{-2}(\partial^2 P/\partial t^2),
\]

where \( P(r,t) \) is the acoustic pressure at position \( r \) and time \( t \), \( \rho(r) \) is the density, and \( c(r) \) is the speed of sound. For time-harmonic motions, with \( P = \text{Re} \{ \rho e^{-i\omega t} \} \), we obtain Eq. (1).

According to Pierce (1990), Eq. (4) was first given by Bergmann (1946). However, the derivation of Eq. (4) does not seem to be well known. In particular, Eq. (4) cannot be derived without mentioning the entropy.

A. Derivation of Bergmann’s equation

The exact equations for the motion of an inviscid compressible fluid are (Batchelor, 1967, Sec. 3.6; Ostashev, 1997, Sec. 2.1.1)

\[
\frac{D\tilde{\rho}}{Dt} + \tilde{\rho} \text{ div } \tilde{v} = 0,
\]

\[
\tilde{\rho} \frac{D\tilde{v}}{Dt} + \text{grad } \tilde{P} = 0,
\]

\[
\frac{D\tilde{S}}{Dt} = 0.
\]

Here, \( \tilde{\rho} \) is the density, \( \tilde{v} \) is the velocity, \( \tilde{P} \) is the pressure, and \( \tilde{S} \) is the entropy per unit mass; all these quantities may depend on \( r \) and \( t \). They will be related to the quantities in Eq. (4) later. The material derivative is defined by \( (Df/Dt)(r,t) = \partial f/\partial t + (\tilde{v} \cdot \nabla) f \). In writing Eq. (6), we have assumed that there are no body forces. Equation (7) means that the flow is isotropic (Batchelor, 1967, p. 156).

We also require an equation of state. As usual, we suppose that

\[
\tilde{P} = \tilde{P}(\tilde{\rho}, \tilde{S}).
\]

It follows that \( \text{grad } \tilde{P} = \tilde{c}^2 \text{ grad } \tilde{\rho} + \tilde{\rho} \text{ grad } \tilde{S} \), where

\[
\tilde{c}^2(\tilde{\rho}, \tilde{S}) = (\partial \tilde{P}/\partial \tilde{\rho})_{\tilde{S}} \quad \text{and} \quad \tilde{h}(\tilde{\rho}, \tilde{S}) = (\partial \tilde{P}/\partial \tilde{S})_{\tilde{\rho}}.
\]

The temperature \( \tilde{T} \) satisfies [Batchelor, 1967, Eq. (3.6.6)]

\[
\tilde{T}^{-1} \frac{D\tilde{T}}{Dt} = \nu \tilde{T} \frac{D\tilde{\rho}}{Dt} = -\nu \tilde{c}^2 \text{ div } \tilde{v},
\]

using Eqs. (5), (7), and (8).

For linear acoustics, we suppose that \( \tilde{P} = P_0 + P, \tilde{\rho} = \rho_0 + \rho, \tilde{h} = h_0 + h, \tilde{S} = \tilde{S}_0 + \tilde{S}, \tilde{T} = T_0 + \tilde{T}, \tilde{c} = c_0 + c, \) and \( \tilde{h} = h_0 + h_1 \), where the ambient flow is denoted by the subscript 0 and the small acoustic disturbance is denoted by the subscript 1. (The quantities of most interest are \( P_1, \rho_1, \rho_0, S_1, \) and \( c_0 \).) We require that the ambient flow satisfies Eqs. (5)–(9) exactly, and then we derive a set of linear equations governing the small acoustic disturbance.

From Eq. (8), we obtain \( P_0 = \tilde{P}(\rho_0, S_0) \) and then

\[
P_1 = c_0^2 \rho_1 + h_0 S_1,
\]

where \( c_0^2 = \tilde{c}^2(\rho_0, S_0) \) and \( h_0 = \tilde{h}(\rho_0, S_0) \).

