Multiple scattering by random configurations of circular cylinders:
Weak scattering without closure assumptions

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Abstract
Acoustic scattering by random collections of identical circular cylinders is considered. Each cylinder is penetrable, with a sound-speed that is close to that in the exterior: the scattering is said to be “weak”. Two classes of methods are used. The first is usually associated with the names of Foldy and Lax. Such methods require a “closure assumption”, in addition to the governing equations. The second class is based on iterative approximations to integral equations of Lippmann–Schwinger type. Such methods do not use a closure approximation. Our main result is that both approaches lead to exactly the same formulas for the effective wavenumber, correct to second-order in scattering strength and second-order in filling fraction. Approximations for the average wavefield are also derived and compared.

1. Introduction
Understanding multiple scattering by random arrays of identical obstacles is a problem of long-standing interest. One approach, going back to Foldy [1], begins with a deterministic model for scattering by \( N \) obstacles, which we write concisely as

\[
\mathbf{u} = \mathbf{u}_{\text{in}} + \sum_{n=1}^{N} \mathcal{K}_n \mathbf{u}_n,
\]

where \( \mathbf{u} \) is the unknown wavefield, \( \mathbf{u}_{\text{in}} \) is the given incident field and \( \mathcal{K}_n \) is an operator. The quantity \( \mathbf{u}_n \) can be viewed as the (unknown) contribution to \( \mathbf{u} \) coming from the scatterer centred at \( \mathbf{r}_n \). Then, the ensemble average, \( \langle \mathbf{u} \rangle \), is calculated over all possible configurations of the scatterers. It follows from the right-hand side of Eq. (1.1) that we have to calculate \( \langle \mathcal{K}_n \mathbf{u}_n \rangle \), but this only exists if there is a scatterer at \( \mathbf{r}_n \); we are forced to introduce a conditional average of \( \mathbf{u}_n \). Thus, we cannot derive an equation for \( \langle \mathbf{u} \rangle \), merely a hierarchy of equations relating various different conditional averages of \( \mathbf{u} \). Breaking this hierarchy requires an additional “closure assumption”. Such assumptions are difficult to justify, in general; for an example, see [2].

For an alternative approach, suppose that we had an explicit formula for \( \mathbf{u}_n \), for any given configuration of the scatterers. Then, we could calculate \( \langle \mathbf{u} \rangle \) directly, in principle, without the use of closure assumptions. Of course, we do not have such an explicit formula for \( \mathbf{u}_n \), but we do have explicit approximations for \( \mathbf{u}_n \), and these could be used. This idea also has a long history (see, for example, [3] and [4, §7.4.2]).
In this paper, we shall compare these two approaches for a simple two-dimensional, time-harmonic, acoustic problem, where detailed explicit calculations can be made. We choose random arrays of identical penetrable circles. The number of circles per unit area is $n_0$. We assume that the area fraction occupied by the circles, $\phi \equiv \pi a^2 n_0$, is small, where each circle has radius $a$. We also assume that the scattering is “weak”, meaning that the strength $m_0 = 1 - (k_0/k)^2$ is small, where $k$ and $k_0$ are the wavenumbers in the exterior and interior, respectively. The assumption of small $n_0$ is common in many theoretical studies (the scatterers are dilute), whereas the assumption of small $m_0$ is convenient for the derivation of good approximations to the deterministic problem. We work to second-order in both $m_0$ and $n_0$. Maurel [5] has made similar comparisons, but with uncorrelated “point scatterers”, meaning that each scatterer is represented by a Dirac delta function; we are interested here in scatterers of finite size (and we do not make small-$ka$ approximations). Comparisons with Maurel’s results are made in Section 5.3.

One output from the analysis is a formula for the effective wavenumber, $K$: the averaged field $\langle u \rangle$ is found to solve the Helmholtz equation, $(\nabla^2 + K^2) \langle u \rangle = 0$, in a fictitious “effective medium”. For small $n_0$, formulas for $K$ look like

$$K^2 \approx k^2 + n_0 \delta_1 + n_0^2 \delta_2,$$

with explicit expressions for $\delta_1$ and $\delta_2$. (For references to the literature, see [6] and [7].) Foldy-type theories use his closure assumption, they are essentially linear in $n_0$, they assume that the scatterers are independent, and they predict that

$$K^2 = k^2 - 4im_0 f(0),$$

where $f$ is the far-field pattern for one scatterer in isolation (see Eq. (2.7)). Also, when $u_{in} = e^{ika}$, $\langle u \rangle = Ae^{i\alpha}$,

$$(1.4)$$

where Foldy theory predicts that the “amplitude”, $A$, is given by

$$A = 1 + im_0 k^{-2} f(0).$$

Observe that if we write $A = |A|e^{i\alpha}$ and $K = K_r + iK_i$, then

$$\text{Re} \{(\langle u \rangle e^{-i\omega t}) = |A|e^{-K_i x} \cos(K_r x - \omega t + \alpha),$$

showing that the quantities $|A|$ and $K_i = \text{Im} K$ are of interest.

For one circular scatterer, we can construct the exact solution by separation of variables. This elementary calculation is outlined in Section 2; it yields an exact formula for $f(0)$, which can then be inserted in Eqs. (1.3) and (1.5).

For our model problem, the fluid density is constant everywhere, both inside and outside the scatterers. Consequently, the scattering problem can be solved using the Lippmann–Schwinger integral equation (see Section 3). (This equation can also be used when $k_0$ is a function of position; if the density inside differs from that outside, a different integral equation must be used [8].) Under certain circumstances, the Lippmann–Schwinger equation can be solved by iteration; we examine the first-order (Born) approximation (linear in the strength $m_0$) and the second-order approximation (quadratic in $m_0$). For one circular scatterer, this is done in Section 4. We confirm that the second-order iterative approximation for the wavefield (both inside and outside the scatterer) agrees with the second-order approximation of the exact solution.

Scattering by random arrangements of scatterers is considered in Section 5. We write down the iterative approximation for scattering by $N$ circles. Then, we calculate the ensemble average; we begin with first-order in both $n_0$ and $m_0$, and end with second-order in both. No closure assumptions are used. At first-order in $n_0$, we find precise agreement with the Foldy estimate for $K$, Eq. (1.3), correct to second-order in $m_0$. For the amplitude, $A$, we find agreement at first-order in $m_0$ with the Foldy estimate (given by Eq. (1.5)), but we find a discrepancy at second-order (see Eq. (5.25)).

Calculations at second-order in $n_0$ are more difficult because one has to introduce conditional probabilities, intended to prevent scatterers overlapping during the averaging process. We use a simple (but standard) pair-correlation function, giving what is known as a “hole correction”, Eq. (5.13). This involves a new parameter, $b$, with $b \geq 2a$. Linton and Martin [6] found a formula for $\delta_2$ in Eq. (1.2), making use of the Lax quasicrystalline approximation for their closure assumption (see Section 2.3). We compare the Linton–Martin formula (approximated to second-order in $m_0$) with our alternative approach: the agreement turns out to be perfect. One could view this agreement as supporting either of the two approaches described above, depending on one’s point of view. We also obtain a new estimate for $A$, and we verify agreement with known results for “point scatterers” in the appropriate limit (see Section 5.3). Thus, our calculations show that good results may be obtained without closure assumptions, and so such methods deserve further investigation. In particular, extensions to three dimensions (random configurations of spheres) and to scatterers with a mass density that differs from that of the surrounding medium (including bubbles or rigid obstacles) should be made.

There is an extensive literature on multiple scattering by random configurations of identical obstacles. For thorough reviews, see, for example, [7] or [9].

2. Scattering by one cylinder: exact solution

Consider one circle of radius $a$, centred at the origin. We have $(\nabla^2 + k^2) u = 0$ for $r > a$ and $(\nabla^2 + k_0^2) u_0 = 0$ for $r < a$, where $r$ and $\theta$ are plane polar coordinates and $k$ and $k_0$ are real constants. The interface conditions are $u = u_0$ and $\partial u / \partial r = \partial u_0 / \partial r$ on
\( r = a \). Outside, we have \( u = u_{\infty} + u_m \) where \( u_{\infty} = e^{ikx} \) and \( u_m \) satisfies the Sommerfeld radiation condition. Throughout, we suppress a time dependence of \( e^{-i\omega t} \).

