Some Applications of the Mellin Transform to Asymptotics of Series
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Abstract
Mellin transforms are used to find asymptotic approximations for functions defined by series. Such approximations were needed in the analysis of a water-wave problem, namely, the trapping of waves by submerged plates. The method seems to have wider applicability.

1 Introduction

It is well known that asymptotic approximations for functions defined by integrals can often be found using Mellin transforms [2], [8]. Analogous results can also be sought for functions defined by series,

\[ f(x) = \sum_{n=1}^{\infty} c_n u(\mu_n x), \]  

(1)

where \( c_n \) and \( \mu_n \) are known constants and \( u(y) \) is defined for all \( y > 0 \). We assume that the series is convergent for all \( x > 0 \), and seek the asymptotic behaviour of \( f(x) \) as \( x \to 0^+ \). Ramanujan [1, Ch. 15] considered some problems of this type, such as \( \mu_n = n^p \) and \( u(x) = e^{-x} \), using the Euler-Maclaurin formula. In (1), \( u(y) \) is sampled at points \( y = \mu_n x \); we describe such points, and the series (1), as separable. Separable series can often be analysed using Mellin transforms; this method was used to confirm some of Ramanujan’s results [1]. More generally, we study non-separable series

\[ f(x) = \sum_{n=1}^{\infty} c_n u(\lambda_n(x)), \]  

(2)
where \( u(y) \) is sampled at non-separable points, \( y = \lambda_n(x) \). We find asymptotic approximations to such series by first finding suitable separable approximations to \( \lambda_n(x) \), and then use Mellin transforms.

Both separable and non-separable series arise in waveguide problems. In this context, the limit \( 1/h \to 0 \) is of interest, where the walls are at \( y = 0 \) and \( y = h \). A typical problem in acoustics concerns an infinite bifurcated waveguide, with a semi-infinite plate (the septum) along \( y = d \), \( x < 0 \) (closed geometry). A waveguide mode is incident from \( x = -\infty \) in the region \( 0 < y < d \); it is partially reflected at the end of the septum and partially transmitted into the rest of the guide. The same problem can be considered when \( h = \infty \). This corresponds to an open-ended waveguide (open geometry). The connection between open-geometry and related closed-geometry problems is of interest because the latter are often easier to solve.

The same closed geometry has been used by Linton & Evans [4] in the context of linear water waves. The governing equation is the modified Helmholtz equation \( (\nabla^2 - l^2)\phi = 0 \), where \( l \) is the positive wavenumber in a direction perpendicular to the \( xy \)-plane. The septum and the bottom \( (y = h) \) are hard, whereas the boundary condition on the free surface \( y = 0 \) is \( K \phi + \partial\phi/\partial y = 0 \), where \( K \) is another positive wavenumber. Two more wavenumbers, \( k \) and \( k_0 \), are defined to be the unique positive real roots of

\[
K = k \tanh kd \quad \text{and} \quad K = k_0 \tanh k_0 h, \tag{3}
\]

respectively, and then \( l \) is chosen to satisfy \( K < k_0 < l < k \). Hence, a surface wave incident from \( x = -\infty \) will be totally reflected by the end of the plate. Linton & Evans [4] gave an explicit formula for the argument of the (complex) reflection coefficient, which they used to estimate the frequencies of waves trapped above a long horizontal submerged plate. We examine their formula below, and extract the limiting formula for deep water \( (h \to \infty) \). Indeed, it was a study of the limiting problem that originally motivated the present analysis.

## 2 Mellin transforms

To find asymptotic approximations for separable series (1), we use the Mellin transform. The Mellin transform of a function \( f \), and its inverse, are

\[
\tilde{f}(z) = \int_{0}^{\infty} f(x)x^{z-1}\,dx \quad \text{and} \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(z)x^{-z}\,dz,
\]

respectively. Typically, \( \tilde{f}(z) \) will be an analytic function of \( z \) within a strip, \( a < \sigma < b \), say, where \( z = \sigma + i\tau \); within this strip, \( |\tilde{f}(z)| \to 0 \) as \(|\tau| \to \infty\); and \( a < c < b \). We can obtain an asymptotic expansion of \( f(x) \) for small
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3