The leading-order equations, governing the ambient flow, follow from Eqs. (5)–(7) and (9)

\[
\frac{\partial \rho_0}{\partial t} + \text{div} (\rho_0 \boldsymbol{v}_0) = 0,
\]

\[
\rho_0 \left( \frac{\partial \boldsymbol{v}_0}{\partial t} + (\boldsymbol{v}_0 \cdot \nabla) \boldsymbol{v}_0 \right) = -\text{grad } P_0
\]

\[
= -c_0^2 \text{ grad } \rho_0 - h_0 \text{ grad } S_0,
\]

\[
\frac{\partial S_0}{\partial t} + \boldsymbol{v}_0 \cdot \text{ grad } S_0 = 0,
\]

\[
\frac{\partial T_0}{\partial t} + (\boldsymbol{v}_0 \cdot \nabla) T_0 = -\nu c_0^2 T_0 \text{ div } \boldsymbol{v}_0.
\]

Having selected an ambient flow, the acoustic disturbance is then governed by

\[
\frac{\partial \rho_1}{\partial t} + \text{div} (\rho_1 \boldsymbol{v}_0 + \rho_0 \boldsymbol{v}_1) = 0,
\]
\[
\rho_0 \left( \frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_0 \right) + \rho_1 \left( \frac{\partial \mathbf{v}_0}{\partial t} + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 \right) = - \nabla P_1. \tag{16}
\]

We could derive an equation for \( T_1 \), but it will not be needed below.

### 1. Homogeneous fluid at rest

This is the textbook case (Lighthill, 1978; Pierce, 1989; DeSanto, 1992), where the ambient flow has \( \mathbf{v}_0 = 0 \), with \( P_0, \rho_0, S_0, T_0, c_0, \) and \( h_0 \) all constant. These choices satisfy Eqs. (11)–(14) identically. Then, Eqs. (15)–(17) reduce to

\[
\frac{\partial P_1}{\partial t} + \rho_0 \text{div} \mathbf{v}_1 = 0, \tag{18}
\]

\[
\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = - \nabla P_1, \tag{19}
\]

\[
\frac{\partial S_1}{\partial t} = 0, \tag{20}
\]

together with Eq. (10). Multiplying Eq. (18) by \( \rho_1^2 \) and Eq. (20) by the thermodynamic coefficient \( h_0 \) and adding the results, gives

\[
\frac{\partial P_1}{\partial t} + \rho_0 \rho_1^2 \text{div} \mathbf{v}_1 = 0. \tag{21}
\]

Then, eliminating \( \mathbf{v}_1 \) between this equation and Eq. (19), we obtain the familiar wave equation, \( \nabla^2 P_1 = \rho_1 c_0^2 \left( \partial^2 P_1 / \partial t^2 \right) \), for the acoustic pressure. The other acoustic quantities, \( \mathbf{v}_1, P_1, \rho_1, \) and \( S_1 \), can then be calculated in terms of \( P_1 \).

In practice, the dependence of entropy on \( t \) is often ignored, so that Eq. (8) is replaced by \( \tilde{P} = \tilde{P}(\tilde{\rho}) \) when \( \tilde{h} = 0 \) and \( \tilde{S} = 0 \). However, we claim that entropy should be retained when the fluid is not homogeneous.

### 2. Constant entropy

Suppose that \( S_0 \) is constant, so that Eq. (13) is satisfied identically. Equation (12) reduces to

\[
\frac{\partial \mathbf{v}_0}{\partial t} + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 = - c_0^2 \rho_0^{-1} \nabla \rho_0. \tag{22}
\]

If we assume that \( \mathbf{v}_0 \) is a constant vector, Eqs. (22) and (11) imply that \( \rho_0 \) is constant. Equation (12) then implies that \( P_0 \) is a function of \( t \) only. But, \( P_0 = \tilde{P}(\rho_0, S_0) \), so that \( P_0 \) is also a constant. Thus, constant \( S_0 \) and constant \( \mathbf{v}_0 \) imply that \( \rho_0 \) and \( P_0 \) are constant too. Also, \( T_0 \) must satisfy Eq. (14) with zero on the right-hand side.

Further remarks on the assumption of constant \( S_0 \) in the context of stratified media can be found in the book by Ostashev (1997, Sec. 2.2.4).