The one-cylinder problem can be solved by separation of variables. We use the expansions [6]

\[
u_{in}(r, \theta) = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in\theta},
\]

\[
u_{sc}(r, \theta) = \sum_{n=-\infty}^{\infty} A_n Z_n H_n(kr) e^{in\theta},
\]

\[
u_0(r, \theta) = \sum_{n=-\infty}^{\infty} B_n J_n(k_0r) e^{in\theta},
\]

where \( J_n \) is a Bessel function, \( H_n \equiv H_n^{(1)} \) is a Hankel function,

\[ Z_n = \frac{|\text{Re} A_n|}{A_n} = Z_n \]

and

\[ A_n = H_n^*(ka)f_n(k_0a) - \frac{(k_0/k)J_n(k_0a)H_n(ka)}{J_n(ka)} \] (2.5)

The interface conditions yield \( A_n = -i^n \) and

\[ B_n = \frac{2i^{n+1}}{(\pi ka A_n)} \] (2.6)

The far-field pattern, \( f(\theta) \), is defined by

\[ u_{sc} \sim \sqrt{2/(\pi kr)} f(\theta) \exp(ikr - i\pi/4) \text{ as } r \to \infty. \] (2.7)

Hence,

\[ f(\theta) = -\sum_{n=-\infty}^{\infty} Z_n e^{in\theta}. \] (2.8)

### 2.1. Approximation for small \( m_0 \)

For weak scattering, \( k_0 \) is close to \( k \). To quantify this, we define the scattering strength, \( m_0 \), by

\[ m_0 = 1 - (k_0/k)^2. \] (2.9)

Then, for weak scattering, we approximate the exact solution, assuming that \( m_0 \) is small. From (2.9), we obtain

\[ k_0/k \simeq 1 - \frac{1}{2} m_0 - \frac{1}{8} m_0^2 \] (2.10)

and

\[ F(k_0a) \simeq F(ka) - \frac{1}{2} m_0 kaF'(ka) - \frac{1}{8} m_0^2 ka F'(ka) - kaF''(ka) \] (2.11)

for any smooth function \( F \). Then, some calculation gives

\[ A_n \sim 2i/(\pi ka) - (m_0/2) kad_n(ka) + (m_0^2/8) U_n, \] (2.12)

where

\[ d_n(x) = J_n(x)H_n^*(x) \]
\[ + (1 - (n/\pi)^2) J_n(x)H_n(x), \]

\[ U_n = ka[d_n(ka)]^* - (J_n - ka f_n + ka f_n^m)H_n = 2ka f_n(ka)H_n(ka) - d_n(ka) - i/\pi + 2in^2/(\pi ka) \] (2.13)

and we have used

\[ H_n^*(x) J_n(x) - J_n(x) H_n^*(x) = 2i/(\pi x). \] (2.15)

Then, from Eq. (2.12), the numerator in Eq. (2.4) is given approximately by

\[ \text{Re} A_n \simeq -(m_0/2) ka f_n(ka) + (m_0^2/8) S_n, \]

where

\[ f_n(ka) = \text{Re} d_n(ka) = f_n^m(ka) - J_{n-1}(ka)J_{n+1}(ka), \]

\[ S_n = \text{Re} U_n = 2ka f_n(ka)J_{n-1}(ka). \] (2.16)

Hence, we obtain the following approximation for \( Z_n \):
\[ Z_n \simeq \frac{\pi}{4} m_0 i (ka)^2 \varphi_n(ka) - \frac{\pi}{16} m_0^2 ka \{ i S_n - \pi (ka)^3 \varphi_n(ka) d_n(ka) \}. \quad (2.18) \]

Similarly, for the interior field, \( u_0 \), Eq. (2.6) gives

\[ i^n B_n \simeq 1 - \frac{im_0}{4} \mu_n + \frac{m_0^4}{16} \left( \pi k a U_n - \mu_n^2 \right) \quad \text{with} \quad \mu_n = \pi (ka)^2 d_n(ka). \]

Also, using Eq. (2.11),

\[ J_n(kr) \simeq J_n(kr) - \left( m_n / 2 \right) kr f_n(kr) + \left( m_n^2 / 8 \right) \left( |n|^2 - (kr)^2 \right) J_n(kr) - 2k r f_n(kr), \]

and then Eq. (2.3) gives

\[ u_0(r, \theta) \simeq e^{ikr} - \frac{im_0}{4} \sum_{n=-\infty}^{\infty} \int_0^1 n^2 u_n(1)e^{im\theta} + \frac{m_0^2}{16} \sum_{n=-\infty}^{\infty} \int_0^1 n^2 u_n(2)e^{im\theta}, \quad (2.19) \]

where

\[ u_n^{(1)} = \mu_n J_n(kr) - 2ikr f_n(kr), \]

\[ u_n^{(2)} = \left( 2 |n|^2 - (kr)^2 \right) + \pi k a U_n - \mu_n^2 \right) J_n(kr) + 2i(\mu_n - 2)kr f_n(kr). \]

### 2.2. Application to Foldy theory

Foldy’s theory predicts that \( (u) = Ae^{ikr} \), where \( K \) is given by Eq. (1.3) and \( A \) is given by Eq. (1.5). Using Eqs. (2.8), (1.3) gives

\[ K^2 - k^2 = 4im_0 \sum_{n=-\infty}^{\infty} Z_n \simeq -m_0 n_0 \pi (ka)^2 \sum_{n=-\infty}^{\infty} \varphi_n(ka) = -m_0 n_0 \pi (ka)^2 \]

To first-order in \( m_0 \). Here, we have used

\[ \sum_{n=-\infty}^{\infty} \varphi_n(x) = \sum_{n=-\infty}^{\infty} f_n^2 - \sum_{n=-\infty}^{\infty} J_n f_{n+2} = 1, \quad (2.21) \]

\[ \sum_{n=-\infty}^{\infty} f_n^2(x) = 1, \quad (2.22) \]

\[ \sum_{n=-\infty}^{\infty} J_n(x)f_{n+m}(x) = 0 \quad \text{for} \quad m = \pm 1, \pm 2, \ldots \quad (2.23) \]

Also, from Eq. (1.5), we obtain \( A \simeq 1 + \frac{1}{4} \pi m_0 n_0 a^2 \).

For an approximation to second-order in \( m_0 \), use Eq. (2.18). The term in \( S_n \) does not contribute to \( f(0) \): \( \sum_{n=-\infty}^{\infty} S_n = 0 \). To see this, use Eqs. (2.17) and (2.23). Thus, we obtain

\[ f(0) = -\frac{1}{4} im_0 \pi (ka)^2 + \frac{1}{4} im_0^2 \pi^2 (ka)^4 \varphi(ka), \quad (2.24) \]

where

\[ \varphi(ka) = \frac{i}{4} \sum_{n=-\infty}^{\infty} \varphi_n(ka)d_n(ka). \quad (2.25) \]

(Another formula for \( \varphi(ka) \) is given in Appendix B.) Then Eq. (1.3) gives

\[ K^2 - k^2 \simeq -m_0 n_0 \pi (ka)^2 + m_0^2 n_0 \pi^2 (ka)^4 \varphi(ka) \]

and Eq. (1.5) gives

\[ A \simeq 1 + \frac{1}{4} m_0 n_0 \pi a^2 - \frac{1}{4} m_0^2 n_0 \pi^2 k^2 a^4 \varphi(ka). \quad (2.27) \]

### 2.3. Application to Linton–Martin theory

According to Linton and Martin \([6, \text{Eq. (80)}]\), the second-order correction in Eq. (1.2) is given by

\[ \delta_2 = 4\pi b^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} Z_n Z_m d_{n-m}(kb). \quad (2.28) \]

where \( b \) is the parameter in the “hole correction”, Eq. (5.13). (In the limit \( kb \to 0 \), \( \delta_2 \) can be written as a certain integral of the far-field pattern; see [6, Eq. (86)].)
As a special case of Eq. (2.28), consider isotropic scattering, meaning that \( Z_n = 0 \) when \( n \neq 0 \), so that \( f(0) = -Z_0 \). Then, we obtain the estimate [6, Eq. (42)]
\[
K^2 = k^2 - 4i\alpha f(0) + 4\pi|\alpha|n_0 N(0)^2 d_{\alpha}(kb).
\]
(2.29)
This is Eq. (5) in [10] when \( k_{\text{eff}} \) is replaced by \( k_0 \) in its right-hand side. (In fact, [10, Eq. (5)] is equivalent to [6, Eq. (40)].)
From Eq. (2.18), the first-order approximation to \( Z_n \) is given by
\[
Z_n \approx (m_0/4)\pi^2 (ka)^2 \mathcal{I}_{n}(ka),
\]
(3.30)
whence Eq. (2.28) gives
\[
\delta_2 \approx \frac{m_0^3}{4i} b^5 (ka)^4 \sum_{m=-\infty}^{\infty} \mathcal{I}_{n}(ka)\mathcal{I}_{m}(ka) d_{n-m}(kb).
\]
(2.31)
Thus, there are no contributions to \( K^2 \) that are proportional to \( m_0^3 m_0 \).

