Antiplane anisotropic elastodynamics: Babinet’s principle

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A R T I C L E I N F O

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A B S T R A C T

Babinet’s principle is established for certain problems in antiplane anisotropic elasticity. For example, the scattered field behind a finite straight crack is exactly cancelled by the total field behind an aperture in an infinite flat rigid screen, where the crack and the aperture have the same length and the incident wave is the same for both problems. The method used involves Fourier integrals and dual integral equations.

1. Introduction

Babinet’s principle is usually phrased as an equivalence between two problems: the solution to one problem gives the solution to the other problem without further effort. Babinet himself envisaged one problem as being light scattering by a finite object (such as sphere) and the second problem being scattering by a thin screen in the shape of the object’s shadow (a disc) [1]. It is known that Babinet’s principle does not hold with this generality, but it is a rigorous result for problems involving thin flat screens with apertures or coplanar screens in free space. This result was proved by Bouwkamp for acoustic problems (see [2] for details) and subsequently for electromagnetic problems (see [3, §V.3.3] and [4, §11.3] for details).

We are interested in analogous results for elastodynamic problems. We are aware of two relevant papers [5,6]. The first of these gives approximate results whereas the second gives numerical results for several configurations including problems that are of interest here: antiplane motions of anisotropic media with cracks and rigid screens. For such problems, the equation of motion is [7, Eq. (2.9)], [8, Eq. (3)],

\[ C_{35} \frac{\partial^2 u}{\partial x^2} + 2C_{45} \frac{\partial^2 u}{\partial x \partial y} + C_{44} \frac{\partial^2 u}{\partial y^2} + \rho \omega^2 u = 0, \]  

(1)

where \( u(x, y) \) is the out-of-plane displacement, \( C_{35}, C_{45} \) and \( C_{44} \) are the stiffnesses, \( \rho \) is the mass density, and \( \omega \) is the frequency. We place various cracks or rigid strips on the x-axis, and there is an incident plane wave (Section 2.1). As the material is not symmetric about \( y = 0 \) (unless \( C_{45} = 0 \)), we consider incident waves that propagate upwards (with the y-axis pointing upwards) or downwards; the solutions of these problems are related, of course, because a change in the sign of \( y \) is equivalent to a change in the sign of \( C_{45} \). For each incident wave (up or down), we examine four problems: an infinite rigid \((u = 0)\) screen along the x-axis with a finite number of finite apertures; an infinite traction-free screen containing apertures; a finite number of finite screens; and a finite number of cracks along the x-axis. This gives a total of eight problems (Section 3).

Each problem can be reduced to a pair of dual integral equations, using Fourier integrals (Section 2.3). It turns that there are only two distinct pairs. Solving one pair gives the solution of four problems, and so these four problems are related: for example, there is close relation between scattering by a crack \( \Gamma \) and scattering by a rigid screen in which there is an aperture \( \Gamma \). From our knowledge of Babinet’s principle for acoustic problems, this is exactly what we might expect in the present context. The dual integral equations may also be solved numerically [7], but that is not our purpose here.

2. Basic equations

The displacement vector is \( u = (u_x, u_y, u_z) \) with respect to Cartesian coordinates \( (x_1, x_2, x_3) \). For special anisotropic materials, antiplane deformations are possible [9, §3.1]. These have the form \( u_1 = u_2 = 0 \) with \( u_3 \) independent of \( x_3 \). For time-harmonic motions, we have

\[ u_3 = \text{Re}(u(x_1, x_2) e^{-i\omega t}), \]

and then the equation of motion is Eq. (1), in which \( x \equiv x_1 \) and \( y \equiv x_2 \). The non-trivial stresses are

\[ \sigma_{31} = C_{35} \frac{\partial u}{\partial x} + C_{45} \frac{\partial u}{\partial y}, \quad \sigma_{32} = C_{35} \frac{\partial u}{\partial y} + C_{45} \frac{\partial u}{\partial x}. \]

(2)

The stiffnesses are constants satisfying

\[ C_{44} > 0, \quad C_{45} > 0 \quad \text{and} \quad C_{44}C_{35} - C_{45}^2 > 0 \]

(3)

(see [9, Eq. (3.3-8)]). When \( C_{45} = 0 \), the solid is said to be orthotropic. In this case, we have symmetry about the lines \( x = 0 \) and \( y = 0 \). If, in addition, \( C_{44} = C_{35} = \mu \), say, the solid is isotropic; \( \mu \) is the shear modulus.

