Long-wave asymptotic approximations are developed for two-dimensional acoustic waves along rigid ducts. The waves are scattered by obstacles, constrictions, bulges and/or bends. Matched asymptotic expansions are used, requiring the calculation of blockage coefficients, which are defined in terms of the solution of related potential-flow problems. The emphasis is on estimating reflection and transmission coefficients, correct to first order in the ratio of the waveguide width to the wavelength. Detailed results are given for sharp bends of arbitrary angle, including right-angled bends and hairpin bends. Applications to multiple scattering by labyrinthine structures are also made.

1. Introduction

Lamb’s *Hydrodynamics* contains an analysis of acoustic scattering by a grating of circular cylinders [1, §307] in which it is assumed that the incident (time-harmonic) waves are long compared to the spacing between the cylinders. A plausible argument is given in which the wave-like character is retained far from the grating (but the geometry is simplified) but, near the grating, the governing Helmholtz equation is replaced by Laplace’s equation (but the exact geometry is retained). The solutions in the two regions are matched to give a long-wave estimate for the reflection coefficient. This approach (which goes back to Rayleigh [2]) can be seen as an intuitive form of the method of matched asymptotic expansions (MAE); according to Lesser & Lewis [3, p. 1665], MAE
‘merely provide a convenient language for expressing one’s physical intuition, much like the man who discovered he’d been speaking prose all his life’.

However, it is known that the intuitive matching described above can lead to significant errors. Quoting Crighton [4, p. 169] (with his emphasis) ‘anything short of a proper mathematical rule for matching is likely to lead to erroneous results, even if the physical basis for heuristic matching may seem clear’. For example, Lamb’s solution for a grating of circular cylinders was known to be erroneous in the 1950s; see [5] for references and details. It turns out that in order to obtain the correct leading-order estimate for the reflection coefficient using MAE, one has to examine second-order contributions, and these contributions are not governed by Laplace’s equation. For more examples of the failure of intuitive matching, see [6].

In this paper, we use MAE for several two-dimensional waveguide problems, extending [5] in two ways. First, we consider perturbations to a parallel-walled sound-hard waveguide. We permit bulges, constrictions and embedded bounded rigid structures, without invoking any geometrical symmetries (contrary to [5]). The goal is to estimate the reflection and transmission coefficients correct to first order in $kb$, where the incident waves have length $2\pi/k$ and the unperturbed waveguide has width $b$. The estimates have two components. One comes by solving an analogous potential-flow problem: this is the intuitive part recognized by Lamb and others. The other component is less intuitive, and turns out to be computable explicitly in terms of a certain area.

Solving the potential-flow problem depends on the geometry but, fortunately, we do not need the complete solution, just the far-field behaviour: the difference in the values of the potential at the two ends of the infinitely long waveguide is a constant, known as the blockage coefficient. This constant can sometimes be found exactly (examples later) and good approximations are available in other situations. One of these concerns a parallel-walled waveguide with a patch of roughness on one of the walls. This is a good example because it permits comparison with an alternative method based on perturbation theory: this theory is given in appendix A. Interestingly, MAE have been used recently to extract good approximations to the electrostatic capacity for a variety of complicated two-dimensional geometries [7].

The first-order estimates obtained do not satisfy energy conservation exactly. Nevertheless, this constraint can be enforced by a simple modification, as shown in §2(b).

Our second extension of [5] concerns waveguides with a sharp bend (§3). Perhaps the simplest examples are right-angled ($90^\circ$, L-shaped) bends and hairpin $180^\circ$ bends. Waves round right-angled bends have a long history, beginning with a 1947 paper by Miles [8]. He set up a scheme that today we would describe as matched eigenfunction expansions. This method is attractive when the geometry is a union of rectangles. For an application to a parallel-walled waveguide with a rectangular bulge, see [9]. For $90^\circ$ bends, see papers by Momoi, such as [10]. (Momoi published about 50 papers in the 1960s in the *Bulletin of the Earthquake Research Institute*, University of Tokyo, many of them on waveguide problems in the context of linear water waves. These papers are freely available.) For $90^\circ$ and $180^\circ$ bends, see [11]. For $90^\circ$ bends, there are later papers in which finite-element methods are used [12,13]. For some other geometries, see [14–16].

For sharp bends, we determine the blockage coefficient exactly using conformal mappings. For $90^\circ$ and $180^\circ$ bends, the results are essentially known; see §4 and 6. For an arbitrary angle, our formula for the blockage coefficient, (5.8), seems to be new. It was obtained using a Schwarz–Christoffel mapping. Although this is a standard technique, there are errors in the literature caused by lack of attention to proper specification of branch cuts. See §5 for details.

One reason for our interest in hairpin bends is because they have been the focus of recent work on ‘labyrinthine metamaterials’, in which a sequence of hairpin bends are joined together [17–22]. For analysis of such structures, finite elements have been used [17–19], as have matched eigenfunction expansions [20] and homogenization theory [21,22]. We regard these labyrinthine problems as examples of multiple scattering, with each bend considered to be a scatterer. In §7, we sketch how long-wave approximations can be obtained, exploiting known results for certain one-dimensional scattering problems.
2. Perturbed straight waveguides

Consider an infinite straight sound-hard waveguide of width $b$. The unknown function $u(x,y)$ is dimensionless and satisfies $(\nabla^2 + k^2)u = 0$ for all $x$ and for $0 < y < b$, with $\partial u/\partial y = 0$ at $y = 0$ and at $y = b$. The wavenumber $k = \omega/c$, where $c$ is the speed of sound and the time dependence $e^{-i\omega t}$ is suppressed. A plane wave $e^{ikx}$ propagates along the waveguide towards $x = +\infty$. This wave together with $e^{-ikx}$ is the only propagating mode if $kb < \pi$; in fact, we shall assume that $kb \ll 1$.

Suppose that there is a scatterer in the vicinity of the origin $O$; it could be a bounded sound-hard obstacle (such as a circle) or a perturbation of the waveguide walls (such as a bulge or a constriction) or a combination of both, as sketched in figure 1. Denote the (unbounded) perturbed domain by $D$ with rigid boundary $\partial D$. The domain $D$ consists of three parts, a finite domain $\Omega$ containing all the perturbations and two semi-infinite waveguides $W_{\pm}$ with parallel walls at $y = 0$ and $y = b$. The boundary of $\Omega$, $\partial \Omega$, consists of three parts: two straight lines, $\ell_0$ at $x = 0$ and $\ell_d$ at $x = d$ for some $d > 0$, together with $\partial \Omega_r$, which consists of all those components of $\partial D$ between $x = 0$ and $x = d$; see figure 1.

The incident wave is partly reflected and partly transmitted by the scatterer. Far away, we have

$$u \sim \begin{cases} T e^{ikx} & \text{as } x \to \infty, \\ e^{ikx} + R e^{-ikx} & \text{as } x \to -\infty, \end{cases}$$

(2.1)

where $R$ and $T$ are the complex reflection and transmission coefficients, respectively. Energy considerations imply that $|R|^2 + |T|^2 = 1$.

Let us seek $R$ and $T$ as power series in $\kappa = kb$, with $\kappa \ll 1$, trying

$$R = \sum_{n=1}^{\infty} \rho_n(i\kappa)^n \quad \text{and} \quad T = 1 + \sum_{n=1}^{\infty} \tau_n(i\kappa)^n,$$

(2.2)

where the coefficients $\rho_n$ and $\tau_n$ do not depend on $\kappa$. (A similar approach is described in [5, Appendix D]; here, we avoid using symmetry properties.) Then, the inner expansion of the outer expansion (2.1) is

$$u \sim \begin{cases} 1 + i\kappa(\bar{x} + \tau_1) - \kappa^2 \left( \frac{\bar{x}^2}{2} + \tau_1\bar{x} + \tau_2 \right) & \text{as } \bar{x} \to 0^+, \\ 1 + i\kappa(\bar{x} + \rho_1) - \kappa^2 \left( \frac{\bar{x}^2}{2} - \rho_1\bar{x} + \rho_2 \right) & \text{as } \bar{x} \to 0^-, \end{cases}$$

(2.3)

where $\bar{x} = x/b$ is an inner variable. It is convenient to define new coefficients by

$$\alpha_n = \frac{1}{2}(\tau_n + \rho_n) \quad \text{and} \quad \beta_n = \frac{1}{2}(\tau_n - \rho_n).$$

Figure 1. Perturbed straight waveguide $D = W_{-} \cup \Omega \cup W_{+}$. The thick lines are rigid boundaries, $\partial D$. A plane wave is incident from the left. The boundary $\partial \Omega$, consists of rigid boundaries between $x = 0$ and $x = d$. 
Then (2.3) becomes

\[ u \sim 1 + i\kappa (\tilde{x} + \beta_1 \text{sgn} \tilde{x} + \alpha_1) - k^2 \left( \frac{\tilde{x}^2}{2} + \alpha_1 |\tilde{x}| + \beta_1 \tilde{x} + \beta_2 \text{sgn} \tilde{x} + \alpha_2 \right) \quad \text{as } |\tilde{x}| \to 0, \tag{2.4} \]

where \( \text{sgn} x = \pm 1 \) for \( \pm x > 0 \).