3. Lippmann–Schwinger equation
In two dimensions, the Lippmann–Schwinger integral equation is [11]
\[
u(r) = u_0(r) - k^2 \int G_0(r, r') m(r') u(r') dV',
\]
(3.1)
where \( u \) is the total field and
\[
G_0(r, r') = (i/4)H_0^0(k|r - r'|).
\]
(3.2)
The governing partial differential equation is
\[
\nabla^2 u + k^2 u = k^2 m(r) u,
\]
(3.3)
with \( m \equiv 0 \) outside the scatterers. Thus, although the integration in Eq. (3.1) can be written as an integration over all space, it is really only over the scatterers.
It is known that Eq. (3.1) is always uniquely solvable. Moreover, the solution can be constructed by iterating the integral equation, under certain circumstances. We do not investigate these circumstances here. We simply accept the second iteration, giving
\[
u(r) = u_0(r) - k^2 \int G_0(r, r') m(r') u_0(r') dV' + k^2 \int G_0(r, r') m(r') \int G_0(r', r'') m(r'') u_0(r'') dV'' dV'.
\]
(3.4)
We assume that the scatterers are \( D_i \), centred at \( r_i \), \( i = 1, 2, \ldots, N \). Each scatterer is a circular disc of radius \( a \) with constant strength \( m_0 \); when Eq. (2.9) is combined with Eq. (3.3), we see that \( (\nabla^2 + k_0^2) u = 0 \) holds inside the disc, as assumed in Section 2. Thus
\[
(\nabla^2 + k^2) u = k^2 m_0 u(r) \sum_{i=1}^{N} \chi_i(r),
\]
(3.5)
where \( \chi_i \) is the characteristic function for \( D_i \): \( \chi_i(r) = 1 \) when \( r \in D_i \) and \( \chi_i(r) = 0 \) when \( r \notin D_i \). The approximation (3.4) becomes
\[
u(r) = u_0(r) - k^2 m_0 \sum_{i=1}^{N} \int_{D_i} G_0(r, r') u_0(r') dV' + k^2 m_0 \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{D_i} G_0(r, r') \int_{D_j} G_0(r', r'') u_0(r'') dV'' dV'.
\]
(3.6)
Using
\[
(\nabla^2 + k^2) \int_{B} G_0(r, r') f(r') dV' = \begin{cases} 0, & r \notin B, \\
-f(r), & r \in B,
\end{cases}
\]
we see that Eq. (3.6) correctly generates a solution of \( (\nabla^2 + k^2) u = 0 \) outside the scatterers. On the other hand, if \( u_i \) denotes the field in \( D_i \), Eq. (3.6) implies that
\[
(\nabla^2 + k^2) u_i = k^2 m_0 u_0(r) - k^2 m_0 \sum_{j=1}^{N} \int_{D_j} G_0(r, r') u_0(r') dV'
\]
for \( r \in D_i \), whereas the exact solution satisfies \( (\nabla^2 + k^2) u_i = k^2 m_0 u_i \).
We shall return to the formula (3.6) in Section 5, but first we consider scattering by one cylinder.
4. Scattering by one cylinder: iterative solution

Let us specialise the second-order solution (3.6) to a single cylinder. We have

\[ u(\mathbf{r}) \simeq u_{m}(\mathbf{r}) + m_{b}I_{1}(\mathbf{r}) + m_{b}^{2}I_{2}(\mathbf{r}), \quad (4.1) \]

where

\[ I_{1}(\mathbf{r}) = -k^{2} \int_{D_{b}} G_{0}(\mathbf{r}, \mathbf{r}')u_{m}(\mathbf{r}')dV', \quad (4.2) \]
\[ I_{2}(\mathbf{r}) = -k^{2} \int_{D_{b}} G_{0}(\mathbf{r}, \mathbf{r}')I_{1}(\mathbf{r}')dV', \quad (4.3) \]

and \( D_{b} \) is the circular disc of radius \( a \), centred at the origin. In this section, we will evaluate \( I_{1} \) and \( I_{2} \). These quantities will be needed when we consider scattering by many circles (in Section 5). Also, as a check on our calculations, we verify that we recover the small-\( m_{b} \) approximations to the exact solution given in Section 2.1.

4.1. Evaluation of \( I_{1} \)

With \( \mathbf{r} = (r, \theta) \) and \( \mathbf{r}' = (r', \theta') \), we use Eq. (2.1) in Eq. (4.2) to give

\[ I_{1}(r, \theta) = -\sum_{n=-\infty}^{\infty} i^{n}L_{n}(r, \theta), \quad (4.4) \]

where

\[ L_{n}(r, \theta) = k^{2} \int_{0}^{a} \int_{0}^{2\pi} G_{0}(\mathbf{r}, \mathbf{r}')J_{n}(kr')e^{in\theta-\theta'}d\theta'dr'. \quad (4.5) \]

We give separate evaluations of \( L_{n}(r, \theta) \) for \( r > a \) and \( r < a \).

4.1.1. Exterior field

For \( r > a \), we use

\[ G_{0}(\mathbf{r}, \mathbf{r}') = \frac{i}{4} \sum_{n=-\infty}^{\infty} H_{n}(kr)J_{n}(kr')e^{in\theta-\theta'}, \quad r > r', \quad (4.6) \]

in Eq. (4.5). This gives

\[ L_{n}(r, \theta) = C_{n}^{(1)}H_{n}(kr)e^{in\theta} \quad (4.7) \]

with

\[ C_{n}^{(1)} = \frac{\pi i}{2} k^{2} \int_{0}^{a} j_{n}^{2}(kr)r'dr = \frac{\pi i}{4} (ka)^{2} \mathcal{J}_{n}(ka), \quad (4.8) \]

where \( \mathcal{J}_{n} \) is defined by Eq. (2.16). Here, we have used [12], 7.14.1(10), p. 90,

\[ \int w_{n}(kr)W_{n}(kr)rdr = \frac{r^{2}}{4} \left\{ 2w_{n}(kr)W_{n}(kr) - w_{n+1}(kr)W_{n-1}(kr) - w_{n-1}(kr)W_{n+1}(kr) \right\}. \quad (4.9) \]

where \( w_{n} \) and \( W_{n} \) are any two Bessel functions. In particular, we note that

\[ \frac{2}{a^{2}} \int_{0}^{a} j_{n}^{2}(kr)rdr = \mathcal{J}_{n}(ka), \quad (4.10) \]

a result that will be useful later.

The result (4.8) agrees with the exact solution (see Eq. (2.30)).

4.1.2. Interior field

For \( r < a \), the calculation is more complicated. We obtain

\[ L_{n}(r, \theta) = (\pi i/8)\Lambda_{n}^{(1)}(kr)e^{in\theta}, \quad r < a \quad (4.11) \]

with

\[ \Lambda_{n}^{(1)}(kr) = 4k^{2}H_{n}(kr) \int_{0}^{a} j_{n}^{2}(kr)r'dr' + 4k^{2}J_{n}(kr) \int_{r}^{a} H_{n}(kr')J_{n}(kr)r'dr'. \quad (4.12) \]

Use of Eq. (4.9) gives
\[ \Lambda_n^{(1)}(kr) = 2(kr)^2 H_n(kr) \left( j_n^2(kr) - j_{n+1}(kr)j_{n-1}(kr) \right) - (kr)^2 j_n(kr) \left( 2j_n(kr)H_n(kr) - j_{n+1}(kr)H_{n-1}(kr) - j_{n-1}(kr)H_{n+1}(kr) \right) + (ka)^2 j_n(kr) \left( 2j_n(ka)H_n(ka) - j_{n+1}(ka)H_{n-1}(ka) - j_{n-1}(ka)H_{n+1}(ka) \right). \]

Suppressing the argument \( kr \), the first line of this formula becomes
\[ (kr)^2 [j_n j_{n+1} H_{n+1} + j_n j_{n-1} H_{n-1} - 2H_n j_{n-1} j_{n+1}] = j_n [j_n - kr f_n''(kr)] [nH_n + kr H'_n] + j_n [j_n + kr f_n''(kr)] [nH_n - kr H'_n] - 2H_n [j_n - kr f_n''(kr)] [nH_n + kr f_n''(kr)] = 2(kr)^2 j_n^2 F_n H_n - j_n H'_n = -(4i/\pi)kr f_n''(kr), \]
whereas the second line reduces to \( 2(ka)^2 j_n(ka) d_n(ka) \) with \( d_n \) defined by Eq. (2.13). Hence,
\[ \Lambda_n^{(1)}(kr) = -(4i/\pi)kr f_n''(kr) + 2(ka)^2 j_n(ka) d_n(ka). \] (4.13)

Combining Eqs. (4.1), (4.4), (4.11) and (4.13), we find agreement with the first-order approximation of the exact solution, Eq. (2.19).

4.2. Evaluation of \( I_2 \)

As \( r' < a \) in Eq. (4.3), we use Eq. (4.11) in Eq. (4.4) to give
\[ I_2(r) = \frac{\pi i k^2}{8} \sum_{m=-\infty}^{\infty} i^m \int G_0(r, r') \Lambda_n^{(1)}(kr') e^{im\theta} dV'. \]

Again, the calculations depend on whether \( r > a \) or \( r < a \).

4.2.1. Exterior field

For \( r > a \), we use Eq. (4.6). This gives
\[ I_2(r, \theta) = - \sum_{n=-\infty}^{\infty} i^n C_n^{(2)} H_n(kr) e^{im\theta} \] (4.14)
with
\[ C_n^{(2)} = \frac{\pi^2 k^2}{16} \int_0^a j_n(kr') \Lambda_n^{(1)}(kr') r' \, dr' \] (4.15)
\[ = (\pi ka/4)^2 \left( (ka)^2 d_n(ka) f_n(ka) - (2i/\pi)j_n(ka) j_{n+1}(ka) \right). \] (4.16)
for the evaluation of the integral, see Eq. (A.7).

The result (4.16) agrees with the exact solution, see Eq. (2.18).

