Explicit energy calculation for a charged elliptical plate

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Abstract

Potential problems for thin elliptical plates are solved exactly with emphasis on computation of the electrostatic energy. Expansions in terms of Jacobi polynomials are used.

1. Introduction

Let $\Omega$ denote a thin flat plate lying in the plane $z = 0$, where Oxyz is a system of Cartesian coordinates. The charge distribution on the plate is $\sigma(x)$, where $x = (x, y)$. The potential on the plate is

$$f(x') = \frac{1}{4\pi} \int_\Omega \frac{\sigma(x)}{|x - x'|} \, dx, \quad x' \in \Omega.$$  

(1)

The electrostatic energy, $I$, is given by

$$I = \int_\Omega f(x') \sigma(x') \, dx' = \frac{1}{4\pi} \int_\Omega \int_\Omega \frac{\sigma(x') \sigma(x)}{|x - x'|} \, dx \, dx',$$

where the overbar denotes complex conjugation. In a recent paper, Laurens and Tordeux [1] showed how to calculate $I$ when $\Omega$ is an ellipse and $\sigma(x, y)$ is a linear function of $x$ and $y$. We generalize their result: we allow arbitrary polynomials in $x$ and $y$, and we incorporate a weight function to represent singular behaviour near the edge of the plate.

2. An elliptical plate

When $f$ is given, the function $\sigma$ is infinite at $\rho = 1$, in general. In fact, there is a general result, known as Galin's theorem, asserting that if $f(x, y)$ is a polynomial, then $\sigma$ is a polynomial of the same degree multiplied by $(1 - \rho^2)^{-1/2}$. In particular, if $f$ is a constant, then $\sigma$ is a constant multiple of $(1 - \rho^2)^{-1/2}$. 

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3. Fourier transforms

We start with a standard Fourier integral representation,

$$\frac{1}{|x - x'|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\xi|^{-1} \exp\{i\xi \cdot (x - x')\} \, d\xi,$$

where $\xi = (\xi, \eta)$. Henceforth, we write $\iint$ when the integration limits are as in (3). Thus

$$f(x') = \frac{1}{4\pi} \iint |\xi|^{-1} U(\xi) \exp(-i\xi \cdot x') \, d\xi \quad \text{and} \quad I = \frac{1}{2} \iint |U(\xi)|^2 \, d\xi,$$

where

$$U(\xi) = \frac{1}{2\pi} \int_{\Omega} \sigma(x) \exp(i\xi \cdot x) \, dx.$$

For an elliptical plate, we write the Fourier-transform variable $\xi$ as

$$\xi = (\lambda/a) \cos \psi \quad \text{and} \quad \eta = (\lambda/b) \sin \psi.$$

Then, using (2), $\xi \cdot x = \lambda, \rho \cos(\phi - \psi)$. Hence,

$$\exp(i\xi \cdot x) = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(\lambda, \rho) \cos n(\phi - \psi),$$

where $J_n$ is a Bessel function, $\epsilon_0 = 1$ and $\epsilon_n = 2$ for $n \geq 1$.

In order to evaluate $U(\xi)$, defined by (6), we suppose that $\sigma$ has a Fourier expansion,

$$\sigma(x) = \sum_{m=0}^{\infty} \sigma_m(\rho) \cos m\phi + \sum_{m=1}^{\infty} \tilde{\sigma}_m(\rho) \sin m\phi.$$

Then, using $dx = ab \rho \, d\rho \, d\phi$ and defining

$$S_n[g_n; \lambda] = \int_0^1 g_n(\rho) f_n(\lambda, \rho) \rho \, d\rho,$$

we obtain

$$U(\xi) = ab \sum_{n=0}^{\infty} i^n S_n[\sigma_n; \lambda] \cos n\psi + ab \sum_{n=1}^{\infty} i^n S_n[\tilde{\sigma}_n; \lambda] \sin n\psi.$$

We have $d\xi = (ab)^{-1} \lambda \, d\lambda \, d\psi$ and $|\xi| = (\lambda/b)(1 - k^2 \cos^2 \psi)^{1/2}$, where $k^2 = 1 - (b/a)^2$; $k$ is the eccentricity of the ellipse.