DeSanto (1992, Appendix 1A) argued that the second term on the left-hand side of Eq. (22) is negligible, and then deduced that \( \mathbf{v}_0 \) cannot be constant in an inhomogeneous fluid. In fact, if the right-hand side of Eq. (22) does not depend on \( t \), we can integrate, thus showing that \( |\mathbf{v}_0| \) must grow linearly with \( t \). This is an unpleasant consequence of neglecting entropy.

The derivation given by Morse and Ingard (1986, p. 408) is flawed. They begin (their first displayed equation on p. 408) with

\[
0 = \frac{D \rho_0}{D t} + \mathbf{v}_0 \cdot \nabla \rho_0 \tag{23}
\]

(in their notation), which is incorrect; cf. Eq. (11). Next, they ‘add a sound wave, with its velocity \( \mathbf{u} \) its pressure \( p \), and its additional density change \( \delta , \)’ so that \( \mathbf{u} = \mathbf{v}_1, P = P_1, \) and \( \delta = \rho_1 \) in our notation. The following equations are erroneous because, in their Eq. (23), they have \( \mathbf{u} \) in place of \( \mathbf{v}_0 \), so that their \( \mathbf{u} \) is both \( \mathbf{v}_0 \) and \( \mathbf{v}_1 \).

### 3. Zero ambient velocity

Suppose, instead, that \( \mathbf{v}_0 = 0 \). Then, Eqs. (11), (13), and (14) imply that \( \rho_0, S_0, T_0, c_0, \) and \( h_0 \) do not depend on \( t \). Equation (12) will also be satisfied, provided that

\[
c_0^2 \rho_0 + h_0 \text{grad} S_0 = 0. \tag{24}
\]

This constraint permits us to have spatial variations in \( c_0^2 \) and \( \rho_0 \) within a stationary fluid.

For the acoustic disturbance, Eqs. (15)–(17) reduce to

\[
\frac{\partial \rho_1}{\partial t} + \text{div}(\rho_0 \mathbf{v}_1) = 0, \tag{25}
\]

\[
\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \nabla P_1 = 0, \tag{26}
\]

\[
\frac{\partial S_1}{\partial t} + \mathbf{v}_1 \cdot \nabla S_0 = 0. \tag{27}
\]

Making use of Eq. (10), we combine Eqs. (25) and (27) to give

\[
\frac{\partial P_1}{\partial t} + c_0^2 \text{div}(\rho_0 \mathbf{v}_1) + h_0 \mathbf{v}_1 \cdot \nabla S_0 = 0. \tag{28}
\]

Using the constraint (24) so as to eliminate \( S_0 \) from Eq. (28), we obtain Eq. (21) again (except now \( \rho_0 \) and \( c_0^2 \) are not required to be constants). Finally, by eliminating \( \mathbf{v}_1 \) between Eqs. (26) and (21), we obtain Bergmann’s equation (4), wherein \( \rho = \rho_0, P = P_1, \) and \( c = c_0 \).

### B. Reduced equations

We can reduce Bergmann’s equation to an equation without first derivatives by introducing a new dependent variable (Bergmann, 1946); thus, define

\[
P = \rho^{1/2} U, \tag{29}
\]

whence \( U \) is found to satisfy

\[
\nabla^2 U + KU = c_0^{-2} (\partial^2 U / \partial t^2), \tag{30}
\]

where
\[ K = \frac{1}{2}p^{-1}\nabla^2 \rho - \frac{1}{2}p^{-2}[\text{grad } \rho]^2 \]  
(31) 
[\rho^{1/2}\nabla^2 (\rho^{-1/2})]. \]  
(32)

Equations (29)–(31) [but not Eq. (32)] can be found in Brekhovskikh (1960, p. 171).

Evidently, Eq. (30) can be reduced to a partial differential equation with constant coefficients if, for example, \( \nabla^2 (\rho^{-1/2}) = \kappa \rho^{-1/2} \) and \( c = c_1 \), where \( \lambda \) and \( c_1 \) are constants.

For time-harmonic problems, we can write \( U = \text{Re}\{ue^{-i\omega t}\} \) and \( v = \text{Re}\{we^{-i\omega t}\} \), whence \( u \) satisfies
\[ \nabla^2 u + (k^2 + K)u = 0, \]  
(33)
where \( k^2 = \omega^2/c^2 \), and then the fluid velocity is given by \( \mathbf{v} = (i\omega)\rho^{-1/2} \text{grad } (\rho^{1/2}u) \).

