Acoustic scattering by a small obstacle in the time domain

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ABSTRACT: Acoustic scattering by small obstacles in the frequency domain has an extensive literature, going back to Lord Rayleigh in the 19th century. However, analogous results in the time domain are rare. A typical problem concerns the interaction of a sound pulse with a small obstacle. The beginning of a theory for such problems is outlined, using time domain boundary integral equations. A key question is “What does it mean for an obstacle to be ‘small’?”

I. INTRODUCTION

Consider the scattering of sound by a three-dimensional obstacle. In the time domain, the three-dimensional wave equation,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2},$$

has to be solved, subject to boundary and initial conditions. In Eq. (1), the independent variables $x$, $y$, and $z$ have the dimensions of length, $t$ is time, and $c$ is the constant speed of sound. It is convenient (but not essential) to assume that $u$ is dimensionless; it can be taken as a dimensionless form of the velocity potential.

This paper is concerned with scattering by small obstacles, but a definition of “small” is required; small compared to what?

A. Scales

Suppose the obstacle has diameter $2a$. Then all lengths can be scaled by $a$, writing $x = \alpha x$, $y = \alpha y$, and $z = \alpha z$. Similarly, introduce a time scale $T$ (to be chosen later), and write $t = \alpha^2 t$. In terms of the dimensionless independent variables $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, and $\tilde{t}$, the wave equation, Eq. (1), becomes

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2} = \beta^2 \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2},$$

where $\beta = a/cT$ is a dimensionless parameter (it is the ratio of two lengths), and $u(x, y, z, t) = \tilde{u}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$.

In the frequency domain, with $u(x, y, z, t) = \text{Re}[u(x, y, z) e^{-i\omega t}],$ where $\omega$ is the frequency, the time scale is $\omega^{-1}$, leading to the Helmholtz equation for $\tilde{u}(\tilde{x}, \tilde{y}, \tilde{z})$. The relevant dimensionless parameter is $ka$ with $k = \omega/c = 2\pi/\lambda$, where $\lambda$ is the wavelength. Then scattering by a small obstacle means $ka \ll 1$; one also speaks of “low-frequency scattering” or “long-wave scattering.” The associated asymptotic approximations began with Rayleigh; for a careful and detailed treatment, see Ref. 1. Pierce (Ref. 2, Secs. 4–7) gave a small-$ka$ analysis using matched asymptotic expansions and then passed to the time domain “using the prescription $-i\omega \to \partial/\partial t$ and using the correspondence of the factor $e^{i\theta}$ to the time shift $t \to t - r/c$”; it is unclear how this prescription could be justified, given that the basic analysis was done under the assumption that $ka = o(a/c) \ll 1$.

In more complicated frequency-domain problems, there could be more than two relevant length scales. For example, consider scattering by a pair of spheres, each of radius $a$, spaced a distance $b$ apart. In this situation, one possibility is $ka \ll 1$ while $kb$ is not small; another is $kb \ll 1$, implying that $ka \ll 1$ (see Sec. 8.2.7 in Ref. 3); in other words, identifying relevant small parameters depends on the problem of interest. The same issue will be seen in time-domain problems; see Sec. VI.

It is perhaps worth noting that solving time-domain problems by passing to the frequency domain (using Fourier or Laplace transforms with respect to $t$) is a standard technique.4

B. This paper

The scalings introduced above suggest that when $\beta \ll 1$, solutions of Laplace’s equation will be needed (just as they are for frequency-domain problems).1 Indeed, that is what will be seen when considering scattering by a sound-soft obstacle in Sec. IV and by a sound-hard obstacle in Sec. V. To check the results, comparisons with known exact results for a sphere generating waves in a spherically symmetric manner are made; these results are collected in Sec. II. Section VI contains some discussion on how to choose the time scale $T$ (or, equivalently, the length scale $cT$) and some concluding remarks.

II. PULSATING SPHERES: EXACT SOLUTIONS

Consider a pulsating sphere of radius $a$, centered at the origin. Let $r$ denote distance from the origin. Two initial
boundary value problems with spherical symmetry will be considered: the unknown function \( u(r, t) \) satisfies a boundary condition at \( r = a \) and zero initial conditions. In the first problem (Sec. II A), \( u(a, t) \) is specified, and in the second (Sec. II B), \( \partial u / \partial r \) is specified.

