Explicit energy calculation for a charged elliptical plate

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Abstract
Potential problems for thin elliptical plates are solved exactly with emphasis on computation of the electrostatic energy. Expansions in terms of Jacobi polynomials are used.

Keywords: Charged elliptical plate, Jacobi polynomials

1. Introduction

Let $\Omega$ denote a thin flat plate lying in the plane $z = 0$, where $Oxyz$ is a system of Cartesian coordinates. The charge distribution on the plate is $\sigma(x)$, where $x = (x, y)$. The potential on the plate is

$$f(x') = \frac{1}{4\pi} \int_\Omega \frac{\sigma(x)}{|x - x'|} \, dx, \quad x' \in \Omega.$$  (1)

The electrostatic energy, $I$, is given by

$$I = \int_\Omega \int_\Omega f(x') \overline{\sigma(x')} \, dx' \, dx = \frac{1}{4\pi} \int_\Omega \int_\Omega \frac{\sigma(x') \overline{\sigma(x)}}{|x - x'|} \, dx \, dx',$$

where the overbar denotes complex conjugation. In a recent paper, Laurens and Tordeux [1] showed how to calculate $I$ when $\Omega$ is an ellipse and $\sigma(x, y)$ is a linear function of $x$ and $y$. We generalize their result: we allow arbitrary polynomials in $x$ and $y$, and we incorporate a weight function to represent singular behaviour near the edge of the plate.

2. An elliptical plate

When $\Omega$ is elliptical, it is convenient to introduce coordinates $\rho$ and $\phi$ so that

$$x = a \rho \cos \phi, \quad y = b \rho \sin \phi, \quad 0 < b \leq a.$$  (2)

Then, $\Omega$ is defined by $\Omega = \{(x, y, z) : 0 \leq \rho < 1, -\pi \leq \phi < \pi, z = 0\}$. Thus, $\rho = 1$ gives the edge of the plate $\Omega$.

Equation (1) can be regarded as an integral equation for $\sigma$ when $f$ is given [2, 3, 4]. Alternatively, (1) can be regarded as a formula for $f$ when $\sigma$ is given; this is the view adopted in [1].

When $f$ is given, the function $\sigma$ is infinite at $\rho = 1$, in general. In fact, there is a general result, known as Galin’s theorem, asserting that if $f(x, y)$ is a polynomial, then $\sigma$ is a polynomial of the same degree multiplied by $(1 - \rho^2)^{-1/2}$. In particular, if $f$ is a constant, then $\sigma$ is a constant multiple of $(1 - \rho^2)^{-1/2}$. 

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3. Fourier transforms

We start with a standard Fourier integral representation,

\[
\frac{1}{|x - x'|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\xi|^{-1} \exp \{i\xi \cdot (x - x')\} \, d\xi,
\]

where \( \xi = (\xi, \eta) \). Henceforth, we write \( \iint \) when the integration limits are as in (3). Thus

\[
f(x') = \frac{1}{4\pi} \iint |\xi|^{-1} U(\xi) \exp (-i\xi \cdot x') \, d\xi,
\]

and

\[
I = \frac{1}{2} \iint |\xi|^{-1} |U(\xi)|^2 \, d\xi,
\]

where

\[
U(\xi) = \frac{1}{2\pi} \int_\Omega \sigma(x) \exp (i\xi \cdot x) \, dx.
\]

For an elliptical plate, we write the Fourier-transform variable \( \xi \) as

\[
\xi = \left( \frac{\lambda}{a} \right) \cos \psi \quad \text{and} \quad \eta = \left( \frac{\lambda}{b} \right) \sin \psi.
\]

Then, using (2), \( \xi \cdot x = \lambda \rho \cos (\phi - \psi) \). Hence,

\[
\exp (i\xi \cdot x) = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(\lambda \rho) \cos n(\phi - \psi),
\]

where \( J_n \) is a Bessel function, \( \epsilon_0 = 1 \) and \( \epsilon_n = 2 \) for \( n \geq 1 \).