From (4), we obtain

$$f(x) = f_0(\rho) + 2 \sum_{n=1}^{\infty} \left[ f_n(\rho) \cos n\phi + \tilde{f}_n(\rho) \sin n\phi \right]$$

where

$$f_n(\rho) = \frac{b}{2\pi} \sum_{m=0}^{\infty} l_n^m(k) \int_0^\infty J_n(\lambda, \rho) \, S_m[\sigma_m; \lambda] \, d\lambda,$$

$$\tilde{f}_n(\rho) = \frac{b}{2\pi} \sum_{m=1}^{\infty} l_n^m(k) \int_0^\infty J_n(\lambda, \rho) \, \tilde{S}_m[\tilde{\sigma}_m; \lambda] \, d\lambda,$$

$$l_n^m(k) = i^n (-i)^n \int_0^\pi \frac{\cos m\psi \cos n\psi}{\sqrt{1 - k^2 \cos^2 \psi}} \, d\psi,$$

$$l_n^m(k) = i^n (-i)^n \int_0^\pi \frac{\sin m\psi \sin n\psi}{\sqrt{1 - k^2 \cos^2 \psi}} \, d\psi.$$
and we have noticed that |ξ| is an even function of ψ. The integrals \( \mathcal{I}_{m}^{(\nu)} \) and \( \mathcal{I}_{m}^{(\mu)} \) can be reduced to combinations of complete elliptic integrals, \( K(k) \) and \( E(k) \). They are zero unless \( m \) and \( n \) are both even or both odd. (See [5, p. 276] for a discussion of similar integrals.) Explicit formulae for a few of these integrals will be given later.

For the energy, \( I \), (5) gives
\[
I = \frac{1}{2\alpha} \int_{0}^{\pi} \int_{-\pi}^{\pi} \frac{d\psi \, d\lambda}{\sqrt{1 - k^2 \cos^2 \psi}}
\]
\[
= a b^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{I}_{m}^{(\nu)}(\xi) \int_{0}^{\infty} s_{m}[\sigma_m; \lambda] \, \delta_{n}[\sigma_n; \lambda] \, d\lambda + a b^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathcal{I}_{m}^{(\mu)}(\xi) \int_{0}^{\infty} s_{m}[\tilde{\sigma}_m; \lambda] \, \tilde{\delta}_{n}[\tilde{\sigma}_n; \lambda] \, d\lambda. \quad (13)
\]

4. Polynomial expansions

To make further progress, we must be able to evaluate \( \delta_n[g_n; \lambda] \), defined by (8). We do this by expanding \( g_n(\rho) \) using the functions
\[
G_{ij}^{(\nu, \rho)}(\rho) = \rho^n (1 - \rho^2)^\nu P_j^{(\nu)}(1 - 2\rho^2),
\]
where \( P_j^{(\nu)} \) is a Jacobi polynomial. The parameter \( \nu \) controls the behaviour near the edge of the ellipse, \( \rho = 1 \). Thus, when \( \nu = 0 \), \( G_{ij}^{(n, 0)}(\rho) \) is a polynomial; this covers the case discussed in [1]. Setting \( \nu = -\frac{1}{2} \) gives appropriate expansion functions when the goal is to solve (1) for \( \sigma \). We note that Boyd [6, Section 18.5.1] has advocated using the polynomials \( G_{ij}^{(n, 0)}(r) \) as radial basis functions in spectral methods for problems posed on a disc, \( 0 \leq r < 1 \).

The functions \( G_{ij}^{(\nu, \rho)} \) are orthogonal. To see this, note that Jacobi polynomials satisfy
\[
\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta P_i^{(\alpha, \beta)}(x) P_j^{(\alpha, \beta)}(x) \, dx = h_i(\alpha, \beta) \delta_{ij},
\]
where \( h \) is known and \( \delta_{ij} \) is the Kronecker delta; see [7, Section 18.3]. Hence, the substitution \( x = 1 - 2\rho^2 \) gives
\[
\int_{0}^{1} G_i^{(\nu, \rho)}(\rho) G_j^{(\nu, \rho)}(\rho) \frac{\rho \, d\rho}{(1 - \rho^2)^\nu} = 2^{-\nu - 2} h_i(\nu, \nu) \delta_{ij}. \quad (14)
\]