In the inner region, where the relevant length scale is \( b \), denote the solution by \( U(\tilde{x}, \tilde{y}) \), where \( \tilde{y} = y/b \). Suppose that \( U \) has the expansion

\[ U = 1 + i\kappa U_1(\tilde{x}, \tilde{y}) - k^2 U_2(\tilde{x}, \tilde{y}) - \cdots. \tag{2.5} \]

Substitution in \((V^2 + k^2)U = 0\) gives

\[ \nabla^2 U_1 = 0 \quad \text{and} \quad \nabla^2 U_2 = 1, \tag{2.6} \]

where \( \nabla^2 = \partial^2/\partial \tilde{x}^2 + \partial^2/\partial \tilde{y}^2 = b^2 \nabla^2 \), together with sound-hard boundary conditions. In order to match with the outer solution, (2.4), we require that

\[ U_1(\tilde{x}, \tilde{y}) \sim \tilde{x} + \beta_1 \text{sgn} \tilde{x} + \alpha_1 \quad \text{and} \quad U_2(\tilde{x}, \tilde{y}) \sim \frac{\tilde{x}^2}{2} + \alpha_1 |\tilde{x}| + \beta_1 \tilde{x} + \beta_2 \text{sgn} \tilde{x} + \alpha_2 \tag{2.7} \]

as \( |\tilde{x}| \to \infty \); these agree with [5, eqns (D7) and (D8)]. Our main goal is to determine \( \alpha_1 \) and \( \beta_1 \).

As \( U_1 \) is harmonic, we introduce the potential \( \phi(x, y) \) satisfying \( \nabla^2 \phi = 0 \) in \( D \), \( \partial \phi/\partial n = 0 \) on the boundary \( \partial D \) and the far-field conditions

\[ \phi(x, y) = x + C \text{sgn} x + o(1) \quad \text{as } |x| \to \infty. \tag{2.8} \]

Comparison with (2.7) gives \( \beta_1 = C/b \). We also see that \( U_1 = b^{-1} \phi + \alpha_1 \).

The constant \( C \) in (2.8) is known as the blockage coefficient. It is uniquely determined by solving the boundary value problem for \( \phi \). It can be interpreted as giving a measure of how much a uniform flow is affected by the presence of the perturbation to the straight-walled geometry. Further comments on \( C \) will be made later.

To find the constant \( \alpha_1 \), we have to examine the second-order term \( U_2 \). (Setting \( \alpha_1 = 0 \) leads to erroneous results, in general; see the discussion in [5] and examples below.) Introduce the function \( \psi \) so that \( \psi(x, y) = (1/2)\tilde{x}^2 + \chi(x, y) \) with \( \partial \psi/\partial n = 0 \) on \( \partial D \). We want \( \nabla^2 \psi = 1 \) implying that \( \nabla^2 \chi = 0 \). Suppose that the potential \( \chi \) satisfies

\[ \chi(x, y) = M_{\mp} x + O(1) \quad \text{as } x \to \pm \infty, \tag{2.9} \]

with constants \( M_+ \) and \( M_- \), together with \( \partial \chi/\partial n = -\partial (\tilde{x}^2/2)/\partial n \) on \( \partial \Omega_r \). (Evidently, \( \partial \chi/\partial n = 0 \) on \( \partial D \setminus \partial \Omega_r \).) Comparison with (2.7) gives \( \beta_1 \pm \alpha_1 = bM_{\pm} \), whence \( 2\alpha_1 = b(M_+ - M_-) \). Fortunately, this quantity can be determined without determining \( \chi \) itself.

Thus, applying Green’s theorem to \( \chi \) in a truncated form of \( D \), \( D_\xi \), (see figure 2) gives \( \int_{\partial D_\xi} \partial \chi/\partial n \, ds = 0 \). Letting \( \xi \to \infty \), and making use of (2.9), we obtain

\[ 2\alpha_1 = b(M_+ - M_-) = - \int_{\partial \Omega_r} \frac{\partial}{\partial n} \left( \frac{\tilde{x}^2}{2} \right) \, ds, \tag{2.10} \]

where the normal derivative is taken into \( D \). For the remaining integral, there are various cases; we consider two.

**Grating problem.** Suppose that the scatterer consists of a bounded sound-hard obstacle \( B \) inside a parallel-walled waveguide (so no bulges in figure 1). Examples of this kind are considered in [5]. Then the integral reduces to one over the boundary of \( B \), an integral that can be evaluated using the divergence theorem; the right-hand side of (2.10) evaluates to \( -|B|/b^2 \), where \( |B| \) is the area of \( B \).

**Deformation problem.** Suppose that the scatterer consists of a deformation of the otherwise parallel walls (so no circle in figure 1). Then, as \( \partial \Omega = \partial \Omega_r \cup \ell_0 \cup \ell_d \) is a
Figure 2. Truncated perturbed straight waveguide $D_\xi$, with boundary $\partial D_\xi$ consisting of two vertical lines $\ell_\pm$ at $x = \pm \xi$ and pieces of the rigid boundary shown in figure 1.

closed curve, we have $-\int_{\partial \Omega_1} = \int_{\ell_0 \cup \ell_d} - \int_{\partial \Omega};$ the integral over $\ell_0 \cup \ell_d$ is elementary whereas the integral over $\partial \Omega$ is amenable to the divergence theorem. After some calculation, the right-hand side of (2.10) evaluates to $(|\Omega| - bd)/b^2$. (Note that this quantity does not change if we increase $d$ so as to include some of $W_+$.)

There are formulae similar to (2.10) for the blockage coefficient $C$, defined by (2.8). Thus, an application of Green’s theorem to $\phi(x, y)$ and $x$ in $D_\xi$ (figure 2) gives

$$C = \frac{1}{2b} \int_{\partial \Omega_1} \phi \frac{\partial x}{\partial n} d s. \quad (2.11)$$

This exact formula relates the far-field quantity $C$ (see (2.8)) to a near-field quantity, the unknown potential $\phi$ on the rigid boundary perturbation $\partial \Omega_1$. It can be used to find approximations to $C$ by, for example, inserting approximations to $\phi$ on $\partial \Omega_1$ in the right-hand side of (2.11); see (2.13) below for one application. For some applications to analogous three-dimensional problems, see [23].

Summarizing, we obtain the estimates

$$R \simeq ikb(\alpha_1 - \beta_1) \quad \text{and} \quad T \simeq 1 + ikb(\alpha_1 + \beta_1), \quad (2.12)$$

where $\alpha_1$ is given by (2.10), $\beta_1 = C/b$ and $C$ is the blockage coefficient. In general, we do not know $C$ explicitly. It is defined from the far fields of a potential-flow problem, and so it depends on the geometry of the junction region $\Omega$: this part of the solution is expected from physical intuition, as described in §1. On the other hand, we do know $\alpha_1$ explicitly, and it also depends on $\Omega$: this part of the solution may be missed.

For grating problems, the estimates (2.12) have been verified against known exact results in [5]. Connections have also been made with the added mass of a single scatterer, and with earlier work by Miles [24]. For deformation problems, the difficulty is to estimate $C$. If the deformation is mild, we may replace $\phi$ in (2.11) by $x$ giving the approximation

$$C \simeq \frac{1}{4b} \int_{\partial \Omega_1} \frac{\partial x^2}{\partial n} d s = -b\alpha_1 \quad (2.13)$$

(see (2.10)) leading to the approximations

$$R \simeq 2ikb\alpha_1 = ik(|\Omega| - bd)/b \quad \text{and} \quad T \simeq 1. \quad (2.14)$$

These approximations agree with an independent analysis, based on perturbation theory for slightly rough waveguide walls; see appendix A.

(a) Incident wave propagating towards $x = -\infty$

The estimates (2.12) are for an incident wave propagating towards $x = +\infty$; denote them by $R_+$ and $T_+$. What happens when the incident wave is in the opposite direction, propagating towards
$x = -\infty$? In that case, we have

$$u \sim \begin{cases} e^{-ikx} + R_- e^{ikx} & \text{as } x \to \infty, \\ T_- e^{-ikx} & \text{as } x \to -\infty, \end{cases} \tag{2.15}$$

where $R_-$ and $T_-$ are the corresponding reflection and transmission coefficients. It is known that $T_+ = T_+ = \mathbb{1}$, say, but $R_- \neq R_+$, in general. Of course, we do have $R_- = R_+ = R$, say, when the scatterer is symmetric about the line $x = 0$, in which case, we also have $|R + T| = 1$.

Note that we have defined the reflection and transmission coefficients with respect to the line $x = 0$ (Figure 1). If the geometry is symmetric about a different line, $x = x_0$, say, then it may be convenient to define $R$ and $T$ with respect to that line so that symmetry can be exploited.

