ON GREEN'S FUNCTION FOR LAPLACE'S EQUATION IN A RIGID TUBE

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Abstract. A classical problem from potential theory (a point source inside a long rigid tube) is revisited. It has an extensive literature but its resolution is not straightforward: standard approaches lead to divergent integrals or require the discarding of infinite constants. We show that the problem can be solved rigorously using classical methods.

1. Introduction. The basic singular solution of Laplace’s equation, \( \nabla^2 u = 0 \), in unbounded three-dimensional space has the form \( u = A/R \), where \( R \) is the distance between the singularity location (the source point) and the field point, and \( A \) is a constant. This defines Green’s function (or fundamental solution) for Laplace’s equation. It can be used for several purposes, including the derivation of boundary integral equations. Also, further singular solutions (multipoles) can be obtained by forming Cartesian derivatives of the basic point-source solution.

Suppose now that the source is placed inside a cylinder: what is Green’s function for this problem? Here, we are looking for a harmonic function that behaves as \( A/R \) near the source but also satisfies a specified boundary condition on the cylinder and specified conditions far from the source.

For simplicity, suppose the cylinder is a tube, with a circular cross-section of radius \( a \). Introduce cylindrical polar coordinates \( r, \theta \) and \( z \), with the \( z \)-axis along the axis of the tube. Place the source at the origin, so that the problem is axisymmetric (no dependence on \( \theta \)). Later, we shall consider the source to be offset from the axis; this makes the problem slightly more complicated but the main mathematical difficulties are present in the axisymmetric problem.

We have to specify the boundary condition at \( r = a \). Suppose first that \( u = 0 \) at \( r = a \). An early treatment of this (Dirichlet) problem was made by Knight \[9\], using [14, 10.43.20]

\[
\frac{1}{R} = \frac{1}{\sqrt{r^2 + z^2}} = \frac{2}{\pi} \int_0^\infty K_0(kr) \cos k z \, dk,
\]  

(1.1)

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leading to the integral representation

\[ u(r, z) = \frac{2A}{\pi} \int_0^\infty \left( K_0(kr) - \frac{K_0(ka)}{I_0(ka)} I_0(kr) \right) \cos kz \, dk. \quad (1.2) \]

Here, \( I_n \) and \( K_n \) are modified Bessel functions. One can view this approach as being equivalent to using a Fourier transform with respect to \( z \). The formula (1.2) can also be found in [1, Eq. (21)], as can the expansion [1, Eq. (27)]

\[ u(r, z) = \frac{2A}{a} \sum_{n=1}^{\infty} \frac{J_0(\mu_n a)}{(\mu_n a) J_1^2(\mu_n a)} e^{-\mu_n z}, \quad z > 0, \quad (1.3) \]

where \( J_n \) is a Bessel function and \( J_0(\mu_n a) = 0 \), that is, \( \mu_n a = j_{0,n} \), the \( n \)th positive zero of \( J_0 \) [14, 10.21]. For derivations of the offset version of (1.3), see [2, Eq. (7)], [1, Sect. 8] and [17, Sect. 5.298]. In fact, the offset versions of both (1.2) and (1.3) can be found in a long paper by Dougall [3, Sect. 20], published in 1900.

Evidently, the expansion (1.3) is most useful when \( z/a \) is not too small, whereas (1.2) is useful near the source plane at \( z = 0 \).

A striking feature of (1.3) is the exponential decay as \( z \to \infty \). (Note that \( u(r, z) \) is an even function of \( z \), so it is sufficient to assume that \( z > 0 \).) This is one reason why the Fourier transform provides an effective tool. Also, note that the boundary condition \( u(a, z) = 0 \) implies that the only admissible separated solutions are \( J_0(\mu_n a) e^{-\mu_n z} \), having discarded solutions that are unbounded as \( z \to \infty \) or on the axis \( (r \to 0) \). Similar features are present if the Dirichlet boundary condition is replaced by the Robin condition \( \partial u/\partial r + \kappa u = 0 \) at \( r = a \), where \( \kappa \) is a positive constant. The details are worked out in a paper by Peskoff [15]. Thus, (1.2) is replaced by

\[ u(r, z) = \frac{2A}{\pi} \int_0^\infty \left( K_0(kr) - \frac{\kappa K_0(ka) - kK_1(ka)}{\kappa I_0(ka) + kI_1(ka)} I_0(kr) \right) \cos kz \, dk \quad (1.4) \]

and (1.3) is replaced by

\[ u(r, z) = \frac{2A}{a} \sum_{n=1}^{\infty} \frac{J_0(\gamma_n a)}{(\gamma_n a)^2 J_1^2(\gamma_n a)} \frac{e^{-\gamma_n z}}{1 + (\gamma_n/\kappa)^2}, \quad z > 0, \quad (1.5) \]

where \( \gamma_n \) are the positive solutions of \( J_0(\gamma_n a) = (\gamma_n/\kappa) J_1(\gamma_n a) \); see [15] Eqs (10) and (13)]. As Peskoff notes, (1.5) was given earlier [4, Eq. (IV.1-10)].