Suppose that \( \tilde{f}(z) \) is analytic in a left-hand plane, \( \sigma \leq a \), apart from poles at \( z = -a_m, m = 0, 1, 2, \ldots \); let the principal part of the Laurent expansion of \( \tilde{f}(z) \) about \( z = -a_m \) be given by

\[
\sum_{n=0}^{N(m)} A_{mn} \frac{(-1)^n n!}{(z + a_m)^{n+1}}.
\]

Assume that \( |\tilde{f}(\sigma + i\tau)| \to 0 \) as \( |\tau| \to \infty \) for \( \sigma \) from \( \sigma \leq a' \leq \sigma \leq a \), and that \( |\tilde{f}(a' + i\tau)| \) is integrable for \( |\tau| < \infty \). Then, if \( a' \) can be chosen so that

\[
-\text{Re}(a_{M+1}) < a' < -\text{Re}(a_M)
\]

for some \( M \), \( f(x) \) has the asymptotic expansion

\[
f(x) \sim M \sum_{m=0}^{N(m)} \sum_{n=0}^{N(m)} A_{mn} x^{a_n} (\log x)^n \text{ as } x \to 0+.
\]

For more information on Mellin transforms, see [2, Ch. 4], [5], [8, Ch. 3].

3 Separable series: a problem of Ramanujan

Example 1. Let \( \nu \) be a real parameter. Find the behaviour of

\[
f_\nu(x) = \sum_{n=1}^{\infty} n^{\nu-1} e^{-nx} \text{ as } x \to 0+.
\]

We can take \( c_n = n^{\nu-1} \), \( \mu_n = n \) and \( u(x) = e^{-x} \). Hence

\[
\tilde{f}_\nu(z) = \zeta(z - \nu + 1) \Gamma(z),
\]

where \( \Gamma(z) \) is the gamma function and \( \zeta(z) \) is the Riemann zeta function. It is known that \( \Gamma(z) \) is an analytic function of \( z \), apart from simple poles at \( z = -N \) with residue \((-1)^N/N!\), for \( N = 0, 1, 2, \ldots \). It is also known that \( \zeta(z) \) is analytic for all \( z \), apart from a simple pole at \( z = 1 \); near \( z = 1 \), \( \zeta(z) \simeq (z-1)^{-1} + \gamma \), where \( \gamma = 0.5772 \ldots \) is Euler’s constant.

Let us suppose that \( 0 < \nu < 1 \). Then, \( \tilde{f}_\nu(z) \) is analytic for \( \sigma > \nu \). We choose the inversion contour along \( \sigma = c \), with \( c > \nu \). Moving the contour to the left, we pick up a residue contribution from the simple pole at \( z = \nu \): this gives the leading contribution as

\[
f_\nu(x) \sim x^{-\nu} \Gamma(\nu) \text{ as } x \to 0+.
\]
the inversion contour further to the left, we formally obtain Ramanujan’s expansion [1, p. 306],

\[ f_\nu(x) \sim x^{-\nu} \Gamma(\nu) + \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} \zeta(1 - \nu - m) \text{ as } x \to 0. \] (5)

The fact that this is an asymptotic expansion follows from Theorem 1 and the known properties of \( \zeta(z) \) and \( \Gamma(z) \) as \(|\tau| \to \infty\). Actually, (5) is valid for all values of \( \nu \), apart from \( \nu = -N \). In these cases, there is a double pole at \( z = -N \), giving a term proportional to \( x^N \log x \).

4 Non-separable series: a model problem

Let us consider some non-separable series involving the roots of the transcendental equation (3)\(_2\). This has real roots \( \pm k_0 \) and an infinite number of pure imaginary roots, \( \pm ik_n, n = 1, 2, \ldots \); thus, \( k_n \) are the positive real roots of

\[ K + k_n \tan k_n h = 0, \quad n = 1, 2, \ldots; \] (6)

they are ordered so that \((n - \frac{1}{2})\pi < k_n h < n\pi\). In the context of water-wave problems, \( h \) is the constant water depth and \( K \) is the positive real wavenumber. We are interested in the deep-water limit, \( h \to \infty \). In dimensionless variables, we define \( x = (Kh)^{-1} \) and \( \lambda_n(x) = k_n h \), so that

\[ \cos \lambda_n(x) + x\lambda_n(x) \sin \lambda_n(x) = 0 \quad \text{with} \quad (n - \frac{1}{2})\pi < \lambda_n(x) < n\pi, \quad n = 1, 2, \ldots. \] (7)