Suppose instead that the scatterer is symmetric about $x = 0$. Then it is clear that $R_+ = R_- = R$ and $T_+ = T_- = T$. Now move the scatterer to $x = X$ with reflection and transmission coefficients $R_X^\pm$ and $T_X^\pm$ for waves incident from the left ($+$) or right ($-$). (In its new location, the scatterer is not symmetric with respect to $x = 0$, in general.) It is known that $R_X^\pm = R e^{\pm 2i k X}$ and $T_X^\pm = T$, exactly. Again, in general, $R_X^+ \neq R_X^-$ but we cannot see this difference if we only know $R \simeq i \kappa \rho_1$ correct to first order in $k \equiv k b$: differences can only arise at second order because $e^{\pm 2ikX} \sim 1 + O(\kappa)$ as $\kappa \to 0$.

To confirm these results, let us estimate $R_- \text{ and } T_- \text{, proceeding as before.}$ Write $R_\sim \simeq i \kappa \rho_1 - \kappa^2 \rho_2^2$ and $T_\sim \simeq 1 + i \kappa \tau_1 - \kappa^2 \tau_2^2$ with coefficients $\rho_\sim$ and $\tau_\sim$. The inner expansion of the outer expansion (2.15) is

$$u \sim \begin{cases} 1 - i \kappa (x - \rho_1) - \kappa^2 \left( \frac{x^2}{2} + \rho_1^{-} x + \rho_2^{-} \right) & \text{as } x \to 0^+, \\ 1 - i \kappa (x^{-} - \tau_1) - \kappa^2 \left( \frac{x^2}{2} - \tau_1^{-} x + \tau_2^{-} \right) & \text{as } x \to 0^-. \end{cases} \tag{2.16}$$

Let $\alpha_\sim = (1/2)(\tau_\sim^+ + \rho_\sim^+)$ and $\beta_\sim = (1/2)(\tau_\sim^- - \rho_\sim^-)$. Then (2.16) becomes

$$u \sim 1 - i \kappa (x + \beta_1^{-} \text{ sgn } x - \alpha_1^-) - \kappa^2 \left( \frac{x^2}{2} + \alpha_1^- |x| - \beta_1^- x + \alpha_2^- - \beta_2^- \text{ sgn } x \right) \quad \text{as } |x| \to 0. \tag{2.17}$$

In the inner region, suppose that $U = 1 - i \kappa U_{1} (\tilde{x}, \tilde{y}) - \kappa^2 U_{2}(\tilde{x}, \tilde{y}) - \cdots$. In order to match with the outer solution, (2.17), we require that

$$U_{1}(\tilde{x}, \tilde{y}) \sim \tilde{x} + \beta_1^{-} \text{ sgn } \tilde{x} - \alpha_1^- \quad \text{and} \quad U_{2}(\tilde{x}, \tilde{y}) \sim \tilde{x}^2/2 + \alpha_1^- |\tilde{x}| - \beta_1^- \tilde{x} + \alpha_2^- - \beta_2^- \text{sgn } \tilde{x} \tag{2.18}$$

as $|\tilde{x}| \to \infty$. Comparing (2.18)1 with (2.7)1 shows that $\beta_1^- = \beta_1$, whereas comparing (2.18)2 with (2.7)2 shows that $\alpha_1^- = \alpha_1$. We conclude that, to leading order, $R_- = R_+$ and $T_- = T_+$, even if the scatterer is not symmetric about $x = 0$.

(b) Energy

Energy conservation implies that $|R|^2 + |T|^2 = 1$. However, if we use the estimates (2.12), we find that $|R|^2 + |T|^2 \simeq 1 \simeq 2 \kappa^2 (\alpha_1^{-} + \beta_1^{-})$, so that energy conservation is satisfied to second order in $\kappa$.

We can enforce energy conservation exactly, subject to (2.12), by starting from

$$R = \frac{ik A}{1 - ik P} \quad \text{and} \quad T - 1 = \frac{ik B}{1 - ik P}$$

where $A = A_0 + i \kappa A_1$, $B = B_0 + i \kappa B_1$, $P = P_0 + i \kappa P_1$, $A_0 = \alpha_1 - \beta_1$ and $B_0 = \alpha_1 + \beta_1$. The constants $A_1, B_1, P_0$ and $P_1$ are to be chosen. Some calculation gives

$$|R|^2 + |T|^2 - 1 = \kappa^2 \frac{N_0 + i \kappa N_1 + \kappa^2 N_2}{|1 - ik P|^2}$$

where
Figure 3. Bent waveguide $D = W_- \cup \Omega \cup W_+$. The thick lines are rigid boundaries, $\partial D$.

where (noting that $A_0$ and $B_0$ are real)
\[
N_0 = A_0^2 + B_0^2 - B_1 - B_1^* - B_0(P_0 + P_0^*),
\]
\[
N_1 = A_0(A_1 - A_1^*) + B_0(B_1 - B_1^*) - B_1P_0^* + B_1^*P_0 - B_0(P_1 - P_1^*)
\]
and
\[
N_2 = |A_1|^2 + |B_1|^2 - B_1P_1^* - B_1^*P_1
\]
and the $*$ denotes complex conjugation. We achieve $N_2 = 0$ by taking $A_1 = 0$ and $P_1 = (1/2)B_1$. Doing this gives $2N_1 = B_0(B_1 - B_1^*) - 2B_1P_0^* + 2B_1^*P_0$. Thus, we also achieve $N_1 = 0$ by assuming that $P_0$ and $B_1$ are real. Finally, these two are related by enforcing $N_0 = 0$: $2B_1 = A_0^2 + B_0^2 - 2B_0P_0$. These relations leave just one real parameter free, which we take to be $P_0$.

As an example, take $P_0 = B_0$. This gives $B_1 = (1/2)(A_0^2 - B_0^2) = -2\alpha_1\beta_1$ and the formulae
\[
R = \frac{ik(\alpha_1 - \beta_1)}{(1 - ik\alpha_1)(1 - ik\beta_1)} \quad \text{and} \quad T = \frac{1 + k^2\alpha_1\beta_1}{(1 - ik\alpha_1)(1 - ik\beta_1)}.
\]
These agree with [5, eqn (63)], where it is noted that these formulae also satisfy $|R \pm T| = 1$, as expected for symmetric scatterers.

Miles [24] gave a similar analysis except he enforced both $|R|^2 + |T|^2 = 1$ and $|R + T| = 1$.

3. Bent waveguides: general results

Consider two semi-infinite rigid waveguides; both have width $b$. The left waveguide, $W_-$, is aligned with the negative $x$-axis; it occupies $x < 0$, $0 < y < b$. The right waveguide, $W_+$, is inclined at an angle $\varphi$ to the positive $x$-axis. In detail, introduce rotated coordinates $x'$, $y'$, defined by
\[
x' = (x - x_0)\cos \varphi + (y - y_0)\sin \varphi \quad \text{and} \quad y' = -(x - x_0)\sin \varphi + (y - y_0)\cos \varphi,
\]
so that $W_-$ occupies $x' > 0$, $0 < y' < b$; here $x_0$ and $y_0$ are constants. The two waveguides are joined together via a bounded region $\Omega$. (For simplicity only, we assume there are no bounded scatterers in $\Omega$, and we assume that $W_-$ and $W_+$ both have the same width $b$; these assumptions may be relaxed.) The union $W_- \cup \Omega \cup W_+$ defines the whole waveguide $D$. See figure 3.

The general configuration shown in figure 3 arises when ducts are joined via a curved region $\Omega$. For an old but useful review, see the NASA report prepared by Rostafinski [25]. See also [26, §V].

Two configurations are of special interest, corresponding to $\varphi = -(1/2)\pi$ and $\varphi = -\pi$. For the first of these, $\Omega$ could be a square ($0 < x < b$, $0 < y < b$, $x_0 = y_0 = 0$), giving an L-shaped domain (see §4), or $\Omega$ could be the quarter of a disc of radius $b$ centred at $(x, y) = (0, 0)$. For $\varphi = -\pi$, one possibility is to take $x_0 = y_0 = 0$ with $\Omega$ as a rectangle, $0 < x < h$, $-b < y < b$, where $h$ is a positive constant. This gives a waveguide of width $2b$ with a rigid divider along $x < 0$, $y = 0$; see §6.
A plane wave $e^{ikx}$ propagates along $W_-$ towards the junction $\Omega$, where it is partly reflected and partly transmitted into $W_+$. Far away, we have

$$u \sim \begin{cases} T e^{ikx} & \text{as } x' \to \infty, \\ e^{ikx} + R e^{-ikx} & \text{as } x \to -\infty, \end{cases}$$

and the goal is to estimate $R$ and $T$.