The limit \( \kappa \to \infty \) (Robin \to Dirichlet) is benign but the limit \( \kappa \to 0 \) (Robin \to Neumann) is singular. We see that immediately in (1.4) because the resulting integral includes

\[ \int_0^\infty \frac{K_1(ka)}{I_1(ka)} I_0(kr) \cos kz \, dk, \quad (1.6) \]

which is divergent: the integrand is \( O(k^{-2}) \) as \( k \to 0 \). The limit \( \kappa \to 0 \) can be analysed using matched asymptotic expansions [16], but we are more interested here in a direct study of the Neumann problem, with \( \partial u/\partial r = 0 \) at \( r = a \). This problem arises naturally...
in the context of the potential flow of an ideal incompressible fluid along a rigid tube, with the flow generated by a point source. The literature on this problem goes back to Lamb in 1926 \[10\]; see Sect. 3.1 for some details.

The evident divergence in (1.6) was overlooked by several authors: see, for example, \[3, \text{Eq. (17)}\] and \[12, \text{Eq. (4)}\]. However, the divergence was noted by Landweber \[11\], and he offered a remedy; see the discussion below (3.17) for some details. Divergence was also noted in \[7\], and a remedy was proposed in terms of generalized functions. In \[7\], Knight’s solution is criticised unfairly; his formula (1.2) does not have convergence difficulties. Such difficulties arise because of the Neumann boundary condition: this condition admits the (even) solution \(u = U|z| + B\), thus making ordinary Fourier transforms problematic.

Here, \(U\) and \(B\) are constants, with \(U\) related to the source strength \(A\); see (2.3) below.

The additive arbitrary constant \(B\) is admissible, and must be eliminated somehow. This is easily done in a direct treatment of the Neumann problem (see (2.2) below), but it can become an embarrassment when the Neumann problem is solved by taking the limit (such as Robin \(\rightarrow\) Neumann): infinitely large constants can emerge before being discarded.

In this paper, we focus on the point-source problem when the tube is rigid (Neumann boundary condition). The source is offset from the tube axis. As we have seen, there are two basic approaches. One involves separation of variables, and leads to an infinite series of modes similar to (1.3). This method was described by Morse and Feshbach for the axisymmetric problem \[13, \text{p. 1262}\]. Extension to the offset problem is straightforward (see Sect. 3); a similar method was used by Smythe \[17, \text{Sect. 5.298}\] when there is a Dirichlet boundary condition \((u = 0)\) on the tube. More interesting, perhaps, is the derivation of a Fourier-type integral representation. We do not do this by a direct Fourier transformation, because of the convergence difficulties already mentioned. Instead, we convert the Morse–Feshbach series into integrals, using a contour-integral method.

We find that Green’s function can be constructed, using classical methods. There is no need for divergent integrals, infinite constants or generalized functions. This opens the door to future applications of Green’s function to a variety of problems involving objects in rigid tubes.

2. The problem. Let \(r, \theta\) and \(z\) be cylindrical polar coordinates. We consider a circular cylindrical tube defined by \(0 \leq r < a\), \(-\pi \leq \theta < \pi\), \(-\infty < z < \infty\). Inside the tube, we have potential flow, \(\nabla^2 u = 0\). The wall of the tube is rigid, \(\partial u / \partial r = 0\) at \(r = a\).

We generate the flow using a point source located at \((r, \theta, z) = (r_0, 0, 0)\),

\[
u(r, \theta, z) \simeq A/R \quad \text{near the source, where} \quad R = (r^2 + r_0^2 - 2rr_0\cos\theta + z^2)^{1/2} \quad (2.1)\]

and \(A\) is a constant. The flow is symmetric, which means \(u(r, \theta, z)\) is an even function of \(z\) and an even function of \(\theta\).

When \(r_0 = 0\), the problem is a little simpler because it is axisymmetric, with no dependence on the angle \(\theta\).

Suppose that

\[
u = Uz + o(1) \quad \text{as} \quad z \to \infty, \quad (2.2)\]

where \(U\) is a constant (velocity). This condition eliminates an arbitrary constant from \(u\). The right-going flux is \(\pi a^2 U\). The flux coming out of the source is \(-4\pi A\), half of which
goes to the right. Conservation gives
\[ \pi a^2 U = -2\pi A. \]  
We assume that \( U \) is given whence \( A = -\frac{1}{2}a^2U \).

3. The Morse–Feshbach approach. As \( u(r, \theta, z) \) is an even function of \( z \), consider a semi-infinite tube with a plane wall at \( z = 0 \) on which the normal velocity is prescribed. Furthermore, \( u(r, \theta, z) \) is an even function of \( \theta \). Then, separation of variables gives
\[ u(r, \theta, z) = u_0(r, z) + 2\sum_{m=1}^{\infty} u_m(r, z) \cos m\theta, \]  
where
\[ u_0(r, z) = c_0z + \sum_{n=1}^{\infty} c^n_0 J_0(\lambda_{0,n}r) \exp^{-\lambda_{0,n}z}, \]  
\[ u_m(r, z) = \sum_{n=1}^{\infty} c^n_m J_m(\lambda_{m,n}r) \exp^{-\lambda_{m,n}z}, \quad m = 1, 2, \ldots, \]  
and \( \lambda_{m,n} \) is given in terms of the positive zeros of \( J'_m' \): \( \lambda_{m,n}a = j'_{m,n} \) where \( J'_m(j'_{m,n}) = 0 \) and \( n = 1, 2, \ldots \) [14, 10.21]. When writing down (3.2) and (3.3), we have discarded solutions that grow exponentially with \( z \), and we have discarded an additive arbitrary constant from (3.2) so as to satisfy (2.2).