It is straightforward to show that \( \lambda_n(x) \) behaves as follows:

\[ \lambda_n(x) \sim n\pi - (n\pi x)^{-1} - (x - \frac{1}{2})(n\pi x)^{-3} \] (9)
as \( n \to \infty \) for fixed \( x \), and

\[ \lambda_n(x) \sim (n - \frac{1}{2})\pi(1 + x + x^2) \] (10)
as \( x \to 0 \) for fixed \( n \). It is this non-uniform behaviour that causes difficulties.

To find some uniform approximations, we rewrite the definition (7) as

\[ \sin \nu_n(x) - x(\mu_n + \nu_n(x)) \cos \nu_n(x) = 0, \quad \text{where} \quad (11) \]

\[ \lambda_n(x) = \mu_n + \nu_n(x), \quad (12) \]

\( \mu_n = (n - \frac{1}{2})\pi \) and \( 0 < \nu_n < \pi/2 \). Discarding the second term inside the braces in (11) (this is certainly reasonable for large \( n \)), we obtain

\[ \nu_n(x) \simeq \tan^{-1}(\mu_n x) = \nu_n^{(1)}(x), \]

(13)
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say, which is a separable approximation to $\nu_n(x)$. The approximation $\lambda_n(x) \simeq \mu_n + \nu_n^{(1)}(x)$ agrees with the first two terms in (9) and with the first two terms in (10). This approximation can be improved by iteration: replace $\nu_n(x)$ by $\nu_n^{(1)}(x)$ inside the braces in (11) to give

$$\nu_n(x) \simeq \tan^{-1}\{\mu_n x + x \tan^{-1}(\mu_n x)\} = \nu_n^{(2)}(x), \quad (14)$$

say. Then, the approximation $\lambda_n(x) \simeq \mu_n + \nu_n^{(2)}(x)$ agrees with the three-term asymptotics in (9) and in (10).

**Example 2.** Find the behaviour of $f(x) = \pi \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n(x)} - \frac{1}{n\pi}\right)$ as $x \to 0^+$. The series converges for all $x \geq 0$; in fact, using the bounds (8), we have $0 < f(x) < 2 \log 2$ for $x > 0$. As $\lambda_n(0) = (n - \frac{1}{2})\pi = \mu_n$, for all $n$, write

$$f(x) = 2 \log 2 + S(x),$$

where $S(x) = \pi \sum_{n=1}^{\infty} s_n(x)$ and $s_n(x) = \frac{1}{\lambda_n(x)} - \frac{1}{\mu_n}$.

We have $S(x) \to 0$ as $x \to 0$ and $S(x)$ is bounded as $x \to \infty$, whence $\tilde{S}(z)$ is analytic in a strip $-\delta < \sigma < 0$, where $\delta > 0$. In fact, we note that $s_n(x) = O(x)$ as $x \to 0$ and is bounded as $x \to \infty$, whence $\tilde{s}_n(z)$ is analytic for $-1 < \sigma < 0$; thus, we expect that $\delta = 1$.

We shall treat $S(x)$ using our separable approximations for $\nu_n(x)$. Since the latter may not be appropriate for small values of $n$, we split the sum:

$$S(x) = \pi \sum_{n=1}^{M} s_n(x) + \pi \sum_{n=M+1}^{\infty} s_n(x) = S_M(x) + S_M^\infty(x), \quad (16)$$

say, where $M$ is fixed. For $S_M$, we can use (10) to give

$$S_M(x) \sim \pi \sum_{n=1}^{M} \mu_n^{-1}\{(1 + x + x^2)^{-1} - 1\} = -\pi x \sum_{n=1}^{M} \mu_n^{-1} + O(x^3) \quad (17)$$