Expand $R$ and $T$ in powers of $\kappa = kb$, (2.2). The inner expansion of the outer expansion (3.2) is

$$u \sim \begin{cases} 1 + i\kappa (\tilde{x}' + \tau_1) - \kappa^2 \left( \frac{\tilde{x}'^2}{2} + \tau_1 \tilde{x}' + \tau_2 \right) & \text{as } \tilde{x}' \to 0+, \\ 1 + i\kappa (\tilde{x} + \rho_1) - \kappa^2 \left( \frac{\tilde{x}^2}{2} - \rho_1 \tilde{x} + \rho_2 \right) & \text{as } \tilde{x} \to 0-, \end{cases}$$

where $\tilde{x}' = x'/b$. In the inner region, we introduce functions $U_1$ and $U_2$, defined by (2.5), (2.6) and sound-hard boundary conditions. In order to match with (3.3), we require

$$U_1 \sim \begin{cases} \tilde{x}' + \tau_1 & \text{as } \tilde{x}' \to \infty, \\ \tilde{x} + \rho_1 & \text{as } \tilde{x} \to -\infty, \end{cases}$$

and

$$U_2 \sim \begin{cases} \tilde{x}^2/2 + \tau_1 \tilde{x}' + \tau_2 & \text{as } \tilde{x}' \to \infty, \\ \tilde{x}^2/2 - \rho_1 \tilde{x} + \rho_2 & \text{as } \tilde{x} \to -\infty. \end{cases}$$

Introduce the potential $\phi(x, y)$ satisfying $\nabla^2 \phi = 0$ in $D$, $\partial \phi/\partial n = 0$ on the boundary $\partial D$ and the far-field conditions

$$\phi(x, y) = \begin{cases} x' + C + o(1) & \text{as } x' \to \infty, \\ x - C + o(1) & \text{as } x \to -\infty. \end{cases}$$

This represents uniform potential flow along $D$; again, $C$ is the blockage coefficient for this flow. Then comparison with (3.4) gives the relation

$$\tau_1 - \rho_1 = \frac{2C}{b}. \quad (3.7)$$

For a second relation, we examine $u_2$.

Introduce a function $\psi$ satisfying $\nabla^2 \psi = b^2 \nabla^2 \psi = 1$ in $D$, $\partial \psi/\partial n = 0$ on $\partial D$, $\psi \sim \tilde{x}'^2/2$ as $x' \to \infty$ and $\psi \sim \tilde{x}^2/2$ as $x \to -\infty$. Recall that $D = W_- \cup W_+ \cup \Omega$; see figure 3. In detail, the closed boundary of $\Omega$ is $\partial \Omega = \ell_0 \cup \ell'_0 \cup \partial \Omega_r$, where $\ell_0$ is the segment $x = 0$, $0 < y < b$, $\ell'_0$ is the segment $x' = 0$, $0 < y' < b$, and $\partial \Omega_r \subset \partial D$ consists of the rigid parts of the junction $\Omega$. Thus, $\ell_0$ is the interface between $\Omega$ and $W_-$ whereas $\ell'_0$ is the interface between $\Omega$ and $W_+$. Assume that the unit normal vector on $\partial \Omega$ points out of $\Omega$.

Starting with $W_-$, put $\psi = \tilde{x}'^2/2 + \chi_-$ so that $\nabla^2 \chi_- = 0$ in $W_-$ with $\chi_- \sim M_- x$ as $x \to -\infty$. Green’s theorem gives

$$\int_{\ell_0} \frac{\partial \chi_-}{\partial n} \, ds = -bM_- = \int_{\ell'_0} \frac{\partial \psi}{\partial n} \, ds,$$

the last equality following from $\partial \psi/\partial n = \partial \chi_-/\partial n$ on $\ell_0$ (where $x = 0$). Similarly, put $\psi = \tilde{x}^2/2 + \chi_+$ in $W_+$ so that $\nabla^2 \chi_+ = 0$ in $W_+$ with $\chi_+ \sim M_+ x'$ as $x' \to \infty$. Hence

$$\int_{\ell'_0} \frac{\partial \chi_+}{\partial n} \, ds = bM_+ = \int_{\ell_0} \frac{\partial \psi}{\partial n} \, ds.$$

As $\partial \psi/\partial n = 0$ on $\partial \Omega_r$ and $b^2 \nabla^2 \psi = 1$ in $\Omega$, we obtain

$$b(M_+ - M_-) = \int_{\partial \Omega} \frac{\partial \psi}{\partial n} \, ds = \int_{\Omega} \nabla^2 \psi \, dA = \frac{|\Omega|}{b^2},$$
where \(|\Omega|\) is the area of \(\Omega\). Comparing \(u_2\) with \(\psi\) shows that
\[
\tau_1 + \rho_1 = b(M_+ - M_-) = |\Omega|/b^2. \tag{3.8}
\]

The first-order corrections, \(\tau_1\) and \(\rho_1\), are given by (3.7) and (3.8). Thus
\[
R \sim -ikb \left( \frac{C}{b} - \frac{|\Omega|}{2b^2} \right) \quad \text{and} \quad T \sim 1 + ikb \left( \frac{C}{b} + \frac{|\Omega|}{2b^2} \right). \tag{3.9}
\]

We shall elaborate on these results using examples. They could be refined to enforce energy conservation using the strategy described in §2(b). Of course, to use (3.9), we have to know the blockage coefficient \(C\); its determination will be a major task in the rest of the paper.

4. Right-angled bend

Two semi-infinite arms of width \(b\) in the \(xy\)-plane, \(W_- = \{(x, y) : x < 0, 0 < y < b\}\) and \(W_+ = \{(x, y) : 0 < x < b, 0 < y < b\}\), are joined via a square \(\Omega = \{(x, y) : 0 < x < b, 0 < y < b\}\). This gives a right-angled bend, an L-shaped channel; see figure 4. Also, from (3.1), with \(x_0 = y_0 = 0\) and \(\varphi = -(1/2)i\pi\), we have \(x' = -y\) and \(y' = x\). Arbitrary sharp bends will be discussed later, in §5.

To determine \(C\) for this geometry, consider potential flow coming along \(W_-\) from \(x = -\infty\) and leaving along \(W_+\) towards \(y = -\infty\). Thus we seek a real potential \(\phi(x, y)\) satisfying (3.6),
\[
\phi(x, y) = \begin{cases} 
  x - C + o(1) & \text{as } x \to -\infty \text{ for } 0 < y < b \\
  -y + C + o(1) & \text{as } y \to -\infty \text{ for } 0 < x < b.
\end{cases} \tag{4.1}
\]

To simplify the geometry, we map to the upper half of the complex \(\zeta\)-plane using a Schwarz–Christoffel mapping. As is well known, it is easier to use an inverse mapping, from \(\zeta\) to \(z\); an appropriate mapping is
\[
z = f(\zeta) = A \int g(\zeta) \, d\zeta + B \quad \text{with} \quad g(\zeta) = \frac{(\zeta - 1)^{1/2}}{\zeta(\zeta + 1)^{1/2}}, \tag{4.2}
\]
where the constants \(A\) and \(B\) and the branches will be specified later. We want the points \(\zeta = \pm 1\) to be mapped to the two corners: \(f(-1) = b(1 + i)\) and \(f(1) = 0\). We also want \(\zeta = 0\) to be mapped to \(x = -\infty\) (far end of \(W_-\)) and \(\zeta = \infty\) to correspond to \(y = -\infty\) (far end of \(W_+\)).

The mapping (4.2) can be found in many places, such as [27, p. 157], [28, p. 445], [29, Example 8.3.4] and [30, Example 11.28]. The integral can be evaluated in terms of inverse trigonometric functions and/or inverse hyperbolic functions. The tricky part of this calculation comes in choosing branches correctly, and, in fact, it is easier to do this by working directly with the integral; see below. Moreover, we also need asymptotic properties of the mapping, and these are easier to extract from the integral. Indeed, Miles [8, eqn (28)] gives the conformal mapping but he refrains from using it, although he does note that the values of the potential at the two ends of the L-shaped channel are different: this difference is essentially the blockage coefficient.

**Figure 4.** Right-angled bend \(D = W_- \cup \Omega \cup W_+\). The thick lines are rigid boundaries, \(\partial D\).
The function \( g(\zeta) \) has branch points at \( \zeta = \pm 1 \) and a pole at \( \zeta = 0 \). Take the cuts in the lower half-plane. For the branch point at \( \zeta = 1 \), put \( \zeta = 1 + ri \) giving \( (\zeta - 1)^{1/2} = \sqrt{r} e^{i\theta/2} \) with \(-1/2) \pi < \theta < (3/2) \pi\). In particular, when \( \zeta = \xi \) is real, we have \( (\zeta - 1)^{1/2} = \sqrt{\xi - 1} \) when \( \xi > 1 \) and \( (\zeta - 1)^{1/2} = i\sqrt{1-\xi} \) when \( \xi < 1 \). Similarly, for the branch point at \( \zeta = -1 \), put \( \zeta = -1 + ri \) giving \( (\zeta + 1)^{-1/2} = r^{-1/2} e^{-i\theta/2} \). In particular, when \( \zeta = \xi \) is real, we have \( (\zeta + 1)^{-1/2} = (\xi + 1)^{-1/2} \) when \( \xi > -1 \) and \( (\zeta + 1)^{-1/2} = -i(-1 - \xi)^{-1/2} \) when \( \xi < -1 \).