Let \( v(r, \theta) = \partial u/\partial z \) evaluated at \( z = 0^+ \). Then, if we write
\[ v(r, \theta) = v_0(r) + 2\sum_{m=1}^{\infty} v_m(r) \cos m\theta \quad \text{with} \quad v_m(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r, \theta) \cos m\theta \, d\theta, \]  
we obtain, for \( 0 \leq r < a \),
\[ v_0(r) = c_0 - \sum_{n=1}^{\infty} \lambda_{0,n}c^n_0 J_0(\lambda_{0,n}r) \quad \text{and} \quad v_m(r) = -\sum_{n=1}^{\infty} \lambda_{m,n}c^n_m J_m(\lambda_{m,n}r) \]  
for \( m = 1, 2, \ldots \). These are Dini–Bessel series; see [20, Sect. 18.3]. The basic orthogonality relation is [14] 10.22.38
\[ \int_0^a r J_m(\lambda_{m,n}r) J_m(\lambda_{m,p}r) \, dr = \frac{\mathcal{J}_{m,n}}{2\lambda_{m,n}^2} \delta_{np}, \]  
with \( \mathcal{J}_{m,n} = \{(\lambda_{m,n}a)^2 - m^2\} J_m^2(\lambda_{m,n}a) \).

We have
\[ c_0 = \frac{2}{a^2} \int_0^a v_0(r) \, r \, dr = \frac{1}{\pi a^2} \int_\mathcal{D} v(r, \theta) \, dA, \]  
where \( \mathcal{D} \) is the tube cross-section (with area \( \pi a^2 \)) and \( dA = r \, dr \, d\theta \). Similarly
\[ c^n_m = -\frac{2\lambda_{m,n}}{\mathcal{J}_{m,n}} \int_0^a v_m(r) J_m(\lambda_{m,n}r) \, r \, dr \]
\[ = -\frac{\lambda_{m,n}}{\pi \mathcal{J}_{m,n}} \int_\mathcal{D} v(r, \theta) J_m(\lambda_{m,n}r) \cos m\theta \, dA, \quad m = 0, 1, 2, \ldots \] (3.6)
Following Morse and Feshbach [13, p. 1262], suppose the flow is generated at a small circular hole in the plane wall, centred on the source location at \((r, \theta) = (r_0, 0)\). The hole is a circular disc of radius \(\varepsilon\), which we denote by \(D_\varepsilon\). We have \(v(r, \theta) = V\), a constant, in \(D_\varepsilon\), with \(v = 0\) in the remainder of \(D\). Matching fluxes give \(V = U(a/\varepsilon)^2\). So, the integrals in (3.5) and (3.6) reduce to integrals over the small disc \(D_\varepsilon\).

As we are interested in the limit \(\varepsilon \to 0\), we can approximate. A simple Taylor-series argument gives

\[
\int_{D_\varepsilon} f(r, \theta) \, dA = \pi \varepsilon^2 f(r_0, 0) + O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0.
\]

(3.7)

Thus, starting with (3.5),

\[
c_0 = \frac{1}{\pi a^2} \int_{D_\varepsilon} V \, dA = \frac{V \varepsilon^2}{a^2} = U,
\]

consistent with (2.2); of course, this calculation is exact and elementary, with no need for (3.7). However, turning to (3.6), we obtain

\[
e_n^m = -\frac{V \lambda_{m,n}}{\pi J_{m,n}} \int_{D_\varepsilon} J_m(\lambda_{m,n} r) \cos m\theta \, dA
\]

\[
= -\frac{\varepsilon^2 V \lambda_{m,n}}{J_{m,n}} \{J_m(\lambda_{m,n} r_0) + O(\varepsilon^2)\}
\]

\[
\to -U a \frac{\lambda_{m,n} a}{J_{m,n}} J_m(\lambda_{m,n} r_0) \quad \text{as} \quad \varepsilon \to 0.
\]

(3.8)

(3.9)

Noting that \(J_{0,n} = (\lambda_{0,n} a)^2\), (3.2) gives

\[
u_0(r, z) = U z - U a \sum_{n=1}^{\infty} \frac{J_0(\lambda_{0,n} r_0) J_0(\lambda_{0,n} r)}{(\lambda_{0,n} a) J_0^2(\lambda_{0,n} a)} e^{-\lambda_{0,n} z}, \quad z > 0.
\]

(3.10)

This formula agrees with [8, Eq. (36)]. It gives the solution for a ring source of radius \(r_0\). Similarly, (3.3) gives

\[
u_m(r, z) = -U a \sum_{n=1}^{\infty} \frac{(\lambda_{m,n} a) J_m(\lambda_{m,n} r_0) J_m(\lambda_{m,n} r)}{\{\lambda_{0,n} a\}^2 - m^2} \frac{J_0^2(\lambda_{m,n} a)}{J_m^2(\lambda_{0,n} a)} e^{-\lambda_{m,n} z}, \quad z > 0.
\]

(3.11)

When substituted in (3.1), we obtain the solution \(u\) due to a point source at an arbitrary position in the tube. The resulting formula can be extracted from a formula obtained by Peskoff [15, Eq. (31)].