In order to evaluate \( U(\xi) \), defined by (6), we suppose that \( \sigma \) has a Fourier expansion,

\[
\sigma(x) = \sum_{m=0}^{\infty} \sigma_m(\rho) \cos m\phi + \sum_{m=1}^{\infty} \tilde{\sigma}_m(\rho) \sin m\phi.
\]

Then, using \( dx = ab \rho \, d\rho \, d\phi \) and defining

\[
S_n[g_n; \lambda] = \int_0^1 g_n(\rho) J_n(\lambda \rho) \rho \, d\rho,
\]

we obtain

\[
U(\xi) = ab \sum_{n=0}^{\infty} i^n S_n[\sigma_n; \lambda] \cos n\psi + ab \sum_{n=1}^{\infty} i^n S_n[\tilde{\sigma}_n; \lambda] \sin n\psi.
\]

We have \( d\xi = (ab)^{-1} \lambda \, d\lambda \, d\psi \) and \( |\xi| = \left( \frac{\lambda}{b} \right) (1 - k^2 \cos^2 \psi)^{1/2} \), where \( k^2 = 1 - (b/a)^2 \); \( k \) is the eccentricity of the ellipse.

From (4), we obtain

\[
f(x) = f_0(\rho) + 2 \sum_{n=1}^{\infty} \left\{ f_n(\rho) \cos n\phi + \tilde{f}_n(\rho) \sin n\phi \right\}
\]

where

\[
f_n(\rho) = \frac{b}{2\pi} \sum_{m=0}^{\infty} \int_{m\pi}^\infty \int_{0}^{\infty} J_n(\lambda \rho) S_m[\sigma_m; \lambda] \, d\lambda,
\]

\[
\tilde{f}_n(\rho) = \frac{b}{2\pi} \sum_{m=1}^{\infty} \int_{m\pi}^\infty \int_{0}^{\infty} J_n(\lambda \rho) S_m[\tilde{\sigma}_m; \lambda] \, d\lambda.
\]
\[ I_{mn}^e(k) = i^m(-i)^n \int_0^\pi \frac{\cos m \psi \cos n \psi}{\sqrt{1 - k^2 \cos^2 \psi}} \, d\psi, \tag{11} \]
\[ I_{mn}^e(k) = i^m(-i)^n \int_0^\pi \frac{\sin m \psi \sin n \psi}{\sqrt{1 - k^2 \cos^2 \psi}} \, d\psi \tag{12} \]

and we have noticed that \(|\xi|\) is an even function of \(\psi\). The integrals \(I_{mn}^e\) and \(I_{mn}^o\) can be reduced to combinations of complete elliptic integrals, \(K(\xi)\) and \(E(\xi)\). They are zero unless \(m\) and \(n\) are both even or both odd. (See [5, p. 276] for a discussion of similar integrals.) Explicit formulas for a few of these integrals will be given later.

For the energy, \(I\), (5) gives
\[
I = \frac{1}{2a} \int_0^\infty \int_{-\pi}^{\pi} |U(\xi)|^2 \frac{d\psi \, d\lambda}{\sqrt{1 - k^2 \cos^2 \psi}}
= ab^2 \sum_{m=0}^\infty \sum_{n=0}^\infty I_{mn}^o(k) \int_0^\pi S_m[\sigma_m; \lambda] S_n[\sigma_n; \lambda] \, d\lambda
+ ab^2 \sum_{m=1}^\infty \sum_{n=1}^\infty I_{mn}^e(k) \int_0^\pi S_m[\sigma_m; \lambda] S_n[\sigma_n; \lambda] \, d\lambda. \tag{13} \]

### 4. Polynomial expansions

To make further progress, we must be able to evaluate \(S_n[\sigma_n; \lambda]\), defined by (8). We do this by expanding \(g_\lambda(\rho)\) using the functions
\[
G_j^{(n,\nu)}(\rho) = \rho^\nu (1 - \rho^2)^{\nu} P_j^{(n,\nu)}(1 - 2 \rho^2),
\]
where \(P_j^{(n,\nu)}\) is a Jacobi polynomial. The parameter \(\nu\) controls the behaviour near the edge of the ellipse, \(\rho = 1\). Thus, when \(\nu = 0\), \(G_j^{(n,0)}(\rho)\) is a polynomial; this covers the case discussed in [1]. Setting \(\nu = -\frac{1}{2}\) gives appropriate expansion functions when the goal is to solve (1) for \(\sigma\). We note that Boyd [6, §18.5.1] has advocated using the polynomials \(G_j^{(n,0)}(r)\) as radial basis functions in spectral methods for problems posed on a disc, \(0 \leq r < 1\).