as $x \to 0$. For $S_M^\infty$, we start with $s_n(x) \simeq -\mu_n^{-2}(\nu_n - \nu_n^2/\mu_n)$, since $|\nu_n/\mu_n|$ is small. Next, we approximate $\nu_n$ by $\nu_n^{(2)}$ and $\nu_n^2$ by $(\nu_n^{(1)})^2$. Finally, since $|\nu_n^{(1)}/\mu_n|$ is small, we can approximate $\nu_n^{(2)}$ using the Taylor approximation

$$\tan^{-1}(X + H) \simeq \tan^{-1}X + H(1 + X^2)^{-1} \quad (18)$$
for small \( H \); the result is
\[
s_n(x) \simeq -\mu_n^{-2} \{ v_n^{(1)}(x) + x v_n^{(1)}(x) [1 + (\mu_n x)^2]^{-1} - \mu_n^{-1} [v_n^{(1)}(x)]^{-2} \} = s_n^{(1)}(x),
\]
say. This is our final separable approximation for \( s_n(x) \). We find that the error, \( |s_n - s_n^{(1)}| \) is \( O(n^{-4}) \) as \( n \to \infty \) for fixed \( x \), and is \( O(x^2) \) as \( x \to 0 \) for fixed \( n \). The Mellin transform of \( s_n^{(1)}(x) \) is given by
\[
s_n^{(1)}(z) = -\mu_n^{-z-2} \tilde{u}_1(z) + \mu_n^{-z-3} \tilde{u}_2(z), \tag{19}
\]
where, by definition,
\[
\tilde{u}_1(z) = \int_0^\infty x^{z-1} \tan^{-1} x \, dx = \frac{\pi}{2z \sin \left[ \pi (z-1)/2 \right]}, \tag{20}
\]
\[
\tilde{u}_2(z) = \int_0^\infty x^{z-1} \{ \tan^{-1} x - x(1 + x^2)^{-1} \} \tan^{-1} x \, dx. \tag{21}
\]
\( \tilde{u}_1(z) \) is analytic for \(-1 < \sigma < 0 \) and \( \tilde{u}_2(z) \) is analytic for \(-4 < \sigma < 0 \).

Summing over \( n \), using (16) and (19), gives
\[
\tilde{S}_M^\infty(z) \simeq -\psi_M(z+2) \tilde{u}_1(z) + \psi_M(z+3) \tilde{u}_2(z), \tag{22}
\]
where, by definition,
\[
\psi_M(z) = \pi \sum_{n=M+1}^\infty \mu_n^{-z} = \pi^{1-\frac{z}{2}} \zeta(z) - \pi \sum_{n=1}^{M} \mu_n^{-z}. \tag{23}
\]
\( \psi_M(z) \) is analytic for all \( z \), apart from a simple pole at \( z = 1 \);
\[
\psi_M(z) \simeq (z - 1)^{-1} + \gamma + \log (4/\pi) - \pi \sum_{n=1}^{M} \mu_n^{-1} \text{ near } z = 1. \tag{24}
\]

To invert \( \tilde{S}_M^\infty(z) \), we start with the inversion contour to the left of \( z = 0 \), and then move it further to the left; thus, we are interested in singularities in \( \sigma < 0 \). Consider the first term on the right-hand side of (22). From (20), we see that \( \tilde{u}_1(z) \) has simple poles at \( z = -1, -3, \ldots \); near \( z = -1 \), we have \( \tilde{u}_1(z) \simeq (z + 1)^{-1} + 1 \). Hence, \( \psi_M(z+2) \tilde{u}_1(z) \) has a double pole at \( z = -1 \), giving terms proportional to \( x \log x \) and \( x \) in \( \tilde{S}_M^\infty(x) \). The next singularity at \( z = -3 \) gives a term in \( x^2 \), but we have already made errors of this order when we replaced \( s_n(x) \) by \( s_n^{(1)}(x) \). The second term on the right-hand side of (22) is analytic for \(-4 < \sigma < 0 \), apart from a simple pole at \( z = -2 \); this gives a term in \( x^2 \). Combining these results gives
\[
\tilde{S}_M^\infty(x) = x \log x - x \left\{ 1 + \gamma + \log (4/\pi) - \pi \sum_{n=1}^{M} \mu_n^{-1} \right\} + O(x^2)
\]
as \( x \to 0 \). Finally, using (16) and (17), we obtain
\[
S(x) = x \log x - x\{1 + \gamma + \log (4/\pi)\} + O(x^2), \quad \text{as } x \to 0; \tag{25}
\]
\( f(x) \) is given by (15). Note that this result does not depend on \( M \); see (16).