Next, define \( f \) by

\[
z = f(\zeta) = A \int_1^\zeta g(\eta) \, d\eta + B.
\]

Imposing \( f(1) = 0 \) gives \( B = 0 \) and then \( b(1 + i) = f(-1) = -A \int_1^\infty g(\eta) \, d\eta \) determines \( A \). In this formula, we indent the contour above the pole at \( \eta = 0 \) using a semicircular contour of radius \( \varepsilon, C_{\varepsilon} \). Thus

\[
\int_{-1}^{-\varepsilon} g(\eta) \, d\eta = \int_{-\varepsilon}^{1} g(\eta) \, d\eta + \int_{C_{\varepsilon}} g(\eta) \, d\eta + \int_{\varepsilon}^{1} g(\eta) \, d\eta.
\]

We have \( g(\eta) \sim i/\eta \) as \( \eta \to 0 \) so that \( \int_{C_{\varepsilon}} g(\eta) \, d\eta \sim \pi \) as \( \varepsilon \to 0 \). Also

\[
\left( \int_{\varepsilon}^{1} + \int_{-\varepsilon}^{-1} \right) g(\eta) \, d\eta = \left( \int_{\varepsilon}^{1} + \int_{-\varepsilon}^{-1} \right) \frac{1}{\sqrt{1 - \eta - \xi}} \frac{d\xi}{\xi} = i \int_{\varepsilon}^{1} \left( \frac{\sqrt{1 - \eta - \xi}}{\sqrt{1 + \xi}} - \frac{\sqrt{1 + \xi}}{\sqrt{1 - \eta - \xi}} \right) \frac{d\xi}{\xi}
\]

\[
= 2i \int_{\varepsilon}^{1} \frac{d\xi}{\sqrt{1 - \eta - \xi}^2} \sim -i\pi \quad \text{as} \; \varepsilon \to 0.
\]

Hence \( b(1 + i) = f(-1) = -A(\pi - i\pi), \) which gives \( A = -i(b/\pi) \).

For the limit \( \zeta \to \infty \), we write

\[
\int_{1}^{\zeta} g(\eta) \, d\eta = \int_{1}^{\zeta} \frac{d\eta}{\eta} + I(\zeta) \quad \text{with} \quad I(\zeta) = \int_{1}^{\zeta} \left( \frac{1}{(\eta - 1)^{1/2}} - 1 \right) \frac{d\eta}{\eta}.
\]

For large \( \zeta \), we have \( I(\zeta) \sim I(\infty) \), where

\[
I(\infty) = \int_{1}^{\infty} \left( \frac{1}{\sqrt{\xi^2 - 1}} - 1 \right) \frac{d\xi}{\xi} = \int_{1}^{\infty} \left( \frac{1}{\sqrt{\xi^2 - 1}} - \frac{1}{\xi} \right) \frac{d\xi}{\xi} - \int_{1}^{\infty} \frac{d\xi}{\xi \sqrt{\xi^2 - 1}}
\]

\[
= \int_{0}^{\infty} (1 - \tanh \alpha) \, d\alpha - \int_{0}^{1} \frac{df}{\sqrt{1 - \alpha^2}} = \int_{0}^{\infty} \frac{2e^{-2\alpha}}{1 + e^{-2\alpha}} - \frac{\pi}{2}
\]

\[
= 2 \sum_{n=0}^{\infty} (-1)^n \int_{0}^{\infty} e^{-2n(1+1)} \, d\alpha - \frac{\pi}{2} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} - \frac{\pi}{2} = \log 2 - \frac{\pi}{2}.
\]

It is easy to show (using l’Hôpital’s rule) that \( I(\zeta) \approx I(\infty) = O(\zeta^{-2}) \) as \( \zeta \to \infty \). A comparable error is incurred when the point at \( \zeta \) is moved onto the real axis, as assumed implicitly in the evaluation of \( I(\infty) \) above. Hence

\[
z = f(\zeta) = -i \frac{b}{\pi} \log 2\zeta + i \frac{b}{2} + O(\zeta^{-2}) \quad \text{as} \; \zeta \to \infty.
\]

This confirms that \( \zeta \to \infty \) corresponds to \( \zeta \to -\infty \).

We make a similar calculation for the limit \( \zeta \to 0 \). Write

\[
\int_{1}^{\zeta} g(\eta) \, d\eta = \int_{1}^{\zeta} \frac{i}{\eta} \, d\eta + I_0(\zeta) \quad \text{with} \quad I_0(\zeta) = \int_{1}^{\zeta} \left( \frac{1}{(\eta + 1)^{1/2}} - i \right) \frac{d\eta}{\eta}.
\]
Figure 5. Bent waveguide $D = W_- \cup \Omega \cup W_+$. The thick lines are rigid boundaries, $\partial D$. The section $W_+$ is angled down from the $x$-axis by an angle $\beta \pi$. The corner $Q$ is at $(x, y) = b(\tan(\beta \pi/2), 1)$.

For small $\zeta$, we have $I_0(\zeta) \sim I_0(0)$, where

$$I_0(0) = i \int_1^0 \left( \frac{\sqrt{1-\xi}}{\sqrt{1+\xi}} - 1 \right) \frac{d\xi}{\xi} = i \int_1^0 \left( \frac{1}{\sqrt{1-\zeta^2}} - 1 \right) \frac{d\xi}{\xi} - i \int_1^0 \frac{d\xi}{\sqrt{1-\xi^2}}$$

$$= -i \int_1^\infty \left( \frac{1}{\sqrt{t^2-1}} - \frac{1}{t} \right) dt + i \frac{\pi}{2} = -i \log 2 + i \frac{\pi}{2}.$$

Hence

$$z = f(\zeta) = -\frac{b}{\pi} \log \frac{2}{\zeta} + \frac{b}{2} + O(\zeta^2) \quad \text{as} \quad \zeta \to 0. \quad (4.7)$$

This confirms that $\zeta \to 0$ corresponds to $x \to -\infty$.

In the mapped domain, the potential is $\Phi(\xi, \eta)$. We claim that $\Phi = (b/\pi) \log \rho$ where $\rho = |\zeta| = \sqrt{\xi^2 + \eta^2}$. We have to check that the conditions (4.1) are satisfied. The real part of (4.7) gives $x \sim \Phi + C$ as $x \to -\infty (\rho \to 0)$, where

$$C = \frac{b}{2} - \frac{b}{\pi} \log 2 \simeq 0.28 b. \quad (4.8)$$

On the other hand, the imaginary part of (4.6) gives $-y \sim \Phi - C$ as $y \to -\infty (\rho \to \infty)$.

The formula for $C$, (4.8), agrees with [31, eqn (30)] and [32, eqn (43)].

The estimates for $R$ and $T$ are (3.9) in which $|\Omega| = b^2$ (figure 4). Thus

$$R \simeq i k b \pi^{-1} \log 2 \simeq 0.22 i k b \quad \text{and} \quad T \simeq 1 + i k b(1 - \pi^{-1} \log 2) \simeq 1 + 0.78 i k b. \quad (4.9)$$

As we have already noted, waves round right-angled bends have a long history, beginning with a 1947 paper by Miles [8]. Lippert wrote several papers in the 1950s [33–35], in which he developed a theory [33] and compared with his experiments [35]. For the right-angled bend, his formulae for $R$ and $T$ [34, eqns (3) and (4)] yield the following approximations for small $k b$:

$$R \simeq \frac{i k b}{6} \simeq 0.17 i k b \quad \text{and} \quad T \simeq 1 + \frac{5 i k b}{6} \simeq 1 + 0.83 i k b. \quad (4.10)$$

These can be compared with (4.9). We note that Lippert’s $\rho_1 + \tau_1 = (1/6) + (5/6) = 1$ in agreement with (3.8), whereas use of (3.7) implies $C/b \simeq 1/3$ rather than the exact formula (4.8). The estimates (4.10) can also be extracted from [36, eqn (91)].

5. A general sharp bend

Suppose the right arm $W_+$ is inclined at an angle $\varphi = -\pi \beta$ (figure 5), where $-1 < \beta < 1$. The special case $\beta = 1/2$ gives the right-angled bend as discussed in §4.
An appropriate Schwarz–Christoffel mapping is

$$z = f(\zeta) = A \frac{g(\zeta) \, d\zeta}{B + g(\zeta)}$$

We want the points $\zeta = \pm 1$ to be mapped to the two corners, O and Q; $f(-1) = b(\tan(\beta \pi / 2) + i)$ and $f(1) = 0$. We define branches as in §4; thus

$$\zeta - 1 = r_1^\beta e^{i \beta \phi_1} \quad \text{and} \quad (\zeta + 1) = r_2^{-\beta} e^{i \beta \phi_2}, \quad -\frac{1}{2} \pi < \theta_j < \frac{3}{2} \pi, \quad j = 1, 2.$$

In particular, when $\zeta = \zeta$ is real, with $|e| < 1$, $g(\zeta) = \xi \zeta^{-1} e^{i \beta \pi} (1 - \xi)^\beta (1 + \xi)^{-\beta}$.