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2.1. Plane waves

Look for solutions of Eq. (1) in the form \( u(x, y) = e^{i\xi_3 x + \eta y} \), where \( \xi \) and \( \eta \) are parameters. Substitution gives the dispersion relation
\[
\xi^2 C_{33} + 2\eta C_{43} + \eta^2 C_{44} - \rho \omega^2 = 0,
\]
which we can regard as a quadratic equation for \( \eta \), given \( \xi \) and \( \omega \). Following Boström [7, §4.1], let
\[
c_1 = (C_{44} C_{33} - C_{43}^2)/C_{44}, \quad c_2 = C_{33}/C_{44}
\]
and let \( k = \alpha \sqrt{\rho / C_{44}} \). Then Eq. (4) becomes
\[
\eta^2 + 2n_c \xi_2 \xi + \xi^2 C_{33} / C_{44} - k^2 = 0
\]
with solutions \( \eta = \pm \xi_2 \pm \beta(\xi) \), where \( \beta = (k^2 - \xi^2 c_1)^{1/2} \) and, from Eq. (3), \( c_1 > 0 \). We write these solutions as
\[
\eta = +h \quad \text{with} \quad h_+(\xi) = -\xi_2 + \beta(\xi),
\eta = -h \quad \text{with} \quad h_-(\xi) = \xi_2 + \beta(\xi).
\]

2.2. Incident and reflected waves

Let us consider an upward incident wave coming from \( y < 0 \). Suppose first that the plane \( y = 0 \) is rigid, which means \( u = 0 \) at \( y = 0 \). The total field is
\[
u_{\text{rigid}}^{\text{up}}(x, y) = \begin{cases} u_{\text{up}}(x, y) - u_{\text{down}}(x, y), & y < 0, \\ 0, & y = 0, \end{cases}
\]
from Eq. (6), we see that \( u_{\text{rigid}}^{\text{up}}(x, 0+) = 0 \), as required. Similarly, for a downward incident wave coming from \( y > 0 \) onto a rigid plane, the total field is
\[
u_{\text{rigid}}^{\text{down}}(x, y) = \begin{cases} 0, & y < 0, \\ u_{\text{down}}(x, y) - u_{\text{up}}(x, y), & y > 0. \end{cases}
\]
Using Eq. (7), the corresponding tractions are given by
\[
t_{\text{rigid}}^{\text{up}}(x, 0-) = -t_{\text{rigid}}^{\text{down}}(x, 0+) = 2i\beta C_{44} e^{i\xi_3 x},
\]
\[
t_{\text{rigid}}^{\text{down}}(x, 0+) = t_{\text{rigid}}^{\text{up}}(x, 0-) = 0.
\]

Suppose next that the plane \( y = 0 \) is (traction) free. For an upward incident wave coming from \( y < 0 \), we use Eq. (7) and find that the total field is
\[
u_{\text{free}}^{\text{up}}(x, y) = \begin{cases} u_{\text{up}}(x, y) + u_{\text{down}}(x, y), & y < 0, \\ 0, & y > 0, \end{cases}
\]
whereas for a downward incident wave coming from \( y > 0 \), the total field is
\[
u_{\text{free}}^{\text{down}}(x, y) = \begin{cases} 0, & y < 0, \\ u_{\text{down}}(x, y) + u_{\text{up}}(x, y), & y > 0. \end{cases}
\]
In particular, using Eq. (6),
\[
u_{\text{free}}^{\text{up}}(x, 0-) = t_{\text{free}}^{\text{down}}(x, 0+) = 2 e^{i\xi_3 x},
\]
\[
u_{\text{free}}^{\text{down}}(x, 0+) = t_{\text{free}}^{\text{up}}(x, 0-) = 0.
\]

Henceforth, we shall choose an incident plane wave Eq. (6) with \( \xi = \xi_0 \).