4.2.2. Interior field

For \( r < a \), we obtain
\[ I_2(r, \theta) = \left( \frac{\pi}{8k} \right)^2 \sum_{n=-\infty}^{\infty} i^n \Lambda_n^{(2)}(kr) e^{im\theta}, \quad r < a \] (4.17)
with
\[ \Lambda_n^{(2)}(kr) = 4k^2 H_n(kr) \int_0^a j_n(kr') \Lambda_n^{(1)}(kr') r' \, dr' + 4k^2 j_n(kr) \int_r^a H_n(kr') \Lambda_n^{(1)}(kr') r' \, dr'. \]

Use of Eq. (4.13) gives
\[ \Lambda_n^{(2)}(kr) = 2(ka)^2 d_n(ka) \Lambda_n^{(1)}(kr) - 8(i/\pi)X_n \] (4.18)
with
\[ X_n = 2k^3 H_n(kr) \int_0^a j_n(ks') j_n'(ks') s^2 \, ds + 2k^3 j_n(kr) \int_r^a H_n(ks') j_n'(ks') s^2 \, ds. \]

To evaluate the remaining integrals, consider
\[ S = \int s^2 \frac{d}{ds} [j_n(ks') H_n(ks')] \, ds. \]

Integrating by parts and then using Eq. (4.9) gives
\[ S = (s^2/2) [j_n H_{n-1} + j_{n+1} H_{n+1}], \]
suppressing the argument \( ks \). Alternatively, we have
\[ \mathcal{J} = k \int s^2 (J_n H_n + J_n' H_n) \, ds = 2k \int s^2 J_n H_n \, ds + \frac{is^2}{\pi}, \]

using the Wronskian, Eq. (2.15). Hence

\[ 2k \int s^2 J_n'(ks) H_n(ks) \, ds = s^2 \left\{ J_n H_{n-1} + J_{n+1} H_{n+1} - \frac{2i}{\pi} \right\} = s^2 \{ J_n H_n - d_n(ks) - i/\pi \}. \]

In particular, the real part of this formula gives

\[ 2k \int s^2 J_n'(ks) J_n(ks) \, ds = s^2 J_{n-1}(ks) J_{n+1}(ks). \] (4.19)

Hence

\[ X_n = X_n^{(1)} + X_n^{(2)}, \] (4.20)

\[ X_n^{(1)} = (kr)^2 \{ J_n'(kr) [d_n(kr) + i/\pi] - H_n(kr) J_n'(kr) \} = (ikr/\pi) \{ J_n(1, kr) - J_{n+1}(kr) + kr J_n(kr) \}, \] (4.21)

\[ X_n^{(2)} = (ka)^2 J_n'(kr) J_n(ka) - d_n(ka) - d_n(ka) - i/\pi = (ka U_n/2 - n^2 i/\pi) J_n(kr). \] (4.22)

\( U_n \) is defined by Eq. (2.14), \( \Lambda_n^{(2)} \) is given by Eq. (4.18), and then \( I_2 \) is given by Eq. (4.17). When \( I_2 \) is substituted in Eq. (4.1), we find complete agreement with the second-order approximation to the exact solution, Eq. (2.19).

5. Scattering by \( N \) cylinders

Let us return to the second-order solution for scattering by \( N \) cylinders, given by Eq. (3.6). Introduce polar coordinates \((r, \theta)\) centred at the origin and \((r_j, \theta_j)\) centred at \( r_j = (x_j, y_j) \), the centre of the \( j \)th scatterer.

5.1. First-order in \( m_0 \)

As in Section 4, using \( u_{in} = e^{ikr} = e^{ikr, \cos \theta_j} \), we obtain

\[ -k^2 \int_{B_r} G_0(r', r'') u_{in}(r') \, dV' = e^{ikr} I_1(r_j, \theta_j), \] (5.1)

with \( r_j \equiv |r - r_j| \), so that, to first-order in \( m_0 \), Eq. (3.6) gives

\[ u(r) \simeq e^{ikr} + m_0 \sum_{j=1}^{N} e^{ikr_j} I_1(r_j, \theta_j). \] (5.2)

Next, we calculate the ensemble average of \( u \), \( \langle u \rangle \). At this stage, we can assume that the scatterers are independent (uncorrelated); they are also indistinguishable. The result is

\[ \langle u(r) \rangle = e^{ikr} + m_0 n_0 \int_{B_r} e^{ikr_j} I_1(r_j, \theta_j) \, dx_j \, dy_j, \] (5.3)

where \( B_N \) is the region occupied by the \( N \) circles and \( n_0 \) is the number of circles per unit area, so that \( B_N \) has area \( N/n_0 \).

In order to do explicit calculations, we now let \( N \rightarrow \infty \). One option would be for \( B_N \) to become a slab, \( 0 < x < L \), say. We prefer to let \( B_N \) become the half-plane \( x > 0 \), but then we have to take care with the convergence of various integrals as \( x \to \infty \); we replace \( e^{ikx} \) by \( e^{ikx} \), with \( \text{Im} \, k > 0 \), and then let \( k \to k \) at the end of the calculation.

Thus, in the limit \( N \to \infty \), the integral in Eq. (5.3) consists of an integral over the circular disc \( r_1 < a \ (r_1^2 = (x_1 - x)^2 + y_1^2) \) plus an integral over the remainder of the half-plane \( x_1 > 0 \); we denote this region by \( \text{disc} \). We can assume here that \( x > a \) so that the disc does not meet the line \( x_1 = 0 \).

Exact calculation gives (see Appendix A)

\[ \int_{r_1, a} e^{ikr_j} I_1(r_j, \theta_j) \, dx_j \, dy_j = -\pi a^2 e^{ikx}(ka^2) \mathcal{K}(ka), \] (5.4)

\[ \lim_{k \to \infty} \int_{\text{disc}} e^{ikr_j} I_1(r_j, \theta_j) \, dx_j \, dy_j = \frac{\pi a^2}{4} e^{-ikx} \{ 1 - 2ikx + 4\pi(ka)^2 \mathcal{K}(ka) \}, \] (5.5)

where \( \mathcal{K} \) is defined by Eq. (2.25). When these two expressions are added together, the terms in \( \mathcal{K} \) cancel, and then Eq. (5.3) gives

\[ \langle u(r) \rangle = e^{ikx} \{ 1 + m_0 n_0 (\pi/4) a^2 (1 - 2ikx) \}, \quad x > a. \] (5.6)

This should be compared with Eq. (1.4), \( \langle u(r) \rangle = A e^{ikx} \). Write

\[ A \simeq 1 + m_0 A_1 + m_0^2 A_2 \quad \text{and} \quad K \simeq k + m_0 k_1 + m_0^2 k_2, \]

so that \( K^2 \simeq k^2 + 2m_0 k k_1 + m_0^2 (k_1^2 + 2k_2) \). Then,
\[ Ae^{ikx} = e^{ikx}\{1 + im_0(A_1 + ik_1x) + m_0^2(A_2 + i|A_1|k_1 + k_2) - k_1^2x^2/2\}; \] (5.7)

the terms involving \( m_0^2 \) will be used later. Comparing Eqs. (5.6) and (5.7) gives \( A_1 = \phi/4 \) and \( k_1 = -k\phi/2 \), where we have defined the area fraction occupied by the scatterers, \( \phi \), by

\[ \phi = \frac{n_0\pi a^2}{L}. \]

Hence, \( A \approx 1 + m_0\phi/4 \) and \( K^2 \approx k^2(1 - m_0\phi) \), in agreement with the Foldy estimates, Eqs. (2.20) and (2.27), correct to first-order in \( m_0 \).

The idea of rewriting an expansion such as Eq. (5.6) in the form Eq. (1.4) is well established (see [4], §7.4.2, for example).

### 5.2. Second-order in \( m_0 \)

At second-order in \( m_0 \), we add the last term in Eq. (3.6) to the right-hand side of Eq. (5.2). Using Eq. (5.1), this term becomes

\[ -k^2m_0^2 \sum_{r=1}^{N} \sum_{j=1}^{N} e^{iky} \int_{D_j} G_0(\mathbf{r}, \mathbf{r'}) l_1(r_j, \theta_j') dV' = u_2(\mathbf{r}), \] (5.8)

say; recall that \( r_j, \theta_j \) are polar coordinates centred at \((x_j, y_j)\), the centre of the \( j \)th disc, \( D_j \).

Evidently, the evaluation of the integrals in Eq. (5.8) will be different if \( i = j \) or \( i \neq j \), and so we write

\[ u_2 = u_2^{(1)} + u_2^{(2)}, \] (5.9)

where

\[ u_2^{(1)}(\mathbf{r}) = -k^2m_0^2 \sum_{j=1}^{N} e^{iky} \int_{D_j} G_0(\mathbf{r}, \mathbf{r'}) l_1(r_j, \theta_j') dV', \] (5.10)

and \( u_2^{(2)} = u_2 - u_2^{(1)} \).

#### 5.2.1. Calculation of \( \langle u_2^{(1)} \rangle \)

From Eq. (4.3), we obtain

\[ u_2^{(1)}(\mathbf{r}) = m_0^2 \sum_{j=1}^{N} e^{iky} I_2(r_j, \theta_j), \]

where \( I_2 \) is given by Eq. (4.14) for \( r_j > a \) and by Eq. (4.17) for \( r_j < a \). Then, proceeding as in Section 5.1, we obtain

\[ \langle u_2^{(1)}(\mathbf{r}) \rangle = e^{iky}m_0^2(\phi/4)\{P_0(ka) + (2ikx - 1)Q_0(ka)\}, \quad x > a. \] (5.11)

where \( P_0(ka) \) is given by Eq. (A.9) and \( Q_0(ka) = ka^2/\pi(ka) \) (see Appendix A for details of the calculation).