3.1. The axisymmetric problem: Point source at the origin. For a point source on the \(z\)-axis at the origin, put \(r_0 = 0\). As \(J_m(0) = \delta_{m,0}\), \(u = u_0\) and (3.10) reduces to

\[
u(r, z) = U z - U a \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{(\lambda_n a) J_0^2(\lambda_n a)} e^{-\lambda_n z}, \quad 0 < r < a, \quad z > 0,
\]

(3.12)

where \(\lambda_n \equiv \lambda_{0,n}\); \(J_1(\lambda_n a) = 0\). In fact, the solution is valid for \(0 \leq r < a, \quad z \geq 0\), except at the source, \(r = z = 0\), where there is a singularity; see (2.1).

The formula (3.12) agrees with the formula for \(\psi\) in [13, p. 1262], but it is older. It was given in a 1926 publication by Lamb [10, Eq. (8)]; he simply wrote down (3.12),
observing that it can be verified that \( u \) satisfies \( \nabla^2 u = 0, \ \partial u / \partial r = 0 \) at \( r = a \),

\[
\int_0^a \frac{\partial u}{\partial z} \bigg|_{z=0} r \ d r = \pi a^2 U, \tag{3.13}
\]

but “\( \partial u / \partial z = 0 \) when \( z = 0 \) for all but infinitesimal values of \( r \)” [10, p. 3]. Equation (3.13) follows readily from

\[
\int_0^a r J_0(\lambda_n r) \ d r = \frac{1}{\lambda_n^2} \int_0^{\lambda_n a} x J_0(x) \ d x = \frac{a}{\lambda_n} J_1(\lambda_n a) = 0,
\]

whereas the last claim is not so clear. We have

\[
v(r) = \frac{\partial u}{\partial z} \bigg|_{z=0} = U + U \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_0^2(\lambda_n a)}.
\]

For this to be zero for \( 0 < r < a \), we want

\[
\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_0^2(\lambda_n a)} = -1, \quad 0 < r < a. \tag{3.14}
\]

A few years after Lamb’s paper, Watson [18, Eq. (9)] used a contour-integral method to show that

\[
\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_0^2(\lambda_n a)} e^{-\lambda_n z} = -1 + \frac{a^2 z}{2(r^2 + z^2)^{3/2}} + O(z) \quad \text{as} \quad z \to 0,
\]

which justifies Lamb’s claim. Here, the \( O(z) \) terms \( \to 0 \) as \( z \to 0 \) for \( 0 \leq r < a \).

4. Integral representations. Our formulas for \( u_m(r, z) \), namely \( 3.10 \) and \( 3.11 \), are exact but they are not very useful when \( z \) is small. It was this property that motivated Watson’s 1930 analysis [18] of Lamb’s axisymmetric solution (3.12). Recall that Watson is famous nowadays for his awesome book on Bessel functions [20] and for the “Watson transformation”, a contour-integral method for recasting a slowly-convergent series into a new series with better convergence properties, a method that he had developed in 1918 [19] for the propagation of radio waves around the Earth. We shall use similar methods for our problem.

The series in (3.10) and (3.11) are similar, but it turns out that \( u_0 \), defined by (3.10), is more difficult to handle; this case includes the solution of the axisymmetric problem with the source on the axis of the tube.

We start by simplifying and standardising the notation. Thus put \( m = \nu, \ j'_\nu,n = \lambda_m,na, \ x = r_0/a, \ X = r/a \) and \( Z = z/a \). Then the series of interest becomes

\[
S_\nu = \sum_{n=1}^{\infty} \frac{j'_\nu,n J_\nu(j'_\nu,n X)J_\nu(j'_\nu,n X)}{(j'_\nu,n)^2 - \nu^2} J_\nu^2(j'_\nu,n) e^{-j'_\nu,n Z}, \tag{4.1}
\]

where \( \nu \) need not be an integer.

It is natural to derive series expansions such as \( S_\nu \) using a contour integration, picking up residues at zeros of \( J'_\nu(w) \) in the complex \( w \)-plane. Define

\[
F^{(m)}_\nu(w) = \left\{ H^{(m)}_\nu(w) J_\nu(Xw) - J^{(m)}_\nu(Xw) \right\} J_\nu(xw) e^{-wZ} \quad m = 1, 2, \tag{4.2}
\]
with $\nu > -\frac{1}{2}$, $Z > 0$ and

$$0 < x < X < 1.$$  \hfill (4.3)

We have

$$F^{(1)}_\nu(w) = i \left\{ Y'_\nu(w) \frac{J_\nu(Xw)}{J'_\nu(w)} - Y_\nu(Xw) \right\} J_\nu(xw) \ e^{-wZ} = -F^{(2)}_\nu(w).$$  \hfill (4.4)

Despite this relation, it is convenient to examine $F^{(1)}_\nu$ and $F^{(2)}_\nu$ separately: it is easy to show that $F^{(1)}_\nu$ decays exponentially as $v = \text{Im } w \to +\infty$ whereas $F^{(2)}_\nu$ decays exponentially as $v \to -\infty$, making use of (4.3). (A similar calculation based on the common formula, (4.4), requires some care because certain cancellations must occur.)