The functions \(G_j^{(n,\nu)}\) are orthogonal. To see this, note that Jacobi polynomials satisfy
\[
\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta P_i^{(\alpha,\beta)}(x) P_j^{(\alpha,\beta)}(x) \, dx = h_i(\alpha, \beta) \delta_{ij},
\]
where \(h_i\) is known and \(\delta_{ij}\) is the Kronecker delta; see [7, §18.3]. Hence, the substitution \(x = 1 - 2 \rho^2\) gives
\[
\int_0^1 G_i^{(n,\nu)}(\rho) G_j^{(n,\nu)}(\rho) \, \frac{\rho \, d\rho}{(1 - \rho^2)^\nu} = 2^{-n-\nu-2} h_i(n, \nu) \delta_{ij}. \tag{14} \]

Next, we use Tranter’s integral [8, 9] to evaluate \(S_n[G_j^{(n,\nu)}; \lambda]\):
\[
\int_0^1 J_n(\lambda \rho) G_j^{(n,\nu)}(\rho) \, \rho \, d\rho = \frac{2^\nu}{\lambda^{\nu+1} j!} \Gamma(\nu + j + 1) J_{2j+n+\nu+1}(\lambda).
\]

Thus, if we write
\[
\sigma_n(\rho) = \sum_{j=0}^{\infty} \frac{j!}{2^\nu \Gamma(\nu + j + 1)} G_j^{(n,\nu)}(\rho), \tag{15} \]
where \(s_j^n\) are coefficients, we find that
\[
S_n[\sigma_n; \lambda] = \sum_{j=0}^{\infty} \frac{s_j^n}{\lambda^{\nu+1}} J_{2j+n+\nu+1}(\lambda). \tag{16} \]
We also expand \( \tilde{s}_n(\rho) \) as (15) but with coefficients \( \tilde{s}_n^\mu \).

If we substitute (16) in (9), we encounter Weber–Schaafheitlin integrals; these can be evaluated. We give a simple example later.

If we substitute (16) in (13), we encounter integrals of the type

\[
\int_0^\infty \lambda^{-2\mu} J_{p+\mu}(\lambda) J_{q+\mu}(\lambda) \, d\lambda
\]

(17)

where \( \mu = \nu + 1 \), and \( p \) and \( q \) are non-negative integers. The integral (17) is known as the critical case of the Weber–Schaafheitlin integral; its value is [7, eqn 10.22.57]

\[
\frac{\Gamma\left(\frac{1}{2}[p + q + 1]\right) \Gamma(2\mu)}{2^{2\mu} \Gamma(2[p + q + p + 1]) \Gamma(\frac{1}{2}[4\mu + p + q + 1])}.
\]

(18)

5. Three examples

We discuss three examples. In the first, we examine the dependence on the parameter \( \nu \) but, for simplicity, we ignore any dependence on the angle \( \phi \). In the second example, we compare with some results of Roy and Sabina [2] for \( \nu = -\frac{1}{2} \). In the third example, we assume that \( \sigma(x, y) \) is a general quadratic function of \( x \) and \( y \) (so that \( \nu = 0 \)); this extends the calculations in [1], where \( \sigma \) was taken as a linear function.

5.1. Example: dependence on \( \nu \)

For a very simple example, suppose that \( \sigma(x) = (1 - \rho^2)^\nu \) for some \( \nu > -1 \). Thus, as \( P_0^{(n, \nu)} = 1 \), (15) gives \( s_0^0 = 2^\nu \Gamma(\nu + 1) \). All other coefficients \( s_j^n \) and \( \tilde{s}_j^n \) are zero. Then, from (16), \( S_0[\sigma_0; \lambda] = s_0^0 \lambda^{-\nu-1} J_{\nu+1}(\lambda) \).