#### 5.2.2. Calculation of \( \langle u_2^{(2)} \rangle \)

When \( i \neq j, r_j > a \) (as the scatterers are not allowed to overlap) and so we can use Eq. (4.7) in Eq. (4.4):

\[ \int_{D_j} G_0(\mathbf{r}, \mathbf{r'}) l_1(r_j, \theta_j') dV' = -\sum_{n=-\infty}^{\infty} i^n C_n^{(1)} \int_{D_j} G_0(\mathbf{r}, \mathbf{r'}) H_n(kr_j) e^{iny} dV'. \]

Here, \( r_j, \theta_j \) are the polar coordinates of the point at \( \mathbf{r'} \) with respect to the centre of \( D_j \). To integrate over \( D_i \), we need Graf's addition theorem to express \( H_n(kr_j) e^{iny} \) in terms of \( J_n(kr_j) e^{iny} \):

\[ H_n(kr_j) e^{iny} = \sum_{m=-\infty}^{\infty} J_m(kr_j) H_{n-m}(kr_j) e^{-i(m-y)\theta_j}, \]

where \( x_i - x_j = R_i \cos x_{ij} \) and \( y_i - y_j = R_i \sin x_{ij} \) (see [6], Fig. 1 for a diagram showing notation). Hence,

\[ -k^2 \sum_{m,n} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} i^n C_n^{(1)} H_{n-m}(kr_j) e^{i(m-y)\theta_j} L_m(r_j, \theta_j), \]

where \( L_n \) is defined by Eq. (4.5) and we have used the short-hand notation

\[ \sum_{m,n} \equiv \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}. \]

Thus, from Eqs. (5.8)–(5.10), we obtain

\[ u_2^{(2)}(\mathbf{r}) = m_0^2 \sum_{m,n} C_n^{(1)} \Omega_{mn}(\mathbf{r}) \] (5.12)
with

$$\Omega_{mn} = i^n \sum_{l=1}^{N} \sum_{j=1}^{N} e^{ik_{r}L_{n-m}(kr_{j})}e^{j(n-m)\gamma_{l}L_{m}(r_{j}, \theta_{l})}.$$  

To compute the ensemble average of $\Omega_{mn}$, we need a conditional probability. We use $p(r_{i}) = n_{0}/N$ (as above) and $p(r_{i}|r_{j}) = (n_{0}/N)H(R_{i} - b)$. This result is exact when $b \geq 2a$ to ensure that the circular scatterers do not overlap during the averaging. Then, as the scatterers are indistinguishable, we obtain

$$\langle \Omega_{mn} \rangle = i^{n}n_{0}^{2} N - \frac{1}{N} \int_{R_{i}} L_{i}(r_{i}, \theta_{i}) \int_{R_{j}} H(R_{j} - b)e^{ik_{r}L_{n-m}(kr_{j})}e^{j(n-m)\gamma_{l}L_{m}(r_{j}, \theta_{l})} dV_{2} dV_{1}.$$  

As before, we replace $e^{ik_{r}L_{i}}$ by $e^{j(n-m)\gamma_{l}L_{m}}$ with $|\gamma_{l}| > 0$, we let $N \to \infty$ so that $B_{N}$ becomes the half-plane $x > 0$, we evaluate the integrals and then we let $\gamma_{l} \to k$. To begin, we note that the inner integral is very similar to Eq. (A.1); we take its value as

$$\frac{2}{ik} (-1)^{n-m} (\kappa + k) e^{i\kappa_{m}} + \pi i e^{i\kappa_{m}} \gamma_{m}^{(n-m)(\kappa \ell)} x^{(n-m)(\kappa \ell)} (\kappa \ell)$$  

with $\gamma_{m}$ defined by Eq. (A.3). This result is exact when $x_{1} > b$ but it is an approximation when $0 < x_{1} < b$ (see the discussion below Eq. (A.2)). Using Eq. (5.14) for all $x_{1} > 0$, we obtain

$$\langle \Omega_{mn} \rangle = \frac{2n_{0}^{2}}{ik(k^{2} - k^{2})} \left( (\kappa + k) \gamma'_{m}(k) + \pi i \gamma_{m-n}^{(n-m)(\kappa \ell)} x^{(n-m)(\kappa \ell)} (\kappa \ell) \right),$$  

where

$$\gamma'_{m}(k) = i^{m} \int_{x_{1} > 0} e^{i\kappa_{m}} L_{m}(r_{1}, \theta_{1}) d\mathbf{x}_{1} d\mathbf{r}_{1} = \frac{2\gamma_{m}^{(1)}(k^{2})}{k^{2} - k^{2}} \left\{ \frac{\kappa + k}{ik} e^{i\kappa_{m}} + \pi i e^{i\kappa_{m}} \gamma_{m}^{(n-m)(\kappa \ell)} x^{(n-m)(\kappa \ell)} (\kappa \ell) \right\} + \frac{\gamma_{m}^{(1)}}{k^{2}} e^{i\kappa_{m}} \int_{0}^{\pi} f_{m}(k \ell_{1}) \gamma^{(1)}_{m}(k \ell_{1}) \ell_{1} d\ell_{1}. \quad (5.15)$$  

Here, we have used Eqs. (4.7) and (4.11), and made calculations similar to Eqs. (A.2) and (A.6). Letting $\kappa \to k$, we obtain

$$\lim_{\kappa \to k} \langle \Omega_{mn} \rangle = n_{0}^{2} k^{-2} \left\{ [1 + \pi i (kb)]^{2} d_{n-m}(kb) \right\} \gamma''_{m}(k) - 2k \gamma''_{m}(k),$$  

where we have used $\gamma''_{m}(kb) = 2i/\pi$ and $\gamma''_{m}(kb) = kbd_{m}(kb)$. Letting $\kappa \to k$ in Eq. (5.15) gives

$$\gamma''_{m}(k) = k^{2} e^{i\kappa} (p'_{m}^{(0)} + (2i\kappa - 1)p'_{m}^{(1)}),$$  

where

$$p'_{m}^{(0)} = \frac{1}{2} \pi (ka)^{2} j_{m-1}(ka) j_{m+1}(ka), \quad p'_{m}^{(1)} = -ic_{m}^{(1)}$$  

and we have used Eq. (A.7). Differentiating Eq. (5.15) with respect to $\kappa$ gives

Fig. 1. The function $\phi_{0}(k\alpha)$: when $b = 2a$, the imaginary part of the effective wavenumber, $K_{r}$, is positive for $0 < \phi < \phi_{0}(ka)$. 


\[ \mathcal{G}'_{m}(\kappa) = \frac{4\pi \mathrm{C}^{(1)}_{m}}{(k^{2} - \kappa^{2})^{2}} \left\{ \frac{k + k}{\kappa} e^{i\kappa x} \nabla'_{m}(\kappa a) + \frac{2\mathrm{C}^{(1)}_{m}}{k^{2} - \kappa^{2}} \left( \frac{i}{\kappa} e^{i\kappa x} \nabla'_{m}(\kappa a) + a \cdot \nabla'_{m}(\kappa a) \right) \right\} + \frac{\pi^{2} e^{i\kappa x}}{4} \int_{0}^{a} |i \nabla_{m}(\kappa r_{1}) + r_{i} \nabla_{m}(\kappa r_{1}) \Lambda^{(1)}_{m}(\kappa) r_{j} r_{1} dr_{1}. \]

In the limit \( \kappa \to k \), we find (after some calculation)

\[ 2k \mathcal{G}'_{m}(k) = k^{2} e^{i\kappa x} \left\{ q^{(0)}_{m} + (2i\kappa x - 1)q^{(1)}_{m} + (\kappa x)^{2}q^{(2)}_{m} \right\}, \]

where \( q^{(1)}_{m} = i e^{i\kappa x} \) and \( q^{(2)}_{m} = 2iC^{(1)}_{m} \) and \( q^{(0)}_{m} = \pi(ka)^{2} C^{(1)}_{m} |d_{m}(ka) - \frac{\nabla'_{m}(ka)}{\kappa^{2}}| + p^{(0)}_{m} + \frac{\pi^{2} k^{2}}{2} \int_{0}^{a} f_{m}(kr) \Lambda^{(1)}_{m}(kr) r^{2} dr; \)

here, we have used Eqs. (4.8) and (A.7). Simplification gives

\[ q^{(0)}_{m} = 2\pi(ka)^{2} C^{(1)}_{m} \langle d_{m}(ka) - f_{m}(ka) H_{m}(ka) + (i/\pi)[1 - m^{2} / (ka)^{2}] \rangle + [1 + \pi(ka)^{2} d_{m}(ka)] p^{(0)}_{m} + 2\pi \int_{0}^{a} [f_{m}(x)]^{2} x^{3} dx. \]