The behaviour near $w = 0$ depends on $\nu$. For $\nu > 0$, we find that $F^{(1)}_\nu(w) = O(1)$ as $w \to 0$. However, when $\nu = 0$, we have

$$F^{(1)}_0(w) = i \left\{ Y_1(w) \frac{J_0(Xw)}{J_1(w)} - Y_0(Xw) \right\} J_0(xw) \ e^{-wZ} \sim -\frac{4i}{\pi w^2} \quad \text{as } w \to 0,$$  \hfill (4.5)

so that $F^{(1)}_0$ has a double pole at the origin. In all cases, $F^{(1)}_\nu(w)$ has simple poles at $w = j'_{\nu,n}$, the positive zeros of $J'_\nu$.

Integrate $F^{(1)}_\nu(w)$ around a contour $C_+$ in the first quadrant of the $w$-plane. The contour $C_+$ has four pieces; one piece, $C_1$, goes from $w = \varepsilon$ to $w = R$, passing below the poles at $w = j'_{\nu,n}$; one piece, $C_R^+$, is a large quarter-circle of radius $R$, going from $w = R$ to $w = iR$; the third piece, $I_+$, goes along the imaginary axis from $w = iR$ to $w = i\varepsilon$; and the fourth piece, $C_\varepsilon^+$, is a small quarter-circle of radius $\varepsilon$, going from $w = i\varepsilon$ to $w = \varepsilon$. The indentation around the origin, $C_R^+$, is redundant when $\nu > 0$ (we can simply put $\varepsilon = 0$) but it will be needed when $\nu = 0$. There is no contribution from $C_R^+$ as $R \to \infty$.

The residue at $w = j'_{\nu,n}$ is

$$i \frac{Y'_\nu(j'_{\nu,n})}{J'_\nu(j'_{\nu,n})} J_\nu(j'_{\nu,n}x)J_\nu(j'_{\nu,n}X) \ e^{-j'_{\nu,n}Z}.$$

Using the Wronskian \textbf{[14]} 10.5.2] $J_\nu(w)Y'_\nu(w) - J'_\nu(w)Y_\nu(w) = 2/(\pi w)$ and $J'_\nu(j'_{\nu,n}) = 0$, we obtain $Y'_\nu(j'_{\nu,n}) = 2/(\pi j'_{\nu,n}J_\nu(j'_{\nu,n})]$. Similarly, as $w^2J'_\nu(w) + wJ''_\nu(w) + (w^2 - \nu^2)J_\nu(w) = 0$, we obtain $J''_\nu(j'_{\nu,n}) = -(j'_{\nu,n})^{-2}((j'_{\nu,n})^2 - \nu^2)J_\nu(j'_{\nu,n})$. Hence the residue at $w = j'_{\nu,n}$ is

$$\frac{2}{\pi i} \frac{J_\nu(j'_{\nu,n}x)J_\nu(j'_{\nu,n}X)}{(j'_{\nu,n})^2 - \nu^2} \ e^{-j'_{\nu,n}Z},$$

which we recognise as the $n$th term in $S_\nu$, (4.1).

On $I_+$, put $w = \rho e^{i\pi/2}$ and use \textbf{[14]} 10.27.6 and 10.27.8

$$J_\nu(\rho e^{i\pi/2}) = e^{i\nu/2} I_\nu(\rho) \quad \text{and} \quad H^{(1)}_\nu(\rho e^{i\pi/2}) = -i(2/\pi) e^{-i\nu/2} K_\nu(\rho),$$  \hfill (4.6)

where $I_\nu$ and $K_\nu$ are modified Bessel functions. Differentiating,

$$J'_\nu(\rho e^{i\pi/2}) = -i e^{i\nu/2} I_\nu(\rho) \quad \text{and} \quad H^{(1)}'(\rho e^{i\pi/2}) = -(2/\pi) e^{-i\nu/2} K'_\nu(\rho).$$  \hfill (4.7)