Next, define $f$ by (4.3). Imposing $f(1) = 0$ gives $B = 0$. Then, requiring $\zeta = -1$ to map to Q gives

$$b \left( i + \tan \frac{\beta \pi}{2} \right) = -A \int_{-1}^{1} g(\eta) \, d\eta,$$

which determines $A$. Proceeding as in §4, we arrive at (4.4). We have $g(\eta) \sim \eta^{-1} e^{i \beta \pi} \eta \to 0$ so that $\int_{c} g(\eta) \, d\eta \sim -i \pi e^{i \beta \pi} \eta \to 0$. Also

$$h(\xi; \beta) = \frac{1}{\xi} \sum_{n=0}^{\infty} \frac{\Gamma(n - 2\beta)}{\Gamma(-2\beta)} \frac{\xi^n}{(1 - (-1)^n) n!} = \sum_{m=0}^{\infty} \frac{2 \Gamma(2m + 1 - 2\beta)}{\Gamma(-2\beta)(2m + 1)!} \xi^{2m}.$$

Then, taking the limit $\zeta \to 0$, we can integrate, using [37, 3.251.1],

$$\int_{0}^{1} \frac{\xi^{2m} \, d\xi}{(1 - \xi^{2})^\beta} = \frac{\Gamma(m + (1/2)) \Gamma(1 - \beta)}{\Gamma(m + (3/2) - \beta)},$$

whence

$$\int_{0}^{1} h(\xi; \beta) \, d\xi = \frac{\Gamma(1 - \beta)}{\Gamma(-2\beta)} \sum_{n=0}^{\infty} c_n \quad \text{with} \quad c_n = \frac{\Gamma(2n + 1 - 2\beta) \Gamma(n + (1/2))}{\Gamma(n + (3/2) - \beta)(2n + 1)!}.$$

We have

$$\frac{c_{n+1}}{c_n} = \frac{(n + 1 - \beta)(n + 2 - \beta)(n + 1)}{(n + (3/2) - \beta)(n + (3/2))(n + 1)},$$

implying that

$$\sum_{n=0}^{\infty} c_n = c_0 \zeta F_2 \left( \frac{1}{2} - \beta, \frac{1}{2}, 1 - \beta; \frac{3}{2}, \frac{3}{2} - \beta; 1 \right),$$

where the $\zeta F_2$ seen here is a well-poised generalized hypergeometric function [37, 16.4.1]; it can be evaluated using Dixon’s formula (put $a = 1 - \beta$, $b = (1/2) - \beta$ and $c = (1/2)$ in [37, 16.4.4]) as

$$\frac{\Gamma(3/2 - (1/2)\beta) \Gamma(3/2 - (1/2)\beta) \Gamma((1/2) + (1/2)\beta)}{\Gamma((3/2) - (1/2)\beta) \Gamma((3/2) - (1/2)\beta) \Gamma((1/2) + (1/2)\beta)} = D,$$

say. Using [37, 5.5.3], $\Gamma(1 + (1/2)\beta) \Gamma(1 - (1/2)\beta) = Y \Gamma(Y) \Gamma(1 - Y) = \pi Y \sin \pi Y$ with $Y = (1/2)\beta$. Similarly, $\Gamma((3/2) - (1/2)\beta) \Gamma((1/2) + (1/2)\beta) = Z \Gamma(1 - Z) \Gamma(Z) = \pi Z \sin \pi Z \sin Z = -\pi Z /
\[ \cos \pi Y \text{ where } Z = (1/2)(\beta - 1). \text{ Then } \mathcal{D} \text{ simplifies to} \]
\[ \mathcal{D} = \frac{\Gamma(3/2) \Gamma((3/2) - \beta)}{\beta \Gamma(1 - \beta)} \tan (\beta \pi/2), \]
followed by
\[ \int_0^1 h(\xi, \beta) \frac{d\xi}{(1 - \xi^2)\beta} = \frac{\Gamma(1 - \beta)}{\beta} \frac{\Gamma(1 - 2\beta)}{\Gamma((3/2) - \beta)} \tan (\beta \pi/2) = -\pi \tan (\beta \pi/2). \]

Hence
\[ \int_{-1}^1 \mathcal{g}(\eta) \, d\eta = -\pi e^{i\beta \pi} \left( i + \tan \frac{\beta \pi}{2} \right) \]
so that (5.2) gives \( A = (b/\pi)e^{-i\beta \pi}. \)

For the limit \( \zeta \to \infty, \) we write
\[ \int_{-1}^1 \mathcal{g}(\eta) \, d\eta = \int_{-1}^1 \frac{d\eta}{\eta} + I(\zeta) \quad \text{with} \quad I(\zeta) = \int_1^\zeta \left( \frac{(\eta - 1)^\beta}{(\eta + 1)^\beta} - 1 \right) \frac{d\eta}{\eta}. \]

For large \( \zeta, \) we have \( I(\zeta) \sim I(\infty), \) where
\[ I(\infty) = \int_1^\infty \left( \frac{(\xi - 1)^\beta}{(\xi + 1)^\beta} - 1 \right) \frac{d\xi}{\xi} = \int_0^1 \left( \frac{(1 - t)^\beta}{(1 + t)^\beta} - 1 \right) \frac{dt}{t} \]
\[ = \int_0^1 (1 - t)^\beta ((1 + t)^{-\beta} - (1 - t)^{-\beta}) \frac{dt}{t} = -\int_0^1 (1 - t)^\beta \mu(t; \beta/2) \, dt \]
\[ = -\sum_{m=0}^\infty \frac{2 \Gamma(2m + 1 + \beta)}{\Gamma(2m + 1)!} \int_0^1 (1 - t)^\beta t^{2m} \, dt = -\sum_{m=0}^\infty \frac{2\beta}{(m + 1)(2m + \beta + 1)} \]
\[ = \sum_{m=0}^\infty \left( \frac{1}{m + \frac{1}{2} + (1/\beta)} - \frac{1}{m + \frac{3}{2}} \right) = \psi \left( \frac{1}{2} \right) - \psi \left( \frac{\beta + 1}{2} \right), \tag{5.5} \]

using (5.3), (5.4) and [38, 8.363.3]. Here, \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the psi function [37, 5.2.2], [38, §8.36]. As a check, when \( \beta = 1/2, \) we can use \( \psi(1/2) = -\gamma - 2 \log 2 \) and \( \psi(3/4) = -\gamma + (1/2)\pi - 3 \log 2 \) (where \( \gamma \approx 0.5772 \) is Euler’s constant) and recover (4.5). Again, it is easy to show that \( I(\zeta) - I(\infty) = O(\zeta^{-2}) \) as \( \zeta \to \infty \) with a comparable error incurred when the point at \( \zeta \) is moved onto the real axis. Hence
\[ z = f(\zeta) = \frac{b}{\pi} e^{-i\beta \pi} \left( \log \zeta + \psi \left( \frac{1}{2} \right) - \psi \left( \frac{\beta + 1}{2} \right) + O(\zeta^{-2}) \right) \quad \text{as} \quad \zeta \to \infty. \tag{5.6} \]

Using (3.1) with \( x_0 = y_0 = 0 \) and \( \varphi = -\beta \pi, \) we find that \( \zeta \to \infty \) corresponds to \( x' \to \infty. \)

We make a similar calculation for the limit \( \zeta \to 0. \) Write
\[ \int_{-1}^1 \mathcal{g}(\eta) \, d\eta = e^{i\beta \pi} \int_{-1}^1 \frac{d\eta}{\eta} + I_0(\zeta) \quad \text{with} \quad I_0(\zeta) = \int_1^\zeta \left( \frac{(\eta - 1)^\beta}{(\eta + 1)^\beta} - e^{i\beta \pi} \right) \frac{d\eta}{\eta}. \]

For small \( \zeta, \) we have \( I_0(\zeta) \sim I_0(0), \) where
\[ I_0(0) = e^{i\beta \pi} \int_0^1 \left( \frac{(1 - \xi)^\beta}{(1 + \xi)^\beta} - 1 \right) \frac{d\xi}{\xi} = -e^{i\beta \pi} I(\infty), \]
see (5.5). Hence
\[ z = f(\zeta) = (b/\pi)(\log \zeta - I(\infty) + O(\zeta^2)) \quad \text{as} \quad \zeta \to 0. \tag{5.7} \]

This confirms that \( \zeta \to 0 \) corresponds to \( x \to -\infty. \)

In the mapped domain, the potential is \( \Phi(\xi, \eta). \) We claim that \( \Phi = (b/\pi) \log \rho \) where \( \rho = |\zeta| = \sqrt{\xi^2 + \eta^2}. \) We have to check that the conditions (3.6) are satisfied. The real part of (5.7) gives
Figure 6. Hairpin bend $D = W_- \cup \Omega \cup W_+$. The thick lines are rigid boundaries, $\partial D$.

$x \sim \Phi + C$ as $x \to -\infty$ ($\rho \to 0$), where

$$C = -\frac{b}{\pi} I(\infty) = \frac{b}{\pi} \left\{ \psi \left( \frac{\beta + 1}{2} \right) - \psi \left( \frac{1}{2} \right) \right\}. \tag{5.8}$$

On the other hand, using (3.1), (5.6) gives $x' \sim \Phi - C$ as $x' \to \infty$ ($\rho \to \infty$).