Substituting Eqs. (5.17) and (5.18) in Eq. (5.16) gives

\[ \lim_{\kappa \to k} \langle \Omega_{mn} \rangle = \frac{\pi^{2} k^{4}}{4} e^{i\kappa x} \left\{ q^{(0)}_{m} + (2i\kappa x - 1)q^{(1)}_{m} + (\kappa x)^{2}q^{(2)}_{m} \right\}, \]

where \( p^{(0)}_{m} = [1 + \pi(i(kb)^{2} d_{n-m}(kb)] p^{(0)}_{m} - q^{(i)}_{m}, j = 0, 1. \)

Next, we sum over \( m \). From Eqs. (2.21) and (4.8), we have

\[ \sum_{m=-\infty}^{\infty} C^{(1)}_{m} = \frac{\pi}{4} (ka)^{2}. \]

Hence, using Eq. (2.23), we obtain

\[ \sum_{m=-\infty}^{\infty} p^{(1)}_{m} = \pi(ka)^{2} + \pi(kb)^{2} \sum_{m=-\infty}^{\infty} C^{(1)}_{m} d_{n-m}(kb). \]

Then, from Eq. (5.12), we find that

\[ \langle u^{(2)}(r) \rangle = e^{i\kappa x} \left\{ m^{0}_{0}(\phi^{2}/4) \left[ P_{1} + (2i\kappa x - 1) Q_{1} + (\kappa x)^{2} R_{1} \right] \right\}, \]

where

\[ P_{1} = \frac{4i}{\pi^{2} (ka)^{4}} \sum_{m} C^{(1)}_{m} p^{(0)}_{m} = \left( \frac{kb}{i(ka)^{2}} \sum_{m} p^{(0)}_{m} / (ka) d_{n-m}(kb) + \frac{1}{\pi(ka)^{2}} \sum_{m} q^{(0)}_{m} \right), \]

\[ Q_{1} = \frac{4i}{\pi^{2} (ka)^{4}} \sum_{m} C^{(1)}_{m} p^{(1)}_{m} = - \frac{1}{2} + \frac{4i}{\pi(ka)^{2}} \sum_{m} C^{(1)}_{m} c^{(1)}_{n} d_{n-m}(kb), \]

\[ R_{1} = - \frac{4i}{\pi^{2} (ka)^{4}} \sum_{m} C^{(1)}_{m} q^{(2)}_{m} = - \frac{1}{2}. \]

We add Eqs. (5.11) and (5.21) to the right-hand side of Eq. (5.6), giving the approximation

\[ \langle u(x) \rangle = e^{i\kappa x} \left\{ 1 + m_{0}(\phi/4)(1 + m_{0}[P_{0} - Q_{0}] + m_{0}\phi[P_{1} - Q_{1}]) - i\pi m_{0}(\phi/2)(1 - m_{0} Q_{0} - m_{0}\phi Q_{1}) - (\kappa x)^{2} m^{2}_{0} \phi^{2}/8 \right\}. \]

When this is compared with Eq. (5.7), we find that \( A_{1} = \phi/4 \) and \( k_{1} = -k\phi/2 \), as before, the terms in \( x^{2} \) agree,

\[ A_{2} = (P_{0} - Q_{0} + \phi[P_{1} - Q_{1}]) \phi/4 \quad \text{and} \quad k_{2}/k = Q_{0}\phi/2 + (4Q_{1} + 1) \phi^{2}/8. \]

Hence, we obtain the approximation

\[ K^{2}/k^{2} \simeq 1 - m_{0}\phi + m^{2}_{0}Q_{0} + m^{3}_{0}\phi^{3}(Q_{1} + 1/2). \]

The last term implies that \( \delta_{2} \) in Eq. (1.2) is given by

\[ \delta_{2} \simeq k^{2} m^{2}_{0}(\kappa a)^{2}/4(Q_{1} + 1/2) = 4\pi m^{3}_{0} b^{2} \sum_{m} C^{(1)}_{m} C^{(1)}_{n} d_{n-m}(kb), \]

where we have used Eq. (5.23). Then, using Eq. (4.8), we find precise agreement with Eq. (2.31). We also find the approximation

\[ A \simeq 1 + m_{0}\phi/4 + m^{2}_{0}(P_{0} - Q_{0})\phi/4 + m^{3}_{0}(P_{1} - Q_{1})\phi^{3}/4. \]

The term involving \( m^{3}_{0}\phi \) in Eq. (5.25) differs from the Foldy approximation, Eq. (2.27), by the presence of \( P_{0} \). This quantity came from an exact calculation of \( \langle u^{(1)} \rangle \) (see Eq. (5.11)). Specifically, \( P_{0} \) came from a certain integral over the interior of a typical scatterer (see Eq. (A.8)). We have checked that \( P_{0} \) does not vanish identically (see Eq. (A.10)).
5.3. Comparison with “point scatterers”

Similar results have been obtained by Maurel [5] for uncorrelated configurations of “point scatterers”. For the deterministic problem, the governing partial differential equation is

\[(\nabla^2 + k^2)u = k^2m_0\pi a^2 u(r) \sum_{i=1}^{N} \delta(r - r_i),\]

where \(\delta\) is the Dirac delta function. The factor \(\pi a^2\) ensures agreement with the right-hand side of Eq. (3.5): \(\int u dV \approx \pi a^2 u(r)\) when \(D\) is small. Maurel [5] found the approximation

\[u(r) = e^{ikr} \left\{ [1 + m_0(\phi/4)][1 - m_0\phi + m_0\phi/2] - i\kappa m_0(\phi/2)[1 - m_0\phi + m_0\phi/2] - (\kappa a)^2 m_0^2 \phi^2 / 8 \right\}, \tag{5.26}\]

where \(\phi = \pi(i/4)(ka)^2 H_0(ka)\).

From (5.26), the estimates

\[A \simeq 1 + m_0\phi/4 - m_0^2(\phi/4)^2 - m_0^2\phi^2/8,\]

\[K/k \simeq 1 - m_0\phi/2 + m_0^2(\phi/2)^2 - m_0^2\phi^2/8\]

follow; squaring the last formula gives \(K^2/k^2 \simeq 1 - m_0\phi + m_0^2\phi^2\), with no term in \(m_0^2\phi^2\).

Intuitively, the point-scatterer limit should correspond to \(ka \to 0\). If we compare Eqs. (5.24) and (5.26), we see that we will have agreement if

\[P_0 - Q_0 \sim -\phi, \quad P_1 - Q_1 \sim \frac{1}{2}, \quad Q_0 \sim \phi \quad \text{and} \quad Q_1 \sim -\frac{1}{2}; \tag{5.27}\]

The third of these follows quickly from the definition of \(Q_0\), whereas the first follows from the fact that \(P_0\) is smaller than \(Q_0\) as \(ka \to 0\) (see Eq. (A.11)). The second and fourth of (5.27) involve \(P_1\) and \(Q_1\), and these depend on \(kb\) as well as \(ka\). From Eq. (5.23), we find that

\[Q_1 + 1/2 \sim -(i/4)(kb)^2 d_0(kb) \quad \text{as} \quad ka \to 0;\]

the limiting value vanishes when \(kb \to 0\), which is the appropriate limit for uncorrelated scatterers. Finally, consider \(P_1\), defined by Eq. (5.22) as a double-sum term plus a single-sum term. In the double sum, the dominant contributions come from \(\mathcal{J}_0 \sim 1, p_0(0) \sim (\phi/8)(ka)^4\) and \(p_{0,1}^{(0)} \sim p_0^{(0)} / 2\) as \(ka \to 0\). Hence, asymptotically, the double-sum term is

\[-i(\phi/8)(ka)^2(kb)^2(d_0(kb) + d_1(kb)) \to 0 \quad \text{as} \quad ka \to 0.\]

Making use of Eqs. (5.19), (5.20),

\[\sum_{m=-\infty}^{\infty} p_m^{(0)} = 0 \quad \text{and} \quad \sum_{m=-\infty}^{\infty} |f_m(x)|^2 = \frac{1}{2},\]

the single-sum term in Eq. (5.22),

\[\frac{1}{\pi(ka)^4} \sum_{m=-\infty}^{\infty} a_m^{(0)} = -\frac{1}{4} (ka)^{2} + \sum_{m=-\infty}^{\infty} \{ \mathcal{J}_m(ka)\gamma_m(ka) + id_m(ka)p_m^{(0)} \}, \tag{5.28}\]

exactly, where \(\gamma_m(x) = (i\pi/2)x^2[d_m(x) - f_m(x)H_0(x)] + m^2/2\). We have \(\gamma_0(x) \sim x^2/2, d_0(x) \sim H_0(x), \gamma_m(x) \sim |m|+m^2/2\) and \(x^2d_m(x) \sim 2i |m| / \pi\) for \(m \neq 0\), so that the largest terms in Eq. (5.28) are \(O((ka)^4)\) as \(ka \to 0\). Thus, all of Eq. (5.27) hold in the limit \(ka \to 0\), provided \(kb \to 0\) too. It follows that we recover the point-scatterer results if we allow the scatterers to shrink \((ka \to 0)\) and to become uncorrelated.

6. Numerical results and summary

In this section, we give some quantitative illustrations of the analytical results above. However, let us first summarise the various theories. We distinguish between Foldy-type theories and iterative theories.