2.3. Fourier integrals

In what follows we shall consider problems in which there are various gaps, cracks or apertures in the plane at \( y = 0 \), such as a rigid plane in which there is a finite aperture \( \Gamma \). We represent the outgoing fields using Fourier integrals,
\[
u_{\text{out}}^{\text{down}}(x, y) = \int_{-\infty}^{\infty} B_x(\xi) e^{i\xi_3 x + \eta y} d\xi, \quad y < 0,
\]
\[
u_{\text{out}}^{\text{up}}(x, y) = \int_{-\infty}^{\infty} B_x(\xi) e^{i\xi_3 x + \eta y} d\xi, \quad y > 0,
\]
where \( B_x \) and \( B_{\eta} \) are to be determined. The corresponding tractions are
\[
t_{\text{out}}^{\text{down}}(x, 0-) = -iC_{44} \int_{-\infty}^{\infty} B_{\eta}(\xi) B_x(\xi) e^{i\xi_3 x} d\xi,
\]
\[
t_{\text{out}}^{\text{up}}(x, 0+) = iC_{44} \int_{-\infty}^{\infty} B_{\eta}(\xi) B_x(\xi) e^{i\xi_3 x} d\xi.
\]

3. Eight problems

The first four problems are concerned with finite apertures in infinite screens. The second four problems concern scattering by finite cracks or screens in an infinite solid. The purpose here is to reduce each problem to a pair of dual integral equations. It is found that there are only two distinct pairs of dual integral equations, and this leads to simple relations between the solutions of the problems.

Problem 1 (Aperture in a Rigid Screen. Upward Incident Wave). Start by writing the total field as
\[
u(x, y) = u_{\text{rigid}}^{\text{up}}(x, y) + u_{\text{out}}^{\text{up}}(x, y), \quad y < 0, \quad u_{\text{out}}^{\text{up}}(x, y), \quad y > 0,
\]
with \( u_{\text{rigid}}^{\text{up}} \) defined by Eq. (8). Denote the aperture by \( \Gamma \) (it could consist of a finite number of disjoint pieces of the \( x \)-axis) and the rest of the \( x \)-axis by \( \Gamma^c \) (with \( c \) for complement). See Fig. 1.

We require \( u = 0 \) on both sides of the (unbounded) rigid piece \( \Gamma^c \), and continuity of both \( u \) and \( \sigma_{xy} \) across the aperture \( \Gamma \). As \( u \) is to be continuous across the whole of the \( x \)-axis, we take \( B_x = B_{\eta} = B_r \), say. Then the boundary condition on \( \Gamma^c \) gives
\[
\int_{-\infty}^{\infty} B_r(\xi) e^{i\xi_3 x} d\xi = 0, \quad x \in \Gamma^c.
\]
For continuity of tractions across the aperture, impose
\[
t_{\text{up}}(x, 0-) + t_{\text{out}}^{\text{down}}(x, 0-) = t_{\text{rigid}}^{\text{up}}(x, 0+) + t_{\text{out}}^{\text{up}}(x, 0+).
\]
Using Eqs. (9) and (13), and simplifying, we obtain
\[
\int_{-\infty}^{\infty} \beta(\xi) B_r(\xi) e^{i\xi_3 x} d\xi = \beta(\xi_0) e^{i\xi_0 x}, \quad x \in \Gamma.
\]
Problem 2 (Aperture in a Rigid Screen. Downward Incident Wave). Write the total field as
\[ u(x, y) = u_{\text{rigid}}(x, y) + \begin{cases} u_{\text{down}}^{\text{out}}(x, y), & y < 0, \\ u_{\text{down}}^{\text{up}}(x, y), & y > 0. \end{cases} \]
Proceeding as before, take \( B_i = B = B_{II}, \) say. Then, imposing the boundary condition on \( \Gamma \) and tractions continuity across \( \Gamma^c, \) we find that \( B_{II} \) satisfies the same pair of dual integral equations as \( B, \) Eqs. (14) and (15), so that \( B_1 = B_{II}. \)