In a homogeneous region, we have \( \rho = \rho_1 \), \( c = c_1 \), and \( k = k_1 = \omega/c_1 \), all constants. Then, \( K = 0 \) and Eq. (33) reduces to the standard Helmholtz equation. If \( \rho \) is constant but \( c \) is not, we still have \( K = 0 \) and then Eq. (33) is usually written as Eq. (2); see, for example, Colton and Kress (1992, Chap. 8).

Finally, we can write Eq. (33) as \( \nabla^2 u + (k_1^2 - V)u = 0 \), where \( k_1^2 \) is a constant and \( V = k_1^2 - k^2 - K \), which we recognize as Schrödinger’s equation with potential \( V \) [Newton, 1982, Eq. (10.59)].

III. SPHERICAL SYMMETRY

Introduce spherical polar coordinates, \( r, \theta, \) and \( \phi \). Assume that the inhomogeneous medium is spherically symmetric. Then, as \( \nabla^2 \{f(r)\} = r^{-2}\{r^2f(r)’\}’ \) [where \( f’(r) = df/dr \)], we find that
\[ K(r) = -r^{-1}(\rho’/\rho) + \frac{1}{2}(\rho''/\rho) - \frac{3}{2}(\rho’/\rho)^2. \]  
Next, we seek solutions of Eq. (33) in the form
\[ u(r, \theta, \phi) = u_n(r)Y_n(\theta, \phi), \]  
(34)
where \( n \) is an integer, \( Y_n \) is a spherical harmonic, and \( u_n(r) \) is to be found by substituting Eq. (34) in Eq. (33). [A typical spherical harmonic is \( Y_n^m(\cos \theta) \varepsilon^{im\theta} \), where \( P_n^m(\cos \theta) \) is an associated Legendre function and \( A_n^m \) is a normalization constant.]

We have
\[ \nabla^2 (u_n Y_n) = u_n \nabla^2 Y_n + 2(\text{grad } u_n) \cdot (\text{grad } Y_n) + Y_n \nabla^2 u_n. \]  
(35)
But \( (\text{grad } u_n) \cdot (\text{grad } Y_n) = 0 \) because \( u_n \) is a function of \( r \) and \( Y_n \) is a function of \( \theta \) and \( \phi \). We also know that \( r^n Y_n \) is a separated solution of Laplace’s equation, so that
\[ 0 = \nabla^2 (r^n Y_n) = r^n \nabla^2 Y_n + f Y_n \]  
by Eq. (35) and \( \nabla^2 \{r^n\} = n(n + 1) r^{n-2} \) whence \( \nabla^2 Y_n = -n(n + 1) r^{-2} Y_n \) and then Eq. (35) gives
\[ \nabla^2 (u_n Y_n) = \{\nabla^2 u_n - n(n + 1) r^{-2} u_n\} Y_n. \]  
Hence, Eq. (33) reduces to
\[ u_n'' + 2r^{-1} u_n’ + [k^2(r) + K(r) - n(n + 1) r^{-2}] u_n = 0, \]  
(36)
which is a linear second-order differential equation for \( u_n(r) \). If we have solutions of this equation, we can then use the method of separation of variables for various scattering problems involving inhomogeneous spheres. Two such problems are described next.

IV. TWO SCATTERING PROBLEMS

Acoustic scattering by spheres, with various boundary conditions, is a textbook topic (Morse and Ingard, 1986, Sec. 8.2). We shall modify the familiar method of separation of variables so as to treat inhomogeneous spheres.