6.1. Foldy-type theories

The simplest is “basic” Foldy theory: it predicts that \(K^2 = k^2 - 4i m_0f(0)\) and that \(A\) is given by Eq. (1.5); it is first order in \(n_0\); it assumes that the scatterers are statistically independent (possible overlaps are ignored at first-order); it uses the forward-scattered far-field pattern for one scatterer, \(f(0)\), calculated (exactly) taking proper account of the interior wavefield and density differences, if present; and it makes essential use of the Foldy closure assumption.

As a special case, we have “weak Foldy”. This occurs when the scattering from each individual scatterer is weak, meaning that each scatterer is penetrable with a sound-speed that is close to that in the exterior; this closeness is measured by the small parameter, \(m_0\). (For simplicity, we do not permit density differences here). The “weak-Foldy” results are given in Section 2.2. Thus, \(K^2\) is given by Eq. (2.26). Also,
where \( u_m = e^{i k a} \) and \( k_i / k = m_0^2 (\phi/2) \operatorname{Im} Q_0 \), where \( k_i = \operatorname{Im} K, \phi = n_0 \pi a^2 \) is the filling fraction or area fraction occupied by the scatterers, \( Q_0(ka) = \pi (ka)^2 \mathcal{F}(ka) \) and \( \mathcal{F}(ka) \) is given by Eq. (2.25) or Eq. (8.1). Note that
\[
\frac{K_i}{k} = m_0^2 (\phi/4) \frac{\pi}{8} \sum_{n=-\infty}^{\infty} J^2_n(k a)
\]  
(6.2)
which is positive, implying attenuation with \( x \).

Going beyond Foldy, we can seek corrections proportional to \( n_0^2 \). This is more difficult because pair-correlations must be used in order to prevent finite-sized scatterers from overlapping during averaging. The “Linton–Martin” correction to basic Foldy for \( K \) is \( n_0^2 \delta_2 \), with \( \delta_2 \) given by Eq. (2.28). It was derived using the Lax quasicrystalline approximation as closure assumption. Again, as with basic Foldy, there is a weak version: it gives a term in \( n_0^2 m_0^2 \), see Section 2.3, especially Eq. (2.31). The Linton–Martin theory does not give any estimate for \( A \).

Note that Foldy and Linton–Martin theories do not assume explicitly that \( ka \) is small. However, letting \( ka \rightarrow 0 \) gives results for very small scatterers, and these can be compared with results for so-called “point scatterers”; this was done in Section 5.3. In particular, the point-scatterer limit gives
\[
|A| = 1 + m_0 \phi/4 - m_0^2 (\phi/4) \operatorname{Re} \mathcal{A} + m_0^2 \phi^2/8
\]  
(6.3)
and \( K_i/k = m_0^2 (\phi/2) \operatorname{Im} \mathcal{A} \), where \( \mathcal{A} = \pi (i/4)(ka)^2 H_0(ka) \) so that
\[
\operatorname{Re} \mathcal{A} = - (\pi/4)(ka)^2 Y_0(ka) \sim - (1/2)(ka)^2 \log ka,
\]
\[
\operatorname{Im} \mathcal{A} = (\pi/4)(ka)^2 J_0(ka) \sim (\pi/4)(ka)^2.
\]  
(6.4)
(6.5)

6.2. Iterative theories

In this paper, we began with the Lippmann–Schwinger equation, which we solved by iteration for weak scattering. At both first and second–order in \( m_0 \), we obtained exactly the same expressions for \( K \) as those obtained by “weak Foldy” and “weak Linton–Martin”. No closure assumptions were used. However, there is one difference in the result for \( A \):
\[
|A| = 1 + m_0 \phi/4 - m_0^2 (\phi/4) \operatorname{Re} (Q_0 - P_0) + m_0^2 (\phi^2/4) \operatorname{Re} (P_1 - Q_1).
\]  
(6.6)
Thus, the term \( P_0 \) is absent from Eq. (6.1). Also,
\[
\frac{K_i}{k} = m_0^2 (\phi/4) \frac{\pi}{8} \left\{ \sum_{n=-\infty}^{\infty} J^2_n(k a) - \phi (b/a)^2 \sum_{m,n} J_m(k a) J_n(k a) J_{n+m}(k b) \right\}
\]  
(6.7)
this expression can become negative (see Section 6.3). Eqs. (6.6) and (6.7) do reduce to the point-scatterer limits when both \( ka \) and \( kb \rightarrow 0 \).

6.3. Numerical results

The various theories described above give numerical predictions that are close, at least for small values of \( \phi \) and \( ka \). Rather than plot curves that are almost superimposed, we give a few numerical values. In Table 1, we give values of \( |A| \) at \( ka = 1 \) for three values of \( \phi \) and three values of \( m_0 \). In each case, the top value is the Foldy prediction (Eq. (6.1)), the middle value is the iterative point-scatterer result (Eq. (6.3) with the approximation (6.4)) and the bottom value is the iterative finite-size result (Eq. (6.6) with hole radius \( b = 2a \)). Corresponding results for \( \operatorname{Im} K/k \) are given in Table 2: the Foldy prediction is Eq. (6.2), the

<table>
<thead>
<tr>
<th>( m_0 = 0.1 )</th>
<th>( m_0 = 0.5 )</th>
<th>( m_0 = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi = 0.1 )</td>
<td>1.0025</td>
<td>1.0118</td>
</tr>
<tr>
<td>( \phi = 0.2 )</td>
<td>1.0025</td>
<td>1.0128</td>
</tr>
<tr>
<td>( \phi = 0.4 )</td>
<td>1.0025</td>
<td>1.0126</td>
</tr>
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<td>( \phi = 0.2 )</td>
<td>1.0049</td>
<td>1.0236</td>
</tr>
<tr>
<td>( \phi = 0.2 )</td>
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<td>1.0262</td>
</tr>
<tr>
<td>( \phi = 0.4 )</td>
<td>1.0050</td>
<td>1.0260</td>
</tr>
<tr>
<td>( \phi = 0.4 )</td>
<td>1.0099</td>
<td>1.0472</td>
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<tr>
<td>( \phi = 0.4 )</td>
<td>1.0102</td>
<td>1.0550</td>
</tr>
<tr>
<td>( \phi = 0.4 )</td>
<td>1.0102</td>
<td>1.0553</td>
</tr>
</tbody>
</table>

In each case, the top value is the Foldy prediction, the middle value is the iterative point-scatterer result, and the bottom value is the iterative finite-size result (for which \( b = 2a \)).
Table 2
Computed values of $\text{Im} K/k$ at $ka = 1$ for three values of $\phi$ and three values of $m_0$

<table>
<thead>
<tr>
<th>$\phi = 0.1$</th>
<th>$m_0 = 0.1$</th>
<th>$m_0 = 0.5$</th>
<th>$m_0 = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.4721 \times 10^{-4}$</td>
<td>$6.1802 \times 10^{-3}$</td>
<td>$2.0024 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>$3.9270 \times 10^{-4}$</td>
<td>$9.8175 \times 10^{-3}$</td>
<td>$3.1809 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>$1.9587 \times 10^{-4}$</td>
<td>$4.8966 \times 10^{-3}$</td>
<td>$1.5865 \times 10^{-2}$</td>
<td></td>
</tr>
</tbody>
</table>

In each case, the top value is the Foldy prediction, the middle value is the iterative point-scatterer result, and the bottom value is the iterative finite-size result (for which $b = 2a$).

The iterative point-scatterer result is given just below Eq. (6.3) and the iterative finite-size result is Eq. (6.7), again with $b = 2a$. We notice that the last of these, which is presumably more accurate, predicts weaker attenuation, especially as $\phi$ and $m_0$ get larger.

The iterative finite-size prediction for $\text{Im} K$, Eq. (6.7), fails for values of $\phi$ that are sufficiently large. To see this, write Eq. (6.7) as

$$K_1/k = m_0^2 \phi(ka)^2 (\pi/8) \Phi(ka, b/a, \phi).$$

Evidently, $\Phi(ka, b/a, 0) > 0$. It turns out that $\Phi$ remains positive for a range of $\phi$. In Fig. 1, we take $b = 2a$ and plot $\phi_0(ka)$, where $\Phi(ka, 2, 0) > 0$ for $0 \leq \phi < \phi_0(ka)$ and $\Phi(ka, 2, \phi_0) = 0$. The two horizontal asymptotes in the figure can be predicted. Thus, $\phi_0(ka) \sim (a/b)^2$ as $ka \to 0$, whereas, when $b = 2a$, $\phi_0(ka) \sim \frac{48}{\pi^2} [\Gamma(3/4)]^{1/4}$ as $ka \to \infty$ [13]. The figure shows that we cannot use Eq. (6.7) for filling fractions that are too large.

Acknowledgments

The genesis of our collaboration was a GDR meeting on guided waves: we thank Michel Destrade for the invitations to participate. Most of P.A.M.’s work was done while he was visiting ENSTA in Paris: he thanks Eric Lunéville and his colleagues for their hospitality and support.