Hence

\[
f(x) = f_0(\rho) = \frac{bs_0^0}{2\pi} J_0(k) \int_0^\infty \lambda^{-\nu-1} J_0(\lambda \rho) J_{\nu+1}(\lambda) \, d\lambda, \quad 0 \leq \rho < 1.
\]

(19)

From (11), we obtain

\[
I_{00}^\nu = 2 \int_0^{\pi/2} \frac{dx}{\Delta} = 2K(k),
\]

(20)

where \( \Delta = (1 - k^2 \sin^2 x)^{1/2} \). From [7, eqn 10.22.56], the integral in (19) evaluates to

\[
\frac{\sqrt{\pi}}{2^{\nu+1} \Gamma(\nu + \frac{1}{2})} F(\frac{1}{2}, -\nu - \frac{1}{2}; 1; \rho^2),
\]

where \( F \) is the Gauss hypergeometric function. Hence

\[
f(x) = \frac{b}{2\pi} K(k) \frac{\sqrt{\pi} \Gamma(\nu + 1)}{\Gamma(\nu + \frac{1}{2})} F(\frac{1}{2}, -\nu - \frac{1}{2}; 1; \rho^2), \quad 0 \leq \rho < 1.
\]

When \( \nu = -\frac{1}{2} \), \( F(\frac{1}{2}, 0; 1; \rho^2) = 1 \) and \( f(x) = \frac{1}{2} bK(k) \), a constant, in accord with Galin’s theorem.

When \( \nu = 0 \), we obtain \( f(x) = (2b/\pi^2) K(k) E(\rho) \) for \( 0 \leq \rho < 1 \), using [7, eqn 19.5.2]. Thus, for this particular \( f \), the solution of the integral equation (1) is \( \sigma = 1 \). Although this solution is bounded, we see that adding a small constant to \( f \) adds a constant multiple of \( (1 - \rho^2)^{-1/2} \) to \( \sigma \). In other words, the integral equation (1) has bounded solutions for some \( f \), but these solutions are not typical: singular behaviour around the edge of \( \Omega \) should be expected.

\[1\text{There are errors in the published version of this Example; see Addendum}\]
It is shown in section 5.3 that \( I_1(k) = 2(K - E)/k^2 \). From [7, eqn 10.22.56], the integral in (21) evaluates to \( \frac{2}{\rho} \sqrt{\frac{\pi}{2}} \). Hence \( f(x) = \pi bx K_1 \), in agreement with [2, eqn (14b)].

5.3. Example: quadratic \( \sigma \)

Suppose that

\[
\sigma(x) = \alpha_0 + \alpha_1(x/a) + \alpha_2(y/b) + 2\alpha_3(x/a)^2 + 2\alpha_4(x/y)/(ab) + 2\alpha_5(y/b)^2
\]

\[
= \{\alpha_0 + \rho^2(\alpha_3 + \alpha_5)\} + \alpha_1 \rho \cos \phi + \alpha_2 \rho \sin \phi + (\alpha_3 - \alpha_5) \rho^2 \cos 2\phi + \alpha_4 \rho^2 \sin 2\phi,
\]

with constants \( \alpha_j \); Laurens and Tordeux [1] have \( \alpha_3 = \alpha_4 = \alpha_5 = 0 \). Then (7) gives

\[
\sigma_0(\rho) = \alpha_0 + (\alpha_3 + \alpha_5) \rho^2,
\]

(22)

\[
\sigma_1 = \alpha_1 \rho, \quad \sigma_2 = (\alpha_3 - \alpha_5) \rho^2 \quad \text{and} \quad \sigma_2 = \alpha_4 \rho^2. \]

All other terms in (7) are absent.

Next, we use \( P_0^{(n,0)}(x) = 1 \) and \( \nu = 0 \). These give \( s_0 = \alpha_1, s_1 = \alpha_2, s_2 = \alpha_3 - \alpha_5 \) and \( s_4 = \alpha_4 \). For \( s_0^1 \), we use \( P_1^{(0,0)}(x) = P_1(x) = x \), giving

\[
\sigma_0(\rho) = s_0^0G_0^{(0,0)} + s_1^0G_1^{(0,0)} = s_0^0 + s_1^0(1 - 2\rho^2).
\]

Comparison with (22) gives \( \alpha_0 = s_0^0 + s_1^0 \) and \( \alpha_3 + \alpha_5 = -2s_1^0 \); these determine \( s_0^0 \) and \( s_1^0 \). Apart from the six mentioned, all other coefficients \( s_j^0 \) and \( s_j^1 \) are zero.