Problem 3 (Aperture in Traction-Free Screen. Upward Incident Wave). Write the total field as
\[ u(x, y) = u_{\text{up}}(x, y) + \begin{cases} u_{\text{down}}^{\text{out}}(x, y), & y < 0, \\ u_{\text{down}}^{\text{up}}(x, y), & y > 0, \end{cases} \]
with \( u_{\text{up}} \) defined by Eq. (10). We require \( \sigma_{yy} = 0 \) on both sides of \( \Gamma^c \) and continuity of both \( u \) and \( \sigma_{yy} \) across the aperture \( \Gamma. \) As we want continuity of \( \sigma_{yy} \) across the whole of the \( x \)-axis, inspection of Eq. (13) shows that we should take \( B_1 = -B_\omega = B_{III}, \) say. Then the boundary condition on \( \Gamma^c \) gives
\[ \int_{-\infty}^{\infty} \beta(\xi) B_{III}(\xi) e^{i\xi y} d\xi = 0, \quad x \in \Gamma^c. \] (16)
For continuity of \( u \) across \( \Gamma, \) we impose
\[ u_{\text{up}}^{\text{free}}(x, 0-) + u_{\text{up}}^{\text{free}}(x, 0+) = u_{\text{down}}^{\text{out}}(x, 0+) + u_{\text{down}}^{\text{out}}(x, 0+). \]
Using Eqs. (11) and (12), we obtain
\[ \int_{-\infty}^{\infty} B_{III}(\xi) e^{i\xi y} d\xi = \delta(\xi), \quad x \in \Gamma. \] (17)
Eqs. (16) and (17) form a pair of dual integral equations for the function \( B_{III}. \)

Problem 4 (Aperture in Traction-Free Screen. Downward Incident Wave). Write
\[ u(x, y) = u_{\text{up}}^{\text{free}}(x, y) + \begin{cases} u_{\text{down}}^{\text{out}}(x, y), & y < 0, \\ u_{\text{down}}^{\text{up}}(x, y), & y > 0, \end{cases} \]
Proceeding as before, take \( B_1 = -B_\omega = -B_{IV}. \) Then imposing the boundary condition on \( \Gamma^c \) and tractions continuity across \( \Gamma, \) we find that \( B_{IV} \) satisfies the same pair of dual integral equations, Eqs. (16) and (17), as \( B_{III}, \) so that \( B_{III} = B_{IV}. \)

Problem 5 (Finite Rigid Screen. Upward Incident Wave). Write the total field as
\[ u(x, y) = u_{\text{up}}(x, y) + u_{\text{down}}(x, y) \] with
\[ u_{\text{up}}(x, y) = \begin{cases} u_{\text{down}}^{\text{out}}(x, y), & y < 0, \\ u_{\text{down}}^{\text{up}}(x, y), & y > 0, \end{cases} \] where \( u_{\text{up}} \) (the incident wave) is defined by Eq. (6) and \( u_{\text{down}} \) is the scattered field. The boundary condition is \( u_{\text{up}} = -u_{\text{down}} \) on \( \Gamma. \) In addition we want \( u \) and \( \sigma_{yy} \) to be continuous across \( \Gamma^c. \) As we want \( u \) to be continuous across the whole \( x \)-axis, we take \( B_1 = -B_\omega = -B_{IV}, \) say.
\[ u_{\text{up}}(x, y) = -\int_{-\infty}^{\infty} B_{IV}(\xi) e^{i(\xi y + \lambda_{\text{down}}/2)} d\xi, \quad y < 0, \] (20)
\[ u_{\text{down}}(x, y) = -\int_{-\infty}^{\infty} B_{IV}(\xi) e^{i(\xi y + \lambda_{\text{down}}/2)} d\xi, \quad y > 0. \] (21)
Hence, using Eq. (13), continuity of the traction across \( \Gamma^c \) gives
\[ \int_{-\infty}^{\infty} \beta(\xi) B_{IV}(\xi) e^{i\xi y} d\xi = 0, \quad x \in \Gamma^c. \]
Similarly, the boundary condition on \( \Gamma \) gives
\[ \int_{-\infty}^{\infty} B_{IV}(\xi) e^{i\xi y} d\xi = u_{\text{up}}(x, 0) = e^{i\theta}, \quad x \in \Gamma. \]
Thus \( B_{IV} \) satisfies the same pair of dual integrals equations, Eqs. (16) and (17), as \( B_{III} \) and \( B_{IV}. \)

Problem 6 (Finite Rigid Screen. Downward Incident Wave). Write the total field as
\[ u(x, y) = u_{\text{down}}(x, y) + u_{\text{down}}(x, y), \] with \( u_{\text{down}} \) defined by Eq. (19). Again, take \( B_1 = -B_\omega = -B_{IV}, \) Some calculation shows that \( B_{IV} = B_{IV}. \)