Hence

$$\int_{I_+} F^{(1)}_\nu(w) \, dw = \frac{2}{\pi} \int_R^\infty \left\{ K'_\nu(\rho) \frac{I_\nu(X\rho)}{I'_\nu(\rho)} - K_\nu(X\rho) \right\} I_\nu(x\rho) \ e^{-iZ\rho} \, d\rho.$$
On $C_\varepsilon^+$ (when we need it), put $w = \varepsilon e^{i\varphi}$. Then, using Cauchy’s residue theorem, and letting $R \to \infty$, we obtain

$$
\frac{2}{\pi} \int_{\infty}^{\varepsilon} \left\{ K'_\nu(\rho) \frac{I_\nu(X\rho)}{I'_\nu(\rho)} - K_\nu(X\rho) \right\} I_\nu(x\rho) \, e^{-iZ\rho} \, d\rho + i \varepsilon \int_{\pi/2}^{0} F^{(1)}_\nu(\varepsilon e^{i\varphi}) e^{i\varphi} \, d\varphi
$$

$$
+ \int_{\varepsilon}^{\infty} F^{(1)}_\nu(w) \, dw = 4 \sum_{n=1}^{\infty} \frac{j_{\nu,n}'(j_{\nu,n}X)J_\nu(j_{\nu,n}X)}{(j_{\nu,n}')^2 - \nu^2} J_\nu^2(j_{\nu,n}) \, e^{-j_{\nu,n}Z}. \tag{4.8}
$$

Similarly, integrate $F^{(2)}_\nu(w)$ around a contour $C_-$ in the fourth quadrant of the $w$-plane. The contour $C_-$ also has four pieces; one piece, $C_1$, is as before, going from $w = \varepsilon$ to $w = R$, passing below the poles at $w = j_{\nu,n}'$ (which are outside $C_-$); one piece, $C_R^-$, is a quarter-circle of radius $R$, going from $w = R$ to $w = -iR$; the third piece, $I_-$, goes along the imaginary axis from $w = -i\varepsilon$ to $w = -i\varepsilon$; and the fourth piece, $C_{-\varepsilon}^-$, is a quarter-circle of radius $\varepsilon$, going from $w = -i\varepsilon$ to $w = \varepsilon$. (Note the direction of integration.) There is no contribution from $C_R^-$ as $R \to \infty$.

On $I_-$, put $w = \rho \ e^{-i\pi/2}$ and use [14] 10.27.6 and 10.27.8

$$
J_\nu(\rho \ e^{-i\pi/2}) = e^{-i\pi\nu/2} I_\nu(\rho), \quad H^{(2)}_\nu(\rho \ e^{-i\pi/2}) = i(2/\pi) e^{i\pi\nu/2} K_\nu(\rho),
$$

$$
J'_\nu(\rho \ e^{-i\pi/2}) = i e^{-i\pi\nu/2} I'_\nu(\rho), \quad H^{(2)}'_\nu(\rho \ e^{-i\pi/2}) = -(2/\pi) e^{i\pi\nu/2} K'_\nu(\rho),
$$

whence

$$
\int_{I_-} F^{(2)}_\nu(w) \, dw = \frac{2}{\pi} \int_{R}^{\varepsilon} \left\{ K'_\nu(\rho) \frac{I_\nu(X\rho)}{I'_\nu(\rho)} - K_\nu(X\rho) \right\} I_\nu(x\rho) \, e^{iZ\rho} \, d\rho.
$$

On $C_\varepsilon^-$ (when we need it), put $w = \varepsilon e^{i\varphi}$. Then, using Cauchy’s theorem, and letting $R \to \infty$, noting that there are no singularities inside $C_-$, we obtain

$$
\frac{2}{\pi} \int_{\infty}^{\varepsilon} \left\{ K'_\nu(\rho) \frac{I_\nu(X\rho)}{I'_\nu(\rho)} - K_\nu(X\rho) \right\} I_\nu(x\rho) \, e^{iZ\rho} \, d\rho + i \varepsilon \int_{-\pi/2}^{0} F^{(2)}_\nu(\varepsilon e^{i\varphi}) e^{i\varphi} \, d\varphi
$$

$$
+ \int_{\varepsilon}^{\infty} F^{(2)}_\nu(w) \, dw = 0.
$$

Adding this result to (4.8), noting that $F^{(1)}_\nu = -F^{(2)}_\nu$, we obtain

$$
S_\nu = \frac{1}{\pi} \int_{\varepsilon}^{\infty} \left\{ K_\nu(X\rho) - K'_\nu(\rho) \frac{I_\nu(X\rho)}{I'_\nu(\rho)} \right\} I_\nu(x\rho) \cos(Z\rho) \, d\rho
$$

$$
- \frac{i \varepsilon}{4} \int_{-\pi/2}^{\pi/2} F^{(1)}_\nu(\varepsilon e^{i\varphi}) e^{i\varphi} \, d\varphi, \tag{4.9}
$$

with $S_\nu$ defined by [4.1]. This holds subject to [14], $0 < x < X < 1$. For $0 < X' < x < 1$, interchange $x$ and $X$ on the right-hand side.
For \( \nu > 0 \), we can let \( \varepsilon \to 0 \) and discard the last integral. Then, reverting to our original notation,
\[
\frac{u_m(r,z)}{U_a} = \frac{a}{\pi} \int_0^\infty \frac{K_m' (ka)}{I_m' (ka)} I_m(kr) I_m(kr_0) \cos kz \, dk
\]
\[- \frac{a}{\pi} \int_0^\infty K_m(kr) I_m(kr_0) \cos kz \, dk,
\]
for \( 0 < r_0 < r < a \) and \( m = 1, 2, 3, \ldots \).