Then, from (16), we obtain

\[
\lambda S_0[\sigma_0; \lambda] = s_0^0J_1(\lambda) + s_1^0J_3(\lambda),
\]

\[
\lambda S_1[\sigma_1; \lambda] = s_1^1J_2(\lambda), \quad \lambda S_2[\sigma_2; \lambda] = s_2^0J_3(\lambda),
\]

\[
\lambda S_2[\sigma_2; \lambda] = s_2^1J_3(\lambda).
\]

We use these to compute the energy, \( I \), given by (13). We will need the integrals (see (18))

\[
J_{pq} = \int_0^\infty \frac{1}{\lambda^2} J_{p+1}(\lambda)J_{q+1}(\lambda) \, d\lambda
\]

\[
= \frac{\Gamma(\frac{1}{2}p + q + 1)}{4\Gamma(\frac{1}{2}3 + p + q)} \frac{\Gamma(\frac{1}{2}3 + q - p)}{\Gamma(\frac{1}{2}5 + p + q)}.
\]

(23)

Thus

\[
\frac{I}{ab^2} = I_0^0 \int_0^\infty |S_0[\sigma_0; \lambda]|^2 \, d\lambda + I_1^1 \int_0^\infty |S_1[\sigma_1; \lambda]|^2 \, d\lambda
\]

\[
+ I_0^2 \int_0^\infty |S_2[\sigma_2; \lambda]|^2 \, d\lambda + 2I_0^2 \, \text{Re} \int_0^\infty S_0[\sigma_0; \lambda]S_2[\sigma_2; \lambda] \, d\lambda
\]

\[
+ I_1^2 \int_0^\infty |S_1[\sigma_1; \lambda]|^2 \, d\lambda + 2I_0^2 \, \text{Re} \int_0^\infty S_1[\sigma_1; \lambda]S_2[\sigma_2; \lambda] \, d\lambda
\]

\[
= I_0^0 \left\{ |s_0^0|^2 J_0 + 2 \text{Re}(s_0^0 S_1^1) J_0 + |s_1^1|^2 J_2 \right\} + I_1^1 |s_0^1|^2 J_{11}
\]

\[
+ I_0^2 |s_0^0|^2 J_{12} + 2I_0^2 \, \text{Re}(s_0^0 S_1^1 + s_1^1 S_1^2) J_{22}
\]

\[
+ I_1^1 |s_0^1|^2 J_{11} + I_2^2 |s_2^0|^2 J_{22}.
\]

(24)
From (23), we obtain

\[ J_{00} = \frac{4}{3\pi}, \quad J_{11} = \frac{4}{15\pi}, \quad J_{22} = \frac{4}{35\pi}, \quad J_{02} = \frac{4}{45\pi}. \]

For \( I_{mm}^c \) and \( I_{nn}^c \), we have \( I_{00}^c = 2K(k) \) (see (20)), \( I_{mm}^c + I_{nn}^c = I_{00}^c \),

\[
I_{11}^c - I_{11}^s = 2 \int_0^{\pi/2} \frac{\cos 2x}{\Delta} \, dx = \frac{2}{k^2}(k^2 - 2)K(k) + \frac{4}{k^2}E(k),
\]

\[
I_{22}^c - I_{22}^s = 2 \int_0^{\pi/2} \frac{\cos 4x}{\Delta} \, dx = \frac{32k^2}{3k^4}K + 2K + \frac{16}{3k^4}(k^2 - 2)E,
\]

where \( k^2 = 1 - k^2 = (b/a)^2 \). Thus

\[
I_{11}^c = 2(K - E)/k^2, \quad I_{11}^s = 2(E - k^2 K)/k^2,
\]

\[
I_{22}^c = 2\{(3k^4 + 8k^2)/K + 4(k^2 - 2)E\}/(3k^4),
\]

\[
I_{s2}^c = 8\{(2 - k^2)E - 2k^2 K\}/(3k^4).
\]

One can check that these all have the correct limiting values as \( k \to 0 \).