As far as we know, the expression (5.8) for the blockage coefficient is new. It can be used in the first-order estimates for $R$ and $T$, (3.9), in which $|\Omega| = b^2 \tan (\beta \pi/2)$, $0 < \beta < 1$.

6. Hairpin bend

Two semi-infinite arms of width $b$ in the $xy$-plane, $W_- = \{(x,y) : x < 0, 0 < y < b\}$ and $W_+ = \{(x,y) : x < 0, -b < y < 0\}$, are connected via a rectangle $\Omega = \{(x,y) : 0 < x < h, -b < y < b\}$ where $h$ is a positive constant. This gives a hairpin bend; see figure 6. Also, from (3.1), with $x_0 = y_0 = 0$ and $\phi = -\pi$, we have $x' = -x$ and $y' = -y$.

To determine $C$ for this geometry, consider potential flow coming along $W_-$ from $x = -\infty$ and leaving along $W_+$ towards $x' = -x = \infty$. Thus we seek a real potential $\phi(x,y)$ satisfying (3.6),

$$\phi(x,y) = \begin{cases} 
  x - C + o(1) & \text{as } x \to -\infty \text{ for } 0 < y < b \\
  -x + C + o(1) & \text{as } x \to -\infty \text{ for } -b < y < 0.
\end{cases} \tag{6.1}$$

We simplify the geometry using the conformal mapping $\zeta = A \sinh K(h - z)$ with $A = (\sinh Kh)^{-1}$ and $K = \pi/(2b)$. With $z = x + iy$ and $\xi = \xi + i\eta$, we have

$$\xi = A \sinh K(h - x) \cos Ky \quad \text{and} \quad \eta = -A \cosh K(h - x) \sin Ky.$$ 

The choice of $K$ ensures that the three-part boundary of the semi-infinite strip is mapped to the whole vertical line $\xi = 0$, with the strip itself mapped to the half-plane $\xi > 0$. In addition, the choice of $A$ implies that the divider along $y = 0$ (with $x < 0$) is mapped to the horizontal semi-infinite line $\eta = 0, \xi > 1$. Note that $0 < y < b$ corresponds to $\eta < 0$.

In the mapped domain, the potential is $\Phi(\xi, \eta)$. It is harmonic in the half-plane $\xi > 0$ apart from the (mapped) divider along $\eta = 0, \xi > 1$. The boundary conditions are $\partial \Phi / \partial \xi = 0$ along $\xi = 0$ and $\partial \Phi / \partial \eta = 0$ on (both sides of) the divider. For $x$ large and negative, $\xi \sim (A/2)e^{K(h-x)} \cos Ky$ and $\eta \sim -(A/2)e^{K(h-x)} \sin Ky$ so that $-Kx \sim \log (2\rho e^{-Kh}/A)$, where $\rho = \sqrt{\xi^2 + \eta^2}$. Thus,

$$\Phi(\xi, \eta) \sim \pm K^{-1} \log \left( \frac{2\rho e^{-Kh}}{A} \right) \quad \text{as } \rho \to \infty, \quad \text{for } \xi > 0 \text{ and } \pm \eta > 0.$$ 

These conditions imply that $\Phi(\xi, \eta)$ is an odd function of $\eta$. Moreover, as $\partial \Phi / \partial \xi = 0$ on $\xi = 0$, we can extend $\Phi(\xi, \eta)$ into $\xi < 0$ as an even function of $\xi$.

Let us formulate a problem for $\Phi(\xi, \eta)$ in the upper half-plane, $\eta > 0$. There are mixed boundary conditions at $\eta = 0$, namely $\partial \Phi / \partial \eta = 0$ for $|\xi| > 1$ and $\Phi = 0$ for $|\xi| < 1$. In the far field, the second
of (6.1) gives

$$\Phi = K^{-1} \log \left( \frac{2 \rho e^{-Kh}}{A} \right) + C + o(1) \quad \text{as } \rho \to \infty. \quad (6.2)$$

We move the logarithmic far-field behaviour to \( \eta = 0 \) using the complex potentials \( \log (\zeta - 1) \) and \( \log (\zeta + 1) \) with branch cuts along the \( \xi \)-axis, from 1 to \( \infty \) and from \(-1\) to \(-\infty\), respectively. Thus, we write

$$\Phi(\xi, \eta) = \frac{1}{4K} \left[ \log \left( (\xi - 1)^2 + \eta^2 \right) + \log \left( (\xi + 1)^2 + \eta^2 \right) \right] + \frac{1}{K} \Psi(\xi, \eta), \quad (6.3)$$

where the potential \( \Psi \) is bounded as \( \rho \to \infty \), together with mixed conditions at \( \eta = 0 \), namely, \( \partial \Psi / \partial \eta = 0 \) for \( |\xi| > 1 \) and \( \Psi = \Psi_0(\xi) \) for \( |\xi| < 1 \), where \( \Psi_0(\xi) = -(1/2) \log (1 - \xi^2) \).

The potential \( \Psi \) can be constructed exactly. One way proceeds as follows [39]. Expand the given function \( \psi_0 \) using Chebyshev polynomials \( T_n \),

$$\psi_0(\xi) = \sum_{n=0}^{\infty} a_n T_n(\xi), \quad |\xi| < 1.$$ 

Then \( \psi \) is given by

$$\Psi(\xi, \eta) = \sum_{n=0}^{\infty} a_n \text{Re}[R_n(\xi)],$$

where \( R_0(\xi) = 1 \) and \( R_n(\xi) = (\xi - (\zeta^2 - 1)^{1/2})^n \) for \( n = 1, 2, \ldots \). For our \( \psi_0 \), we can calculate \( a_n \) using the known expansion [38, 1.441.2]

$$\log (1 \pm \xi) = \sum_{n=0}^{\infty} b_n T_n(\pm \xi),$$

where \( b_0 = -\log 2 \) and \( b_n = -(2/n)(-1)^n \) for \( n = 1, 2, \ldots \). Hence, as \( T_n(-\xi) = (-1)^n T_n(\xi) \),

$$\log (1 - \xi^2) = 2b_0 + 2 \sum_{m=1}^{\infty} b_{2m} T_{2m}(\xi)$$

so that \( a_0 = \log 2, a_{2m} = 1/m \) and \( a_{2m-1} = 0 \) for \( m = 1, 2, \ldots \).

We are interested in \( \psi \) for large \( \rho \). As \( R_n(\zeta) \sim (2\zeta)^{-n} \) as \( \zeta \to \infty \) for \( n = 1, 2, \ldots \) and \( R_0 = 1 \), the dominant term comes from \( a_0 \): \( \psi \sim \log 2 \). Hence (6.3) gives

$$\Phi(\xi, \eta) = K^{-1} \log \rho + K^{-1} \log 2 + o(1)$$

$$= K^{-1} \log \left( \frac{2 \rho e^{-Kh}}{A} \right) + h + K^{-1} \log A + o(1) \quad \text{as } \rho \to \infty.$$

Comparison with (6.2) then gives

$$C = h + \frac{1}{K} \log A = h - \frac{2b}{\pi} \log \frac{\pi h}{2b} \quad (6.4)$$

(We note that a simpler calculation should be possible because we do not need \( \psi \) exactly, just its far-field behaviour.) The formula (6.4) can be compared with [40, eqn (3.21)], where a similar problem is solved using a sequence of conformal mappings; see also [11, §3].

The formula for \( C, (6.4) \), can then be used in our estimates for \( R \) and \( T, (3.9) \), in which \( |\Omega| = 2bh \) (figure 6). Thus

$$R \simeq ikb \frac{2}{\pi} \log \sinh \frac{\pi h}{2b} \quad \text{and} \quad T \simeq 1 + ikb \left( \frac{2h}{b} - \frac{2}{\pi} \log \sinh \frac{\pi h}{2b} \right). \quad (6.5)$$

### 7. Labyrinths

The first-order estimates (6.5) can be refined as described in §2(b) so as to enforce energy conservation. Let us assume that this has been done, but retain the notation \( R \) and \( T \). We know
that $R$ and $T$ do not depend on the direction of the incident wave and they do not depend on the location of the bend. If we also ignore evanescent modes, we can approximate reflection by a sequence of $N$ bends by the problem of waves on a straight string with $N$ point scatterers. As an example, figure 7 shows a labyrinthine structure composed of five hairpin bends. The geometry plays a minor role: bends of arbitrary angles could be used (using $R$ and $T$ from §5), but the analysis is simpler if all the bends have the same angle.