### A. Dirichlet problem

For this problem, the boundary condition is \( u(a, t) = d(t) \) for \( t > 0 \), where \( d(t) \) is a specified (dimensionless) function with \( d(0) = 0 \) (for consistency); it is also convenient to take \( d(t) = 0 \) for \( t < 0 \). The problem for \( u \) can be solved exactly. Thus, from Eq. (6.25) in Ref. 4,

\[
\frac{d}{dt} w(r, t) = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} w(r, t) \right) + \frac{1}{r^2} w(r, t)
\]

for \( r > a \) and \( u(r, t) = 0 \) otherwise [which fits with the assumption \( d(t) = 0 \) for \( t \leq 0 \)]. In other words, there is a wavefront at \( r = a + ct \), with \( u = 0 \) ahead of the wavefront and \( u(r, t) \) given by Eq. (3) behind the wavefront, where \( a < r < a + ct \).

In the far field, Eq. (3) can be approximated, giving

\[
\frac{d}{dt} w(r, t) \approx \left( a/r \right) \frac{d}{dr} \left( r \frac{d}{dr} w(r, t) \right) \quad \text{for} \quad r \gg a.
\]

This approximation satisfies the wave equation, but the wavefront has moved slightly.

For an alternative approximation, suppose \( \beta \ll 1 \) \((a \ll cT)\). Then, scaling \( r \) and writing \( r = \tilde{r}, t = (r-a)/c, \) \( c = \tilde{c}(\tilde{r} - \beta \tilde{r}) \tilde{T} \tilde{t} = t \) when \( \beta \ll 1 \) and \( \tilde{r} \) is bounded. Hence, Eq. (3) gives the harmonic approximation

\[
\frac{d}{dt} w(r, t) \approx \left( a/r \right) d(t) \quad \text{for} \quad cT \gg a.
\]

This approximation is useful near the sphere (it satisfies the boundary condition), but the expected wavelike behavior has been lost.

### B. Neumann problem

For this problem, the boundary condition is \( \partial u / \partial r = a^{-1} v(t) \) at \( r = a \), where the given function \( v \) is dimensionless. The exact solution is given by Eq. (6.16) in Ref. 4,

\[
\frac{d}{dt} w(r, t) = \frac{1}{r} \int_0^a e^{-\eta/a} v(\eta) \left( a - \eta \right) d\eta
\]

for \( a < r < a + ct \) and \( u(r, t) = 0 \) otherwise. The substitution \( a - \eta = ct \) gives

\[
\frac{d}{dt} w(r, t) \approx \left( a/r \right) w(t - (r - a)/c) \quad \text{for} \quad a < r < a + ct
\]

with

\[
w(t) = - \frac{c}{a} \int_0^t e^{-c(t-t)/a} v(t) \, dt.
\]

Setting \( w(t) = 0 \) for \( t < 0 \) gives \( u(r, t) = 0 \) for \( r > a + ct \), as desired.

In the far field where \( r \gg a \), the approximation \( u(r, t) \approx (a/r) w(t - r/c) \) is obtained.

### III. SCATTERING BY AN OBSTACLE: PRELIMINARIES

Consider a bounded obstacle \( B \) with smooth boundary \( S \) and diameter \( 2a \). There is an incident field \( u_\text{in}(r, t) \), where \( r = (x, y, z) \) locates an arbitrary point in space.

It is convenient to suppose that the incident wave is a sound pulse, meaning that there is a wavefront with \( u_\text{in} \equiv 0 \) ahead of the wavefront. It is also convenient to arrange that the wavefront first reaches \( S \) at \( t = 0 \), implying that the scattered field \( u \) solves an initial boundary value problem for \( t > 0 \) with zero initial conditions.

Look for a solution in the form of a single-layer potential (see Sec. 9.4.1 in Ref. 4),

\[
u(r, t) = \int_S \mu(r', t - R/c) G_0(R) \, ds(r'),
\]

where \( R = |r - r'| \) and \( G_0(R) = -(2\pi R)^{-1} \) is a fundamental solution for Laplace’s equation. Zero initial conditions are enforced by requiring \( \mu(r', t) = 0 \) for \( t < 0 \) and for all \( r' \in S \).

The density function \( \mu \) has the dimensions of \( (length)^{-1} \) and is to be determined by imposing the boundary condition on \( S \).

In the far field, where \( r = |r| \gg a \),

\[
u(r, t) \approx (a/r) f(t - r/c)
\]

with

\[
f(t) = - \frac{1}{2\pi a} \int_S \mu(r', t) \, ds(r').
\]

Note that \( f \) is dimensionless.

In Secs. IV and V, boundary conditions are applied to Eq. (7), leading to time domain boundary integral equations for \( \mu \). These equations are then approximated, exploiting the fact that the obstacle \( B \) is small.

### IV. SOUND-SOFT OBSTACLE (DIRICHLET PROBLEM)

The boundary condition is \( u = d \) on \( S \), where \( d = -u_\text{in} \).