Next, we use Tranter’s integral [8,9] to evaluate \( \delta_n[G_j^{(\nu, \rho)}; \lambda] \):
\[
\int_{0}^{1} j_n(\lambda, \rho) G_j^{(\nu, \rho)}(\rho) \, \rho \, d\rho = \frac{2^\nu}{\lambda^{\nu + 1} b^j} \Gamma(\nu + j + 1) J_{2j-n+\nu+1}(\lambda).
\]

Thus, if we write
\[
\sigma_n(\rho) = \sum_{j=0}^{\infty} \frac{j! \, s_j^n}{2^j \Gamma(\nu + j + 1)} \, G_j^{(\nu, \rho)}(\rho), \quad (15)
\]
where \( s_j^n \) are coefficients, we find that
\[
\delta_n[\sigma_n; \lambda] = \sum_{j=0}^{\infty} \frac{s_j^n}{\lambda^{j+1}} J_{2j-n+\nu+1}(\lambda). \quad (16)
\]

We also expand \( \tilde{\sigma}_n(\rho) \) as (15) but with coefficients \( s_j^n \).

If we substitute (16) in (9), we encounter Weber–Schafheitlin integrals; these can be evaluated. We give a simple example later.

If we substitute (16) in (13), we encounter integrals of the type
\[
\int_{0}^{\infty} \lambda^{-2\mu} J_{p+\mu}(\lambda) J_{q+\mu}(\lambda) \, d\lambda \quad (17)
\]
where \( \mu = n + 1 \), and \( p \) and \( q \) are non-negative integers. The integral (17) is known as the critical case of the Weber–Schafheitlin integral; its value is [7, Eq. 10.22.57]
\[
\frac{\Gamma\left(\frac{1}{2}[p + q + 1]\right) \, \Gamma(2\mu)}{2^{2\mu} \, \Gamma\left(\frac{1}{2}[2\mu + p - q + 1]\right) \, \Gamma\left(\frac{1}{2}[2\mu + q - p + 1]\right) \, \Gamma\left(\frac{1}{2}[4\mu + p + q + 1]\right)}. \quad (18)
\]
5. Three examples

We discuss three examples. In the first, we examine the dependence on the parameter ν but, for simplicity, we ignore any dependence on the angle φ. In the second example, we compare with some results of Roy and Sabina [2] for ν = −1/2. In the third example, we assume that σ(x, y) is a general quadratic function of x and y (so that ν = 0); this extends the calculations in [1], where σ was taken as a linear function.

5.1. Example: dependence on ν

For a very simple example, suppose that σ(x) = (1 − ρ^2)^ν for some ν > −1. Thus, as \( b^{(n, ν)}_0 = 1 \), (15) gives \( s^0_0 = 2^{-\nu} \Gamma(ν + 1) \). All other coefficients \( s^0_j \) and \( \tilde{s}^0_j \) are zero. Then, from (16), \( s_0[σ_0; λ] = s^0_0 \lambda^{−\nu − 1} J_{ν+1}(λ) \). Hence

\[
I(x) = f_0(ρ) = \frac{b s^0_0 j^1_0(k)}{2π} \int_0^∞ \lambda^{−\nu − 1} A(λ) dλ, \quad 0 \leq ρ < 1.
\]

(19)

From (11), we obtain

\[
l_0' = 2 \int_0^{π/2} \frac{dx}{Δ} = 2K(k),
\]

(20)

where \( Δ = (1 − k^2 \sin^2 x)^{1/2} \). From [7, Eq. 10.22.56], the integral in (19) evaluates to

\[
\frac{√π}{2^{ν+1} \Gamma\left(1 + \frac{1}{2} ν\right)} \Gamma\left(\frac{1}{2} + \frac{1}{2} ν\right) F\left(1, -\frac{1}{2} ν; 1, ρ^2\right),
\]

where \( F \) is the Gauss hypergeometric function. Hence

\[
I(x) = \frac{b}{2π} K(k) \frac{√π}{Γ\left(1 + \frac{1}{2} ν\right)} \Gamma\left(\frac{1}{2} - \frac{1}{2} ν\right) F\left(1, -\frac{1}{2} ν; 1, ρ^2\right), \quad 0 \leq ρ < 1.
\]

When ν = −1/2, \( F(1/2, 0; 1; ρ^2) = 1 \) and \( I(x) = \frac{1}{2} bK(k) \), a constant, in accord with Galin's theorem.