Appendix A. Evaluation of some integrals

A.1. Evaluation of (5.5)

From Eqs. (4.4) and (4.7), we have

$$\int_{\text{disc}} e^{ikx_1} I_{1}\, dx_1\, dy_1 = - \sum_{n=-\infty}^{\infty} \int_{\text{disc}} H_n(kr_1) e^{i\theta x_2} e^{ikx_1} \, dx_1\, dy_1 = - \frac{2i}{k} \sum_{n=-\infty}^{\infty} C_n^{(1)} \frac{(k + k)e^{ikx} + \pi k e^{i\kappa x} \text{J}_n(ka)}{k^2 - \kappa^2},$$

where $\text{Im} \kappa > 0$ and

$$\text{J}_n(ka) = k a H'_n(ka) f_n(ka) - \kappa a H_n(ka) f'_n(ka).$$

Here, we have evaluated the integral on the right-hand side of (A.1) as on p. 3419 of [6]: the result is exact when $x > a$. Recall that the region disc consists of that part of the half-plane $x_1 > 0$ that is outside the circle $r_1^2 \equiv (x_1 - x)^2 + y_1^2 = a^2$. The method of evaluation in [6] uses Green’s theorem to reduce the double integral to the sum of an integral along $x_1 = 0$ and an integral around the circle $r_1 = a$. When $0 < x < a$, this circle cuts the $y_1$-axis, and then Eq. (A.2) should be regarded as an approximation. More precisely, we have

$$\int_{\text{disc}} H_n(kr_1) e^{i\theta x_2} e^{ikx_1} \, dx_1\, dy_1 = \frac{2}{ik} (-i)^n \frac{(k + k)e^{ikx} + \pi k e^{i\kappa x} \text{J}_n(ka)}{k^2 - \kappa^2} + E_n(x),$$

where $E_n(x) = 0$ for $x \geq a$,

$$E_n(x) = \int_{\gamma} H_n(kr_1) e^{i\theta x_2} e^{ikx_1} \, dx_1\, dy_1, \text{ for } 0 < x < a$$

and $\gamma$ is the segment of the disc $r_1 < a$ with $x_1 < 0$. 

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Now, we let \( \kappa \to k \). Put \( \kappa = k + i \varepsilon \) with \( \varepsilon > 0 \). We have \( e^{\pm \varepsilon x} \approx \pm (1 - i \varepsilon x) e^{i \kappa x} \), \( k^2 - \kappa^2 \approx -2i k \kappa \) and \( J_n(\kappa a) \approx 2i/\pi + i \kappa a^2 d_n(\kappa a) \), with \( d_n \) defined by Eq. (2.13). Hence, in the limit \( \varepsilon \to 0^+ \), the right-hand side of Eq. (A.2) becomes

\[
\frac{e^{i\kappa x}}{ik^2} \sum_{n=-\infty}^{\infty} C_n^{(1)} (1 - 2i k x + \pi i (ka)^2 d_n(ka)).
\] (A.4)

Finally, use of Eqs. (2.21), (2.25) and (4.8) gives the result (5.5).

In a similar way, using Eqs. (4.14), (4.16) and (2.23), we obtain

\[
\lim_{\varepsilon \to 0} \int_{\text{disc}} e^{i\kappa x} I_2 \, dx \, dy_1 = \frac{e^{i\kappa x}}{ikr} \sum_{n=-\infty}^{\infty} C_n^{(2)} \{1 - 2i k x + \pi i (ka)^2 d_n(ka)\}
\]

\[= -\frac{\pi a^2}{4} (ka)^2 e^{i\kappa x} (1 - 2i k x) \mathcal{H}(ka) + \pi a^2 e^{i\kappa x} \sum_{n=-\infty}^{\infty} d_n(ka) C_n^{(2)}.\] (A.5)

A.2. Evaluation of (5.4)

Next, we consider the integral over the disc,

\[
\int_{j_1 a} e^{i\kappa x} I_1 (r, \theta) \, dx \, dy_1 = \frac{\pi^2}{4} e^{i\kappa x} \sum_{n=-\infty}^{\infty} \int_{0}^{a} \Lambda_n^{(1)} (kr) r \, dr,
\] (A.6)

where we have used \( x_1 = x - r \cos \theta_1 \), Eqs. (2.1), (4.4) and (11.1), and integrated over \( \theta_1 \). Substituting Eq. (4.13), and then using Eqs. (4.10) and (4.19), we obtain

\[
\int_{0}^{a} \Lambda_n^{(1)} (kr) J_n (kr) r \, dr = a^2 (ka)^2 d_n (ka) \mathcal{H}_n (ka) - \frac{2ia^2}{\pi} J_n (-1) (ka) J_{n+1} (ka).
\] (A.7)

Finally, making use of Eq. (2.23), we obtain Eq. (5.4).

Similarly, using Eq. (4.17),

\[
\int_{j_1 a} e^{i\kappa x} I_2 (r, \theta) \, dx \, dy_1 = \frac{\pi^2}{32} e^{i\kappa x} \sum_{n=-\infty}^{\infty} \int_{0}^{a} \Lambda_n^{(2)} (kr) r \, dr = -\pi a^2 e^{i\kappa x} \sum_{n=-\infty}^{\infty} d_n (ka) C_n^{(2)} + \frac{\pi^2 i}{4k^2} e^{i\kappa x} \sum_{n=-\infty}^{\infty} F_n (ka).
\] (A.8)

where we have used Eqs. (4.15) and (4.18),

\[\mathcal{F}_n (ka) = k^2 \int_{0}^{a} X_n (r) J_n (kr) r \, dr\]

and \( X_n (r) \) is given by Eq. (4.20).

Adding Eqs. (A.5) and (A.8), the sums containing \( C_n^{(2)} \) cancel leaving

\[
\lim_{\varepsilon \to 0} \int_{\text{disc}} e^{i\kappa x} I_2 \, dx \, dy_1 = \frac{\pi a^2}{4} e^{i\kappa x} \{P_0 (ka) + (2i k x - 1) Q_0 (ka)\}
\]

for \( x > a \), where

\[P_0 (ka) = \frac{\pi i}{(ka)^2} \sum_{n=-\infty}^{\infty} \mathcal{F}_n (ka) \quad \text{and} \quad Q_0 (ka) = (ka)^2 \mathcal{H} (ka).
\]

The expression for \( P_0 \) simplifies a little. From Eq. (4.20), \( X_n = X_n^{(1)} + X_n^{(2)} \), with \( X_n^{(1)} \) and \( X_n^{(2)} \) given by Eqs. (4.21) and (4.22), respectively. We have

\[
k^2 \int_{0}^{a} r \sum_{n=-\infty}^{\infty} X_n^{(1)} (kr) r \, dr = k^2 \int_{0}^{a} r \frac{\sin^2 \pi r}{\pi} \, dr = \frac{i}{4\pi} (ka)^4,
\]

\[
k^2 \int_{0}^{a} X_n^{(2)} (kr) r \, dr = \frac{1}{2} (ka)^4 \mathcal{H}_n (ka) H_n (ka) - d_n (ka) - i/\pi,
\]

\[
k^2 \sum_{n=-\infty}^{\infty} \int_{0}^{a} X_n^{(2)} (kr) r \, dr = \frac{1}{2} (ka)^4 \left[ \sum_{n=-\infty}^{\infty} \mathcal{H}_n H_n + 4i \mathcal{H} - \frac{i}{\pi} \right].
\]

Hence

\[P_0 (ka) = (ka)^2 \left[ \frac{1}{4} - 2i \mathcal{H} (ka) + \frac{\pi i}{2} \sum_{n=-\infty}^{\infty} \mathcal{F}_n (ka) J_n (ka) H_n (ka) \right].\] (A.9)

Notice that
\[ \text{Im} P_0 = \frac{\pi}{2} (ka)^2 \sum_{n=-\infty}^{\infty} J_n(\rho)J_{n+1}(ka) + \frac{\pi}{8} (ka)^4 \]  \hspace{1cm} (A.10)

as \( ka \to 0 \), implying that \( P_0 \) does not vanish identically. In fact, a more detailed calculation shows that

\[ P_0(ka) \sim \frac{1}{4} (ka)^4 \log ka \quad \text{as} \quad ka \to 0. \]  \hspace{1cm} (A.11)

**Appendix B. An integral representation for \( \mathcal{H}(ka) \)**

We give an integral representation for \( \mathcal{H}(ka) \), defined by Eq. (2.25); it is

\[ \mathcal{H}(ka) = \frac{1}{|D_0|^2} \int_{D_0} \int_{D_0} e^{ik \cdot (r-r')} G_0(r,r') \, dV \, dV', \]  \hspace{1cm} (B.1)

where \( |D_0| = \pi a^2 \) is the area of the disc \( D_0 \) and \( k \) is a constant vector with \( |k| = k \). To see this, we put \( k = (k \cos \alpha, k \sin \alpha) \) and then we find that

\[ \int_{D_0} e^{ik \cdot r} G_0(r,r') \, dV = \frac{\pi i}{8k^2} \sum_{n=-\infty}^{\infty} i^n A_n^{(1)}(kr') e^{in(\theta - \theta')} , \]

where we have used Eqs. (2.1), (4.6) and (4.12). (This calculation is similar to that in Section 4.1.2.)

Then, using Eq. (4.15), the right-hand side of Eq. (B.1) becomes

\[ \frac{4i}{\pi^2 (ka)^4} \sum_{n=-\infty}^{\infty} C_n^{(2)} , \]

this reduces to \( \mathcal{H}(ka) \), once Eqs. (4.16) and (2.23) are used.

**References**