Hence, the governing integral equation is obtained from Eq. (7),

\[
- \frac{1}{2\pi} \int_S \mu(r', t - c^{-1}|r - r'|) \frac{ds(r')}{|r - r'|} = d(r, t)
\]

for \( r \in S \) and \( t > 0 \). This equation is exact. To obtain an approximate solution, it is assumed now that the scatterer \( B \) is “small.” Introducing dimensionless quantities, \( t - c^{-1}|r - r'| = T(\tilde{r} - \beta \tilde{r}) \tilde{T} = t \), provided \( \beta \ll 1 \). Here, \( r = a \tilde{r}, \) \( r' = a \tilde{r}' \), and it has been noted that \( 0 \leq \tilde{r} - \tilde{r}' \leq 2 \) (because \( r \) and \( r' \) are both points on \( S \), and \( 2a \) is the diameter of \( S \)). The condition \( \beta \ll 1 \) defines what is meant by \( B \) being small: its diameter is small compared to the length scale \( cT \), where \( T \) is an appropriate time scale (see Sec. VI).

Under the same condition, \( \beta \ll 1, d(r, t) \) on the right-hand side of Eq. (10) can be approximated by \( d(r_0, t) \), where \( r_0 \) locates the center of \( B \); the error is approximately
\( |r - r_0| \approx a \). Of course, this assumes that \( d(r_0, t) \) is defined: at the outset, \( d(r, t) \) is defined for \( r \in S \) only. This issue does not arise when \( d(r, t) \) does not depend on \( r \) (as in Sec. II A) or when \( d(r, t) = -u_{in}(r, t) \) (because \( u_{in} \) is defined in the absence of \( B \)).

Thus, writing \( \mu(r', t - c^{-1}|r - r'|) \approx \mu_+(r', t) \) for some function \( \mu_+ \) and using \( d(r, t) \approx d(r_0, t) \), Eq. (10) becomes

\[
\int_S \mu_+(r', t) \frac{dS'}{|r - r'|} = -2\pi d(r_0, t), \quad r \in S, \ t > 0. \tag{11}
\]

This is closely related to a standard boundary integral equation coming from classical potential theory. Thus, from Eq. (8.8) in Ref. 3, \( \mu_+(r, t) = -2\pi d(r_0, t) \mu_0(r) \), where \( \mu_0 \) solves

\[
\int_S \mu_0(r') \frac{dS'}{|r - r'|} = 1, \quad r \in S.
\]

Moreover, the corresponding electrostatic capacity of \( S \) is given by \( C = \int_S \mu_0 \, dS \); see Eq. (8.11) in Ref. 3. Hence, the far field is given by Eq. (8).

\[
\mu(r, t) \approx (C/r) d(r_0, t - r/c) \quad \text{for } r \gg a. \tag{12}
\]

For a sphere, \( C = a^2 \); when the sphere is centered at the origin, we obtain agreement with Eq. (4).

It is instructive to compare with a recent paper by Sini et al.\(^5\) Their Theorem 2.5 contains a formula similar to Eq. (12). In detail, their \( q_0 = 4\pi \mu_0 \), and their \( C_0 = 4\pi C \). Their incident field is generated by a simple source located at a point \( r_* \) [see Eq. (1.1) in Ref. 5] so that

\[
u_{in}(r, t) = \frac{\lambda(t - c^{-1}|r - r_*|)}{4\pi |r - r_*|} \tag{13}\]

for some function \( \lambda \), with \( \lambda(t) = 0 \) for \( t < 0 \). Then, given that \( |r - r_0| \approx r \) in the far field, agreement is found between Eq. (12) and the leading term in Eq. (2.27) from Ref. 5. On the other hand, the physical relevance of the definition of “small” scatterer used in Ref. 5 (and in Sec. 7.1 of Ref. 6) is questionable: in the present paper, the criterion \( \beta \ll 1 \) (\( a \ll cT \)) is used, whereas \( \varepsilon \ll 1 \) is used in Ref. 5 (p. 1084), where \( \varepsilon = a/l \) and \( l \) is a certain fixed length scale. Choices for \( T \) (or \( l = cT \)) will be discussed in Sec. VI, based on properties of the incident field.

V. SOUND-HARD OBSTACLE (NEUMANN PROBLEM)

The boundary condition is \( \partial u/\partial n = a^{-1}v \) on \( S \), where \( v = -a \partial u_{in}/\partial n \) is dimensionless, and the normal vector on \( S, n \), points out of \( B \). Look for a solution as a single-layer potential, Eq. (7), before. Applying the boundary condition gives Eq. (9.41) in Ref. 4,

\[
\mu(r, t) + \int_S \left\{ \frac{R}{c} \mu_+(r', t - R/c) + \mu(r', t - R/c) \right\} \frac{\partial G_0}{\partial n} \, ds(r') = \frac{1}{a} \, v(r, t)
\]

for \( r \in S \) and \( t > 0 \), where \( R = |r - r'|, \ \mu = \partial u/\partial t = T^{-1} \partial u/\partial t \), and

\[
\frac{\partial G_0}{\partial n} = \frac{1}{2\pi R^2} \frac{\partial R}{\partial n} = \frac{1}{2\pi R^3} (r - r') \cdot n(r).
\]