When ν = 0, we obtain \( I(x) = (2b/π^2) K(k) E(ρ) \) for \( 0 \leq ρ < 1 \), using [7, Eq. 19.5.2]. Thus, for this particular f, the solution of the integral equation (1) is σ = 1. Although this solution is bounded, we see that adding a small constant to f adds a constant multiple of \( (1 − ρ^2)^{-1/2} \) to σ. In other words, the integral equation (1) has bounded solutions for some f, but these solutions are not typical: singular behaviour around the edge of Ω should be expected.

5.2. Example: comparison with Roy and Sabina

Roy and Sabina [2] consider \( σ(x) = (1 − ρ^2)^{-1/2} g(x, y) \) where \( g(x, y) \) is a quadratic in x and y. As a particular example, let us take \( g(x, y) = 4πx = 4πaρ \cos φ \). Thus, \( n = 1, ν = −1/2 \) and \( j = 0 \) in (15), giving \( s^0_0 = 4πa√π/2; all other coefficients s^0_j \) are zero. Then, from (16), \( s^0_1[σ_1; λ] = s^0_1 λ^{−1/2} J_3/2(λ) \). Hence

\[
I(x) = 2f_1(ρ) \cos φ = \frac{b s^0_1}{π} J^1_0(k) \cos φ \int_0^∞ j_1(λ, ρ) J^{3/2}(λ) dλ, \quad 0 \leq ρ < 1.
\]

(21)

It is shown in Section 5.3 that \( l^1_0(k) = 2(K/E)/k^2 \). From [7, Eq. 10.22.56], the integral in (21) evaluates to \( \frac{1}{2} ρ √π/2 \). Hence \( I(x) = π b x^0 l^1_1(κ) \), in agreement with [2, Eq. (14b)].

5.3. Example: quadratic σ

Suppose that

\[
σ(x) = α_0 + α_1(x/a) + α_2(y/b) + 2α_3(x/a)^2 + 2α_4(xy)/(ab) + 2α_5(y/b)^2
\]

\[
= (α_0 + ρ^2(α_3 + α_5)) + α_1 ρ \cos φ + α_2 ρ \sin φ + (α_3 − α_5) ρ^2 \cos 2φ + α_4 ρ^2 \sin 2φ,
\]

with constants \( α_j \); Laurens and Tordeux [1] have \( α_3 = α_4 = α_5 = 0 \). Then (7) gives

\[
α_0 = α_0 + (α_3 + α_5) ρ^2,
\]

(22)

\[
σ_1 = α_1 ρ, \quad σ_1 = α_2 ρ, \quad σ_2 = (α_3 − α_5) ρ^2 \quad and \quad σ_1 = α_4 ρ^2. \quad All \quad other \quad terms \quad in \quad (7) \quad are \quad absent.
\]

Next, we use \( P^{(n, ρ)}_0 = 1 \) and \( ν = 0 \). These give \( s^0_0 = α_1, s^0_1 = α_2, s^0_2 = α_3 − α_5 \) and \( s^0_2 = α_4 \). For \( s^0_j \), we use \( P^{(0, 0)}_1(x) = P_1(x) = x \), giving

\[
σ_0(ρ) = s^0_0 ρ^0 + s^0_1 G^1_0(0, 0) = s^0_0 + s^0_1(1 − 2ρ^2).
\]
Comparison with (22) gives α_0 = s_0 + s_1 and α_3 + α_5 = -2s_1; these determine s_0 and s_1. Apart from the six mentioned, all other coefficients s_j and s_i are zero.