Suppose that the distance between consecutive bends is $d$. We assume that $kd$ is large enough to justify evanescent modes being ignored: we are making a ‘wide-spacing approximation’ [41, §8.5]. Then, we use known results for one-dimensional multiple scattering by a periodic row of $N$ identical scatterers. The reflection and transmission coefficients, $R_N$ and $T_N$, respectively, are given by [42, eqn (21)]

$$R_N = \frac{RU_{N-1}(X)}{U_{N-1}(X) - Te^{ikd}U_{N-2}(X)} \quad \text{and} \quad T_N = \frac{Te^{-i(N-1)kd}}{U_{N-1}(X) - Te^{ikd}U_{N-2}(X)},$$

(7.1)

where $X$ is a real number defined by

$$2X = \text{Re}\{T^{-1}e^{-ikd}\}$$

and $U_m$ is a Chebyshev polynomial of the second kind, defined by

$$U_{m-1}(\cos \theta) = \frac{\sin m\theta}{\sin \theta}, \quad m = 0, 1, 2, \ldots \quad (7.2)$$

The formulae (7.1) are exact if (i) $R$ and $T$ are known exactly and (ii) evanescent contributions are negligible. Then it can be shown that we are in a passband for the periodic structure if $|X| < 1$ and we are in a stopband if $|X| > 1$ [42, §3].

The simple analysis given here could be extended to include the effects of the mouth of the labyrinth, where the waveguide connects to a two-dimensional half-plane. This extension would require solutions of two problems. First, an incident plane wave from the half-plane generates a wave along the waveguide, propagating away from the mouth with a transmission coefficient $T_{\text{mouth}}$, say. This is the problem of an inlet in a reflecting wall discussed in [6, §4], where first-order and second-order estimates of $T_{\text{mouth}}$ are derived [6, eqns (4.31) and (4.50)]. For some numerical results, see [43,44].

The second problem has an incident plane wave in the waveguide propagating towards the mouth. The problem is to calculate the reflection coefficient for the wave propagating away from the mouth along the waveguide, $R_{\text{mouth}}$. For calculations of $R_{\text{mouth}}$, see, for example, [45–48].

Combining $R_{\text{mouth}}$ and $T_{\text{mouth}}$ with $R_N$ and $T_N$ is straightforward, provided the mouth is not too close to the first bend. In effect, one just has two scatterers (the mouth and the whole labyrinth) interacting via propagating plane waves: this is another application of wide-spacing approximations.

Figure 7. A labyrinthine structure with five hairpin bends. The thick lines are rigid boundaries. A plane wave is incident from the top left.
8. Discussion

Let us summarize what we have achieved. We considered long waves in a waveguide, interacting with ‘scatters’; the scatterers could be obstacles in the waveguide, and/or bulges, constrictions or bends. In our earlier work [5], we made essential use of symmetry; that assumption is removed here, so that asymmetric problems could be studied. We went beyond classical heuristic matching: solutions involve a potential flow problem and a second problem involving Poisson’s equation. Each problem contributes a certain constant: for the potential flow problem, we have to extract the so-called blockage coefficient, C, and this constant depends on the geometry. The other constant can be calculated easily, using Green’s theorem. This part of the analysis mimics what was done in [5] (and was extended to three-dimensional problems in [23]). Next, we calculated C for a few geometries, using conformal mapping. In particular, the formula for an arbitrary-angled bend, (5.8), seems to be new. Finally, we considered long acoustic waves in labyrinthine structures. The result here is that, under certain stated conditions, solving the acoustic problem is exactly equivalent to the one-dimensional problem of waves along a string with each corner in the labyrinth replaced by a ‘point scatterer’ on the string, with reflection and transmission coefficients computed by the methods described in the earlier part of the paper.

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Authors’ contributions. P.A.M.: conceptualization, formal analysis, investigation, writing—original draft, writing—review and editing.

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Appendix A. Reflection by a patch of roughness on a waveguide wall

Consider a waveguide with parallel rigid walls at $y = 0$ and $y = b$. The incident plane wave is $e^{ikx}$ with $kb < \pi$. If the walls are not straight for all $x$, there will be a reflected wave. Define the scattered field $u_{sc}$ by $u = e^{ikx} + u_{sc}$ where $u$ is the total field and $u_{sc}$ is outgoing:

$$u_{sc} \sim \begin{cases} (T - 1) e^{ikx} & \text{as } x \to \infty, \\ R e^{-ikx} & \text{as } x \to -\infty, \end{cases} \quad (A1)$$

see (2.1). Let us consider a small perturbation of the wall at $y = 0$, so that the rigid wall is specified by $y = \varepsilon f(x)$ for $0 < x < d$, with $f(x) = 0$ otherwise. Here, $f$ is given and $\varepsilon$ is a parameter. We seek an approximate solution valid for $0 < |\varepsilon| \ll 1$. We shall assume that the wall perturbation is continuous: $f(0) = f(d) = 0$.

The strategy is familiar: expand in powers of $\varepsilon$ and linearize the boundary condition on the perturbed wall about the flat wall, $y = 0$. Put $u_{sc} = \varepsilon U_1 + \varepsilon^2 U_2 + \cdots$, $T \simeq 1 + \varepsilon T_1$ and $R \simeq \varepsilon R_1$. The boundary condition $\partial u / \partial n = 0$ on the walls becomes $\partial U_1 / \partial y = 0$ at $y = b$ and $\partial U_1 / \partial y = ik f'(x) e^{ikx}$ at $y = 0$. In the far field, $U_1(x, y) \sim T_1 e^{ikx}$ as $x \to \infty$ and $U_1(x, y) \sim R_1 e^{-ikx}$ as $x \to -\infty$. The goal is to determine $R_1$ and $T_1$.

We use a Fourier transform with respect to $x$, defined by

$$\tilde{U}_1(\xi, y) = \int_{-\infty}^{\infty} U_1(x, y) e^{-i \xi x} dx.$$ 

To secure convergence, we suppose $k$ is complex with a small positive imaginary part; hence the integrand decays exponentially as $|x| \to \infty$, using (A1). Fourier transforming $(\nabla^2 + k^2)U_1 = 0$ and
the boundary conditions gives

\[ \frac{\partial^2 \hat{U}_1}{\partial y^2} + (k^2 - \xi^2) \hat{U}_1 = 0, \quad 0 < y < b, \]

\[ \partial \hat{U}_1 / \partial y = 0 \text{ at } y = b \text{ and } \partial \hat{U}_1 / \partial y = G \text{ at } y = 0, \]

where

\[ G(\xi) = i k \int_0^d f'(x) e^{ik(\xi - x)} \, dx. \]

Solving for \( \hat{U}_1 \) followed by inversion, we obtain, formally,

\[ U_1(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \mu(b - y)}{\mu \sin \mu b} G(\xi) e^{i\mu x} \, d\xi, \quad (A 2) \]

where \( \mu^2 = k^2 - \xi^2 \). The integrand is an even function of \( \mu \), which means that we do not need to define branches for \( \mu \) itself. There are simple poles whenever \( \xi \) is such that \( \mu \sin \mu b = 0 \). Near \( \mu = 0 \), we have \( (\mu \sin \mu b)^{-1} \sim (1 - 1/\mu^2) \) implying simple poles at \( \xi = \pm k \). There are also simple poles when \( \mu b = \pm n\pi \), that is, when \( \xi^2 = k^2 - (n\pi/b)^2 \), where \( n \) is a positive integer; for long waves \( (kb < \pi) \), these poles are pure imaginary, and so correspond to evanescent contributions.

Returning to the poles at \( \xi = \pm k \), we indent the contour in (A 2) below the pole at \( \xi = k \) and above the pole at \( \xi = -k \); this will ensure that we generate outgoing waves, as seen below.

For \( x > 0 \), we close the contour in the upper half of the \( \xi \)-plane, and pick up a residue contribution from the pole at \( \xi = k \). This gives

\[ U_1(x, y) \sim \frac{1}{2\pi} 2\pi i \frac{b}{2k} G(k) e^{ikx} \quad \text{as} \quad x \to \infty. \]

But \( G(k) = 0 \) (using \( f(0) = f(d) = 0 \)), implying that \( T = 1 + O(\epsilon^2) \) as \( \epsilon \to 0 \).

For \( x < 0 \), we close the contour in the lower half of the \( \xi \)-plane, and pick up a residue contribution from the pole at \( \xi = -k \). This gives

\[ U_1(x, y) \sim \frac{1}{2\pi} (-2\pi i) \frac{b}{2k} G(-k) e^{-ikx} \quad \text{as} \quad x \to -\infty. \]

Hence

\[ R \sim \frac{e^b}{2ik} G(-k) = \frac{e^b}{2} \int_0^d f'(x) e^{2ikx} \, dx = -ikb \epsilon \int_0^d f(x) e^{2ikx} \, dx, \]

after an integration by parts.

If \( kd \ll 1 \) (the rough patch is not too long), we can use \( e^{2ikx} \approx 1 \) in the integrand. Then, as \( \epsilon \int_0^d f(x) \, dx \) is the (signed) area occupied by the wall perturbation, the remaining area of the waveguide is \( |\Omega| = bd - \epsilon \int_0^d f(x) \, dx \). The resulting approximations to \( R \) and \( T \) agree with (2.14).

The method used here is similar to that used in [49]. A very similar method was used earlier by Davies [50] for an analogous water-waves problem, in which the motion is governed by Laplace’s equation and the boundary condition at \( y = b \) is \( \partial u / \partial y - Ku = 0 \) where \( K = \omega^2 / g \) and \( g \) is the acceleration due to gravity.

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