For \( \nu = 0 \), we have to examine the limit \( \varepsilon \to 0 \). Start with the small semicircular indentation made from \( C_\varepsilon^+ \) and \( C_\varepsilon^- \) (having taken due account of the directions of integration); this is the last term in (4.9). We have to refine the estimate (4.5); some calculation gives
\[
F_0^{(1)}(w) = -\frac{4i}{\pi} \left( \frac{1}{w^2} - \frac{Z}{w} \right) + O(1) \quad \text{as} \quad w \to 0.
\]
So, for small \( \varepsilon \),
\[
-\frac{i\varepsilon}{4} \int_{\pi/2}^{\pi/2} F_\nu^{(1)}(\varepsilon e^{i\varphi}) e^{i\varphi} \, d\varphi \sim -\frac{1}{\pi} \int_{\pi/2}^{\pi/2} \left( \frac{e^{-i\varphi}}{\varepsilon} - Z \right) \, d\varphi = Z - \frac{2}{\pi\varepsilon}.
\]

The term in \( Z = z/a \) will cancel with the \( Uz \) term on the right-hand side of (3.10). The term in \( e^{-1} \) must cancel with a contribution from the first term on the right-hand side of (4.9), because the left-hand side of that equation does not depend on \( \varepsilon \).

The first term on the right-hand side of (4.9) splits as
\[
\frac{1}{\pi} \int_{\varepsilon}^\infty K_0(X\rho) I_0(x\rho) \cos (Z\rho) \, d\rho + \frac{1}{\pi} \int_{\varepsilon}^\infty \frac{K_1(\rho)}{I_1(\rho)} I_0(X\rho) I_0(x\rho) \cos (Z\rho) \, d\rho.
\]
We can let \( \varepsilon \to 0 \) in the first integral, but the second integral (which we denote by \( \mathbb{I}_2 \)) diverges as \( \varepsilon \to 0 \): we have \( I_0(\rho) = 1 + O(\rho^2) \) and \( K_1(\rho)/I_1(\rho) = 2\rho^{-2}(1 + O(\rho^2 \log \rho)) \) as \( \rho \to 0 \), so that the integrand is \( O(\rho^{-2}) \) as \( \rho \to 0 \). We subtract the offending term, but there are (at least) two options. One way is to write the second integral as
\[
\mathbb{I}_2 = \frac{1}{\pi} \int_{\varepsilon}^\infty \left( \frac{K_1(\rho)}{I_1(\rho)} I_0(X\rho) I_0(x\rho) - \frac{2}{\rho^2} \right) \cos (Z\rho) \, d\rho + \frac{2}{\pi} \int_{\varepsilon}^\infty \frac{\cos Z\rho}{\rho^2} \, d\rho.
\]
Again, we can let \( \varepsilon \to 0 \) in the first of these integrals, whereas an integration by parts shows that the second integral is
\[
\frac{2}{\pi} \left( \frac{\cos Z\varepsilon}{\varepsilon} - Z \int_{\varepsilon}^\infty \frac{\sin Z\rho}{\rho} \, d\rho \right) = \frac{2}{\pi\varepsilon} - Z + O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0.
\]
Alternatively, we can write
\[
\mathbb{I}_2 = \frac{1}{\pi} \int_{\varepsilon}^\infty \left( \frac{K_1(\rho)}{I_1(\rho)} I_0(X\rho) I_0(x\rho) \cos (Z\rho) - \frac{2}{\rho^2} \right) \, d\rho + \frac{2}{\pi} \int_{\varepsilon}^\infty \frac{1}{\rho^2} \, d\rho,
\]
and then the last integral evaluates to \( 2/(\pi\varepsilon) \). With either option, the singular term cancels with the singular term in (4.11) (as it must).

Assembling the various pieces, we obtain
\[
\frac{u_0(r,z)}{U_a} = \frac{z}{a} - S_0 = -\frac{1}{\pi} \int_0^\infty K_0(X\rho) I_0(x\rho) \cos (Z\rho) \, d\rho - \mathbb{I}_2 + \frac{2}{\pi\varepsilon}.
\]
If we choose (4.12) with (4.13) for \( I_2 \), and revert to our original notation, we obtain

\[
\frac{u_0(r, z)}{Ua} = \frac{z}{a} - \frac{a}{\pi} \int_0^\infty \left\{ \left( K_0(kr) + \frac{K_1(ka)}{I_1(ka)} I_0(kr) \right) I_0(kr_0) - \frac{2}{(ka)^2} \right\} \cos k z \, dk, \tag{4.16}
\]

for \( 0 \leq r_0 < r < a \). The integral term is a (convergent) Fourier (cosine) transform. When the source is on the axis \((r_0 = 0)\), (4.16) agrees with [7, Eq. (7)], obtained there using generalized functions.

Alternatively, if we use (4.14) for \( I_2 \) in (4.15), we obtain

\[
\frac{u_0(r, z)}{Ua} = -\frac{a}{\pi} \int_0^\infty \left\{ \left( K_0(kr) + \frac{K_1(ka)}{I_1(ka)} I_0(kr) \right) I_0(kr_0) \cos k z - \frac{2}{(ka)^2} \right\} \, dk, \tag{4.17}
\]

for \( 0 \leq r_0 < r < a \). This formula seems to be new. An advantage over (4.16) is that the free-stream term, \( Uz \), has been absorbed into the integral.