Consider an inhomogeneous sphere of radius \( a \) centered at the origin. Without loss of generality, we can take the incident pressure field as
\[ \frac{p_{\text{inc}}}{\rho_0 c_0^2} = e^{ik_0 r} \sum_{n=0}^{\infty} (2n + 1) i^n j_n(k_0 r) P_n(\cos \theta), \]
where \( j_n(w) \) is a spherical Bessel function. Then, we can write the total pressure field outside the sphere, in \( r > a \), as
\[ p_0(r, \theta) = \rho_0 c_0^2 \sum_{n=0}^{\infty} (2n + 1) i^n \{j_n(k_0 r) + A_n h_n(k_0 r)\} P_n(\cos \theta), \]
(37)
where \( h_n(w) = h_n^{(1)}(w) \) is a spherical Hankel function and the dimensionless coefficients \( A_n \) are to be found. This expression for \( p_0 \) satisfies the Helmholtz equation and, moreover, \( p_0 - p_{\text{inc}} \) satisfies the Sommerfeld radiation condition at infinity.

A. An inhomogeneous sphere

Inside the sphere \( (r < a) \), we write
\[ u(r, \theta) = \rho_0^{1/2} \sum_{n=0}^{\infty} (2n + 1) i^n B_n u_n(r) P_n(\cos \theta), \]
(38)
where the dimensionless coefficients \( B_n \) are to be found and \( u_n(r) \) is a solution of Eq. (36) that is regular at \( r = 0 \); some explicit solutions will be given later.

We find \( A_n \) and \( B_n \) by enforcing the transmission conditions across the interface at \( r = a \). Let
\[ \rho_0 = \lim_{r \to a^-} \rho(r) \]  
and \( \kappa_0 = \lim_{r \to a^-} [\rho’(r)/\rho(r)] \),
so that \( \rho_0 \) is the surface value of the interior density. Then, the interface conditions are
\[ p_0 - \rho_0^{1/2} u' = \rho_0^{-1} \frac{\partial p_0}{\partial r} = \rho_0^{-1/2} \left( \frac{\partial u}{\partial r} + \kappa_0 u \right) \]  
(39)
on \( r = a \). Substituting Eqs. (37) and (38), making use of the orthogonality of the Legendre polynomials, gives
\[ j_n(k_0 a) + A_n h_n(k_0 a) = \sigma B_n u_n(a), \]
\[ k_0 j_n(k_0 a) + A_n h_n(k_0 a) = \sigma^{-1} B_n \{u_n'(a) + \kappa_0 u_n(a)\} \]
for \( n = 0, 1, 2, \ldots \), where \( \sigma = (\rho_a/\rho_0)^{1/2} \). These two equations can be solved for \( A_n \) and \( B_n \)
\[ A_n = (k_0 \sigma)^{-1} j_n(k_0 a) \{u_n(a) + \kappa_0 u_n(a)\} - \sigma j_n(k_0 a) u_n(a) \]
and \( B_n = i(k_0 a)^{-2} \), where
\[ \Delta = \sigma h_n'(k_0 a) u_n(a) - (k_0 \sigma)^{-1} h_n(k_0 a) \{u_n'(a) + \kappa_0 u_n(a)\}. \]
B. A homogeneous sphere with an inhomogeneous coating

Suppose that the sphere \( r < a \) consists of a homogeneous core \( r < b \) (with density \( \rho_c \) and sound speed \( c_s \)) with an inhomogeneous concentric coating, \( b < r < a \).

In the coating, we can write

\[
u(r, \theta) = \rho_0^{1/2} \sum_{n=0}^{\infty} (2n+1) \times i^n \{ B_n u_n(r) + C_n v_n(r) \} P_n(\cos \theta),
\]

where \( u_n(r) \) and \( v_n(r) \) are solutions of Eq. (36). We suppose that \( u_n(r) \) is regular at \( r = 0 \), whereas \( v_n(r) \) is singular at \( r = 0 \). In the homogeneous core, the pressure field is

\[
p(r, \theta) = \rho_0^{1/2} \sum_{n=0}^{\infty} (2n+1)i^n D_n A_n(k_c r) P_n(\cos \theta),
\]

where \( k_c = \omega / c_s \).