Problem 7 (Finite Crack. Upward Incident Wave). Write the total field as Eq. (18). The boundary condition is \( t_n = -t_{\text{up}} \) on \( \Gamma. \) As we want \( \sigma_{yy} \) to be continuous across the whole \( x \)-axis, we take \( B_1 = -B_\omega = -B_{IV}, \) say.
\[ u_{\text{up}}(x, y) = -\int_{-\infty}^{\infty} B_{IV}(\xi) e^{i(\xi y + \lambda_{\text{down}}/2)} d\xi, \quad y < 0, \] (23)
\[ u_{\text{down}}(x, y) = -\int_{-\infty}^{\infty} B_{IV}(\xi) e^{i(\xi y + \lambda_{\text{down}}/2)} d\xi, \quad y > 0. \] (24)
Continuity of \( u \) across \( \Gamma^c \) gives
\[ \int_{-\infty}^{\infty} B_{IV}(\xi) e^{i\xi y} d\xi = 0, \quad x \in \Gamma^c, \]
whereas the boundary condition on \( \Gamma \) gives
\[ \int_{-\infty}^{\infty} \beta(\xi) B_{IV}(\xi) e^{i\xi y} d\xi = \beta(\xi_0) e^{i\theta}, \quad x \in \Gamma, \]
so that \( B_{IV} = B_1. \)

Problem 8 (Finite Crack. Downward Incident Wave). Again, write the total field as Eq. (22), and take \( B_1 = -B_\omega = -B_{IV}, \) say. Some calculation shows that \( B_{IV} = B_{IV}. \)

4. Discussion
We have seen that the eight problems break into two groups of four. Each group breaks into two pairs, with upgoing and downgoing plane waves in each pair; it turns out that changing the direction of the incident wave does not have a large effect. But each pair in the group represents a different physical problem: this is a hallmark of Babinet’s principle.

Let us consider the group associated with Problems 1, 2, 7 and 8. These lead to the same pair of dual integral equations,
\[ \int_{-\infty}^{\infty} B(\xi) e^{i\xi y} d\xi = 0, \quad x \in \Gamma^c, \]
\[ \int_{-\infty}^{\infty} \beta(\xi) B(\xi) e^{i\xi y} d\xi = \beta(\xi_0) e^{i\theta}, \quad x \in \Gamma. \]
These equations can be solved for \( B \) using numerical methods [7], but we do not need to do that in order to make interesting inferences. Using \( B_1 \) define \( u_{\text{down}} \) and \( u_{\text{down}} \) by Eq. (12) with \( B_1 = B_\omega = B \) therein.
We collect the solutions found above for inspection and comparison.


\[ u_I(x, y) = \begin{cases} u_{up}(x, y) - u_{down}(x, y) + u_{out}^{up}(x, y), & y < 0, \\ u_{up}^{out}(x, y), & y > 0, \end{cases} \]

with plane waves \( u_{up} \) and \( u_{down} \) defined by Eq. (6).

Solution II: Aperture in a rigid screen. Downward incident wave.

\[ u_{II}(x, y) = \begin{cases} u_{out}^{down}(x, y), & y < 0, \\ u_{down}(x, y) - u_{up}(x, y) + u_{out}^{up}(x, y), & y > 0. \end{cases} \]


\[ u_{VII}(x, y) = u_{up}(x, y) + \begin{cases} u_{out}^{down}(x, y), & y < 0, \\ -u_{out}^{up}(x, y), & y > 0. \end{cases} \]

Solution VIII: Finite crack. Downward incident wave.

\[ u_{VIII}(x, y) = u_{down}(x, y) + \begin{cases} u_{out}^{down}(x, y), & y < 0, \\ -u_{out}^{up}(x, y), & y > 0. \end{cases} \]

In these four solutions, \( u_{out}^{out} \) and \( u_{out}^{up} \) are the same, and so they can be eliminated to find connections between the solutions of two problems. For example, Solution VII says that the scattered field \( u_{sc} = u_{VII} - u_{up} \) behind a crack \( y > 0 \) equals \( -u_{out}^{up} \). But, from Solution I, \( u_{out}^{up} \) is also the total field behind an aperture in a rigid screen. This is a classic statement of Babinet’s principle.

Extensions to in-plane and three-dimensional problems are desirable.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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