This completes the computation of all the quantities required in (24). In the special case considered by Laurens and Tordeux [1], we have \( s_0^0 = \alpha_0 \), \( s_0^1 = \alpha_1 \), \( s_0^2 = \alpha_2 \) and \( s_0^3 = \tilde{s}_0^3 = 0 \), whence

\[
I / (ab^2) = |\alpha_0|^2 I_{00}^c J_{00} + |\alpha_1|^2 I_{11}^c J_{11} + |\alpha_2|^2 I_{11}^c J_{11}
\]

\[
= \frac{8}{15\pi} \left\{ 5|\alpha_0|^2 K + |\alpha_1|^2 \frac{K - E}{k^2} + |\alpha_2|^2 \frac{E - k^2 K}{k^2} \right\},
\]

in agreement with [1, Theorem 1.1].

6. Discussion

The (weakly singular) integral equation (1) arises when Laplace’s equation holds in the three-dimensional region exterior to a thin flat plate \( \Omega \) with Dirichlet boundary conditions on both sides of \( \Omega \). There are analogous (hypersingular) integral equations when a Neumann boundary condition is imposed. Explicit formulas for \( \sigma \) in terms of \( f \) are known when \( \Omega \) is circular; for a review, see [10].

Expansion methods of the kind used above for problems involving elliptical plates, screens or cracks have a long history. The author’s 1986 paper [5] gives references for Neumann problems, in the context of crack problems. For Dirichlet problems, see [2, 3, 4]. Similar expansion methods have been used recently for the problem of internal wave generation in a continuously stratified fluid by an oscillating elliptical plate [11].

References

Addendum: corrections to Example 5.1

The formula for $f_0(\rho)$, (19), is correct but other Fourier components of $f(x)$ are also non-zero, in general. Thus, it is easy to see that $f_{2m+1}$ and $\tilde{f}_n$ are all zero, leaving

$$f(x) = f_0(\rho) + 2 \sum_{m=1}^{\infty} f_{2m}(\rho) \cos 2m\phi$$

with $f_{2m}$ given by (9),

$$f_{2m}(\rho) = \frac{b s_0^0}{2\pi} f_{0,2m}(k) \int_0^\infty \lambda^{-\nu-1} J_{2m}(\lambda \rho) J_{\nu+1}(\lambda) \, d\lambda, \quad 0 \leq \rho < 1.$$  \hfill (25)

From [7, eqn 10.22.56], the integral in (25) evaluates to

$$\frac{\rho^{2m} \Gamma(m + \frac{1}{2})}{2^{\nu+1}(2m)! \Gamma(\nu - m + \frac{3}{2})} F(m + \frac{1}{2}, m - \nu - \frac{1}{2}; 2m + 1; \rho^2) = \mathcal{I}_m^\nu(\rho),$$  \hfill (26)

say. This gives the stated result when $m = 0$.

When $\nu = -\frac{1}{2}$, $\mathcal{I}_m^{-1/2}(\rho) = 0$ for $m = 1, 2, 3, \ldots$ (because of the $\Gamma$ function in the denominator). Then, $f(x) = f_0(\rho) = \frac{\pi}{2} h K(k)$, a constant, in accord with Galin’s theorem.

When $\nu = 0$, $\mathcal{I}_m^0(\rho) = (2/\pi)E(\rho)$ for $0 \leq \rho < 1$, using [7, eqn 19.5.2]. For $m \geq 1$, $\mathcal{I}_m^0(\rho)$ is given by (26) but the hypergeometric function does not seem to simplify. However, we find that

$$\lim_{\rho \to 1^-} \mathcal{I}_m^0(\rho) = (2/\pi)(-1)^m/(1 - 4m^2),$$

implying that $f(x)$ is bounded around the edge of $\Omega$. Having constructed $f$ is this way, the last three sentences of Example 5.1 remain valid.

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