We have to enforce two transmission conditions at \( r = a \) and two at \( r = b \). Let

\[
p_b = \lim_{r \to b^+} p(r) \quad \text{and} \quad \kappa_b = \lim_{r \to b^+} \left[ \frac{\rho' (r)}{\rho(r)} \right].
\]

Then, the interface conditions are Eq. (39) and

\[
p_c = p_b^{1/2} \quad \text{and} \quad \rho_c^{-1} \frac{\partial p_c}{\partial r} = p_b^{-1/2} \left( \frac{\partial u}{\partial r} + \kappa_b u \right)
\]
on \( r = b \). These four conditions can be used to determine \( A_n \), \( B_n \), \( C_n \), and \( D_n \) in a straightforward way.

V. EXPONENTIAL VARIATIONS IN \( \rho \)

Let us assume specific functional forms for \( \rho(r) \) and \( k(r) = \omega / (c_s(r)) \), namely

\[
\rho(r) = \rho_1 e^{2\beta r} \quad \text{and} \quad [k(r)]^2 = k_1^2 + 2 \alpha r^{-1}.
\]

Here, \( \rho_1, \beta, k_1, \text{ and } \alpha \) are four adjustable constants. We find that Eq. (36) becomes

\[
u_n'' + 2r^{-1} u_n' + \left[ k_1^2 - \beta^2 + 2(\alpha + \beta) r^{-1} - n(n+1) r^{-2} \right] u_n = 0.
\]

Equation (41) has a regular singularity at \( r = 0 \), an irregular singularity at \( r = \infty \), and no others. Therefore, it can be transformed into the confluent hypergeometric equation. Make the substitution

\[
u_n(r) = r^{-1} w_n(x) \quad \text{with} \quad x = \beta r
\]
in Eq. (41), giving

\[
w_n''(x) + \left[ k_1^2 - \beta^2 + \frac{2 \alpha + \beta}{x} - \frac{n(n+1)}{x^2} \right] w_n(x) = 0,
\]

where \( \delta \) is a parameter at our disposal. There are now three cases, depending on the relative sizes of \( k_1^2 \) and \( \beta^2 \).

A. Case I (\( k_1^2 > \beta^2 \))

Choose \( \delta^2 = k_1^2 - \beta^2 \) and set \( \eta = - (\alpha + \beta) / \delta \). Then, Eq. (42) becomes

\[
w_n''(x) + \left[ 1 - 2 \eta x^{-1} - n(n+1) x^{-2} \right] w_n(x) = 0,
\]

which is the Coulomb wave equation (Abramowitz and Stegun, 1965, Chap. 14). Its general solution is

\[
w_n(x) = A_n F_n(\eta x) + B_n G_n(\eta x),
\]

where \( A_n \) and \( B_n \) are arbitrary constants, \( F_n \) is the regular Coulomb wave function (bounded at \( x = 0 \)), and \( G_n \) is the irregular Coulomb wave function. These functions arise in nuclear physics (Biedenharn and Brussard, 1965, Chap. 3, Sec. 4).

Unsurprisingly, Coulomb wave functions are wave-like, in the sense that

\[
G_n(\eta x) + i F_n(\eta x) \sim e^{i(x-x')} \quad \text{as} \quad x \to \infty,
\]

where \( \varphi = \eta \log 2x + \frac{\eta}{2} \mu \pi - \sigma_n \) and \( \sigma_n \) is known [Abramowitz and Stegun, 1965, Eq. (14.6.5)]. Moreover,

\[
F_n(0,x) = j_n(x) \quad \text{and} \quad G_n(0,x) = -xy_n(x),
\]

where \( j_n \) and \( y_n \) are spherical Bessel functions, so that the known solutions for homogeneous media are recovered.

B. Case II (\( k_1^2 = \beta^2 \))

Choose \( \delta = 8(\alpha + \beta) \), and then Eq. (42) becomes

\[
w_n''(x) + \left[ \frac{1}{2} x^{-1} - n(n+1) x^{-2} \right] w_n(x) = 0,
\]

which is related to Bessel’s equation; the general solution of Eq. (43) is

\[
w_n(x) = \sqrt{x} \left[ A_n J_{2n+1}(\sqrt{x}) + B_n Y_{2n+1}(\sqrt{x}) \right].
\]