Now, as \( 0 \leq R \leq 2a \), the term \( (R/c)\dot{\mu} \) in the integrand is smaller than the next term \( \mu \) by a factor of \( \beta \). Furthermore, as in Sec. IV, write \( \mu(r', t - c^{-1}|r - r'|) \approx \mu_+(r', t) \) for some function \( \mu_+ \), assuming that \( \beta \ll 1 \). With these approximations and simplifications, it follows that \( \mu_+ \) solves the integral equation

\[
\mu_+(r, t) + \int_S \mu_+(r', t) \frac{\partial G_0}{\partial n} \, ds(r') = \frac{1}{a} \, v(r, t) \tag{14}
\]

for \( r \in S \) and \( t > 0 \). Again, this boundary integral equation occurs in classical potential theory.

The far field is given by Eq. (8) with \( f \) defined by Eq. (9). Hence,

\[
f(t) \approx -\frac{1}{4\pi a^2} \int_S \mu_+(r, t) \, ds(r) = -\frac{1}{4\pi a^2} \int_S v(r, t) \, ds(r). \tag{15}
\]

The second equality comes by integrating Eq. (14) over \( S \) using \( \int_S \frac{\partial G_0}{\partial n} \, ds(r) = 1 \) (Gauss’s integral).

When the approximation \( v(r, t) \approx v(r_0, t) \) can be made in Eq. (15), the estimate

\[
f(t) \approx -\frac{|S|}{4\pi a^2} \, v(r_0, t) \tag{16}
\]

is obtained, where \( |S| \) is the surface area of \( S \). For the spherically symmetric Neumann problem in Sec. II B, where \( v(r, t) \) does not depend on \( r \), \( |S| = 4\pi a^2 \) and

\[
f(t) \approx -v(r_0, t). \tag{16}
\]

This result can be compared with the exact result for a sphere at the origin, Eq. (5), in terms of the function \( w \) defined by Eq. (6). Integrating by parts,

\[
w(t) = -v(t) + v(0) e^{-c/t} + \int_0^t e^{-c(t-t')/a} v'(t) \, dt.
\]

The first term gives agreement with Eq. (16), and the second term is exponentially small. Further integrations by parts show that the integral term is also smaller than the leading term.

Suppose now that \( v = -a \partial u_{in}/\partial n \), where \( u_{in} \) solves the wave equation, Eq. (1), in \( B \). Then the right-hand side of Eq. (15) becomes

\[
\frac{1}{4\pi a} \int_S \frac{\partial u_{in}}{\partial n} \, ds = \frac{1}{4\pi a} \int_B \nabla^2 u_{in} \, dV = \frac{1}{4\pi a^2} \int_B \frac{\partial^2}{\partial t^2} \, u_{in} \, dV,
\]

which is \( O(\beta^2) \) as \( \beta \rightarrow 0 \): as in the frequency domain, soft obstacles scatter more strongly than hard obstacles.
VI. DISCUSSION AND CONCLUSIONS

It has been shown how to estimate scattering by a small obstacle in the time domain, with “small” defined by the criterion

$$\beta = \frac{a}{cT} \ll 1,$$

where the obstacle has diameter $2a$; but what is the time scale $T$ or, equivalently, the length scale $cT$? These scales should come from the incident wave field $u_{\text{in}}(r,t)$. If $u_{\text{in}}$ is dominated by a component with frequency $\omega_0$, say, the choice $T = \omega_0^{-1}$ can be made, much as is done for frequency-domain problems. For sound pulses that are switched on at time $t = t_0$ and then switched off at a later time $t = t_1$, the natural choice is $T = t_1 - t_0$. For the example discussed in Ref. 5, Eq. (13), with a simple source located at $r_*$, another option would be $cT = |r_* - r_0|$, the distance from the source to the obstacle. If there are several obstacles (as in Ref. 5), the geometry of the configuration could be used: putting $cT$ equal to the minimum distance between the obstacles would be an option, leading to approximations for scattering by a cluster of widely spaced obstacles. Clearly, other options are possible, depending on the details of the physical problem of interest. As noted in Sec. IA, related choices of length scales may arise in frequency-domain problems.

As far as the author knows, very little has been done on scattering by a small obstacle in the time domain. The results presented herein are somewhat crude: leading-order approximations have been obtained, but no attempts to develop asymptotic expansions, with associated error estimates, have been made—that remains to be done.

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