Then, from (16), we obtain
\[ \lambda \delta_0[\sigma_0; \lambda] = s_0 f_1(\lambda) + s_1 f_3(\lambda), \]
\[ \lambda \delta_1[\sigma_1; \lambda] = s_0 f_2(\lambda), \quad \lambda \delta_1[\sigma_1; \lambda] = s_1 f_4(\lambda), \]
\[ \lambda \delta_2[\sigma_2; \lambda] = s_0 f_5(\lambda), \quad \lambda \delta_2[\sigma_2; \lambda] = s_1 f_6(\lambda). \]

We use these to compute the energy, I, given by (13). We will need the integrals (see (18))
\[ \mathcal{J}_{pq} = \int_0^\infty \frac{1}{\lambda^2} I_{p+1}(\lambda) I_{q+1}(\lambda) \, d\lambda \]
\[ = \frac{\Gamma \left( \frac{1}{2}[p + q + 1] \right)}{\Gamma \left( \frac{1}{2}[3 + p - q] \right) \Gamma \left( \frac{1}{2}[3 + q - p] \right) \Gamma \left( \frac{1}{2}[5 + p + q] \right)} \]
(23)

Thus
\[ \frac{I}{ab^2} = I_{00} + 2 \Re \int_0^\infty |s_0|^2 \mathcal{J}_{00} + 2 \Re \left( s_0^2 \mathcal{J}_{02} + |s_1|^2 \mathcal{J}_{11} \right) + I_{11} \mathcal{J}_{11} + I_{22} \mathcal{J}_{22}. \]
(24)

From (23), we obtain
\[ \mathcal{J}_{00} = \frac{4}{3\pi}, \quad \mathcal{J}_{11} = \frac{4}{15\pi}, \quad \mathcal{J}_{22} = \frac{4}{35\pi}, \quad \mathcal{J}_{02} = \frac{4}{45\pi}. \]

For \( I_{nn}' \) and \( I_{nn}'' \), we have \( I_{00}' = 2K(k) \) (see (20)), \( I_{11}' = I_{11}'' = I_{22}'. \)

\[ I_{11}' - I_{11}'' = I_{02}' = 2 \int_0^{\pi/2} \frac{\cos 2x}{\Delta} \, dx = \frac{2}{k^2}(k^2 - 2)K(k) + \frac{4}{k^2}E(k), \]
\[ I_{22}' - I_{22}'' = 2 \int_0^{\pi/2} \frac{\cos 4x}{\Delta} \, dx = \frac{32k^2 - 2K + 2K + 16}{3k^4}(k^2 - 2)E, \]
where \( k^2 = 1 - k^2 = (b/a)^2. \) Thus
\[ I_{11}' = 2(K - E)/k^2, \quad I_{11}'' = 2(E - k^2K)/k^2, \]
\[ I_{22}' = 2(8k^2 - 1)K + 4(k^2 - 2)E/(3k^4), \]
\[ I_{22}'' = 8(2 - k^2)E - 2k^2K/(3k^4). \]

One can check that these all have the correct limiting values as \( k \to 0. \)

This completes the computation of all the quantities required in (24). In the special case considered by Laurens and Tordeux [1], we have \( s_0^0 = \alpha_0, s_1^1 = \alpha_1, s_2^0 = \alpha_2 \) and \( s_0^1 = s_1^2 = s_2^0 = 0, \) whence
\[ I/(ab^2) = \frac{1}{15\pi} \left[ 5|\alpha_0|^2K + |\alpha_1|^2K - E/k^2 + |\alpha_2|^2E - k^2K/k^2 \right], \]
in agreement with [1, Theorem 1.1].

6. Discussion

The (weakly singular) integral equation (1) arises when Laplace’s equation holds in the three-dimensional region exterior to a thin flat plate \( \Omega \) with Dirichlet boundary conditions on both sides of \( \Omega. \) There are analogous (hypersingular) integral equations when a Neumann boundary condition is imposed. Explicit formulae for \( \sigma \) in terms of \( f \) are known when \( \Omega \) is circular; for a review, see [10].

Expansion methods of the kind used above for problems involving elliptical plates, screens or cracks have a long history. The author’s 1986 paper [5] gives references for Neumann problems, in the context of crack problems. For Dirichlet problems,
see [2–4]. Similar expansion methods have been used recently for the problem of internal wave generation in a continuously stratified fluid by an oscillating elliptical plate [11].

References