C. Case III (\( k_1^2 < \beta^2 \))

Choose \( \delta^2 = 4(\beta^2 - k_1^2) \) and set \( \kappa = 2(\alpha + \beta) / \delta \) and \( \mu = n + 1/2 \). Then, Eq. (42) becomes

\[
w_n''(x) + \left[ -\frac{1}{4} + \kappa x^{-1} + (\frac{1}{2} - \mu^2) x^{-2} \right] w_n(x) = 0,
\]

which is known as Whittaker’s equation. Its general solution is given by

\[
w_n(x) = A_n M_{\kappa, \mu}(x) + B_n W_{\kappa, \mu}(x),
\]

where \( M_{\kappa, \mu} \) and \( W_{\kappa, \mu} \) are Whittaker functions; these are discussed by Whittaker and Watson (1927, Chap. 16), by Erdélyi et al. (1953, Sec. 6.9), by Abramowitz and Stegun (1965, Chap. 13), and by Buchholz (1969). The occurrence of Whittaker functions is a little surprising, because these functions do not exhibit wave-like behavior. Thus

\[
M_{\kappa, \mu}(x) - x^{-\kappa} e^{x/2} \quad \text{and} \quad W_{\kappa, \mu}(x) - x^{-\kappa} e^{-x/2}
\]
as \( x \to \infty \). Moreover [Buchholz, 1969, Sec. 2, Eqs. (11a) and (29a)]

\[
M_{0,0}(x) = \sqrt{\pi x} I_\mu(x) \quad \text{and} \quad W_{0,0}(x) = \sqrt{\pi x} K_\mu(x),
\]

where \( I_\mu \) and \( K_\mu \) are modified Bessel functions. We remark that Whittaker functions also occur when solving the steady-state heat-conduction equation, \( \text{div} \left( k(r) \text{grad} u \right) = 0 \), when \( k(r) \) varies exponentially with \( r \); see Martin (2002). The solutions described above can be inserted into the formulas obtained by the method of separation of variables in Sec. IV.
VI. GAUSSIAN SPHERES

One drawback of the functional forms in Eq. (40) is that the corresponding sound speed satisfies $c(0) = 0$ (unless $\alpha = 0$), so that Eq. (40) may not be suitable for an inhomogeneous sphere. (This objection does not apply if the sphere has a homogeneous core, as described in Sec. IV B.)

As an alternative, we can make progress by supposing that the density is a Gaussian and that $k^2(r)$ is linear in $r^2$. Thus, we suppose that

$$\rho(r) = \rho_1 e^{-\beta r^2}$$

where $\rho_1, \beta, k_1$, and $\gamma$ are adjustable constants. We find that Eq. (36) becomes

$$u_n'' + 2r^{-1}u_n' + \left[ (\gamma - \beta^2) r^2 + (k_1^2 - 3 \beta) - n(n+1)r^{-2} \right] u_n = 0.$$  

To simplify this equation, make the substitution $u_n(r) = r^{-3/2}w_n(x)$ with $x = \delta r^2$, where $\delta$ is a disposible parameter.

This gives

$$w_n''(x) + \left[ \frac{\gamma - \beta^2}{\delta^2} + \frac{k_1^2 - 3 \beta}{4 \delta x} - \left( n(n+1) - \frac{3}{16} \right) x^{-2} \right] w_n(x) = 0.$$  

As in Sec. V, there are now three cases, depending on the sign of $\gamma - \beta^2$. For example, we obtain the Coulomb wave equation if $\gamma > \beta^2$ and Whittaker’s equation if $\gamma < \beta^2$. Explicit solutions follow readily, but are not recorded here.

VII. CONCLUSIONS

In this paper, we have done two things. First, we have given a derivation of Bergmann’s equation for sound waves in inhomogeneous media, where both the ambient density and sound speed can vary with position (but not time). Second, we have studied the scattering of waves by an inhomogeneous sphere; for certain exponential variations in the density, such scattering problems can be solved by the method of separation of variables, where the radial dependence involves some less-familiar but well-studied special functions. Perhaps the main value of these solutions is to provide non-trivial benchmarks against which numerical schemes (based, for example, on volume integral equations) can be tested.

ACKNOWLEDGMENTS

Thanks are due to C. J. Chapman and J. A. DeSanto for their constructive comments on the derivation in Sec. II A.