The term \( 2(ka)^{-2} \) in the integrands of (4.16) and (4.17) can be seen as serving a regularization purpose. It could be replaced by other functions of \( ka \), with exactly the same behaviour as \( ka \to 0 \) (but not affecting convergence as \( k \to \infty \)). Thus, instead of \( 2(ka)^{-2} \), Landweber [11, Eq. (6)] has \( I_0(ka)K_1(ka)/I_1(ka) \) in his version of (4.17). He arrived at this complicated form because he regularized (1.6) by removing a constant term: he replaced \( I_0(kr) \cos k z \) by \( I_0(kr) \cos k z - I_0(ka) \).

For a \( z \)-directed dipole at the origin, we put \( r_0 = 0 \) and then differentiate (4.17) with respect to \( z \), giving the solution

\[
\frac{Ua^2}{\pi} \int_0^\infty \left( K_0(kr) + \frac{K_1(ka)}{I_1(ka)} I_0(kr) \right) k \sin k z \, dk, \tag{4.18}
\]

in agreement with [11, Eq. (10)] and [7, Eq. (12)]. Note that (4.18) is an ordinary improper integral (despite the comments below [7, Eq. (12)]).

We remark that both (4.16) and (4.17) contain the integral

\[
\int_0^\infty K_0(kr)I_0(kr_0) \cos k z \, dk = \frac{1}{R} \mathcal{K}\left( \frac{2\sqrt{rr_0}}{R} \right), \tag{4.19}
\]

where \( R = \sqrt{(r + r_0)^2 + z^2} \) and \( \mathcal{K}(k) \) is the complete elliptic integral of the first kind \([6, 6.672.8]\). When (4.19) is used in (4.16), we find agreement with [8, Eq. (A25)], obtained there by discarding a “diverging constant”.

More generally, if we construct \( u(r, \theta, z) \) from (3.1), using (4.16) or (4.17), together with (4.10), we encounter

\[
\int_0^\infty \left\{ K_0(kr)I_0(kr_0) + 2 \sum_{m=1}^\infty K_m(kr)I_m(kr_0) \cos m\theta \right\} \cos k z \, dk
= \int_0^\infty K_0\left( k\sqrt{r^2 + r_0^2 - 2rr_0 \cos \theta} \right) \cos k z \, dk = \frac{\pi}{2R},
\]

where we have used (1.1) and [20, Sect. 11.3, Eq. (8)], and \( R \) is defined by (2.1). This makes the source term explicit, and so is useful for near-source calculations. In detail,
we have
\[
    u(r, \theta, z) = -\frac{Ua^2}{2R} - \frac{Ua^2}{\pi} \int_0^\infty \left\{ \frac{K_1(ka)}{I_1(ka)} I_0(kr) I_0(kr_0) \cos k z - \frac{2}{(ka)^2} \right\} \mathrm{d}k
    + \frac{2Ua^2}{\pi} \sum_{m=1}^\infty \cos m \theta \int_0^\infty \frac{K_m'(ka)}{I_m(ka)} I_m(kr) I_m(kr_0) \cos k z \mathrm{d}k.
\]

It is unlikely that the remaining integrals can be evaluated analytically but they can be approximated by, for example, replacing \( \cos k z \) by its Maclaurin expansion. This is essentially what Watson did \cite{watson18} when \( r_0 = 0 \) for \cite{fulling72}. If we make a similar calculation, using \( I_0(w) = 1 + \frac{1}{2} w^2 + \cdots \) and \( \cos w = 1 - \frac{1}{2} w^2 + \cdots \), we obtain
\[
    \frac{2u(r, z)}{Ua^2} = \frac{1}{\sqrt{r^2 + z^2}} + \frac{2}{\pi} \int_0^\infty \left\{ \frac{K_1(ka)}{I_1(ka)} I_0(kr) \cos k z - \frac{2}{(ka)^2} \right\} \mathrm{d}k
    \approx \frac{1}{\sqrt{r^2 + z^2}} + \frac{2}{\pi a} \int_0^\infty \left( \frac{K_1(t)}{I_1(t)} - \frac{2}{t^2} \right) \mathrm{d}t + \frac{r^2 - z^2}{2\pi a^3} \int_0^\infty \frac{t^2 K_1(t)}{I_1(t)} \mathrm{d}t.
\]

The integrals seen here define computable numbers; recall that \( K_1(t) \) decays exponentially with \( t \) whereas \( I_1(t) \) grows exponentially with \( t \).

5. Conclusions. We have seen how to calculate Green’s function for Laplace’s equation in a rigid tube, \( G \). There are two classes of formulas for \( G \). One class is modal expansions, using Dini–Bessel series. Such formulas are essentially known, and they converge quickly unless the evaluation point is close to the source plane at \( z = 0 \). In those cases, we need an integral representation; they are given in Sect. \cite{fulling72}. Our derivations are entirely classical, with no divergent integrals. We note that if Green’s function is to be used in the context of a boundary integral method, then it is inevitable that \( G \) will have to be evaluated for small values of \( z \).

References
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