5 A problem of Linton and Evans

In this section, we consider a water-wave problem described in §1 and solved by Linton & Evans [4]. They calculate a certain complex reflection coefficient; its argument is proportional to
\[
E(h) = \tan^{-1}\left(\alpha^{-1}\sqrt{l^2 - k_0^2}\right) - \tan^{-1}\left(l/\alpha\right) - \frac{1}{2}\pi - (\alpha/\pi)L_0 + T, \tag{26}
\]
where \( L_0 = c \log (h/c) + d \log (h/d) \) and \( T \) is the sum
\[
\sum_{n=1}^{\infty} \left\{ \tan^{-1} \frac{\alpha}{\sqrt{l^2 + n^2 \pi^2 / c^2}} - \tan^{-1} \frac{\alpha}{\sqrt{l^2 + k_n^2}} + \tan^{-1} \frac{\alpha}{\sqrt{l^2 + \kappa_n^2}} \right\}.
\]
The parameters \( d, l, K \) and \( \kappa_n (n = 1, 2, \ldots) \) are fixed. \( k_0 \) is defined by (3)_2 and \( \alpha = \sqrt{k^2 - l^2} \), where \( k \) is defined by (3)_1. We have \( c = h - d > 0 \) and \( K < k_0 < l < k \).

Example 3. Find
\[
\lim_{h \to \infty} E(h) = E_\infty, \tag{27}
\]
say. This corresponds, physically, to solving the problem of Linton & Evans [4] when the water is infinitely deep.

Note that, as \( h \) varies, so too do \( k_0, k_n \) and \( c \); all other parameters remain unchanged. To begin with, (3)_2 shows that \( k_0 h \sim Kh(1 + 2e^{-2Kh}) \) as \( Kh \to \infty \), so we can replace \( k_0 \) by \( K \) in the first term of \( E(h) \), as \( h \to \infty \).

It is elementary to show that \( L_0 = d(\log h + 1 - \log d) + o(1) \) as \( h \to \infty \). For \( T \), we note that the arguments of the three inverse tangents behave like \( \alpha c/(n\pi) \), \( \alpha h/(n\pi) \) and \( \alpha d/(n\pi) \), respectively, as \( n \to \infty \), and so we can write \( T = T_1 - T_2 + T_3 \), where
\[
T_1 = \sum_{n=1}^{\infty} \left\{ \tan^{-1} \frac{\alpha}{\sqrt{l^2 + n^2 \pi^2 / c^2}} - \frac{\alpha c}{n\pi} \right\}, \tag{28}
\]
\[
T_2 = \sum_{n=1}^{\infty} \left\{ \tan^{-1} \frac{\alpha}{\sqrt{l^2 + k_0^2}} - \frac{\alpha h}{n\pi} \right\}, \tag{29}
\]
\[
T_3 = \sum_{n=1}^{\infty} \left\{ \tan^{-1} \frac{\alpha}{\sqrt{l^2 + \kappa_n^2}} - \frac{\alpha d}{n\pi} \right\}. \tag{30}
\]
using $c - h + d = 0$. Note that $T_1$ is a separable series, $T_2$ is a non-separable series and $T_3$ is independent of $h$. So, at this stage, we have

$$E(h) = -\tan^{-1}[\alpha(l^2 - K^2)^{-1/2}] - \tan^{-1}(l/\alpha)$$

$$- (\alpha d/\pi)(\log h + 1 - \log d) + T_1 - T_2 + T_3 + o(1) \quad \text{as } h \to \infty.$$  

**Deep-water behaviour of $T_1$.** From (28), we have $T_1 = f(\pi/c)$, where $f(x)$ is defined by (1) with $c_n = 1$,

$$u(x) = \tan^{-1}\left(\frac{\alpha}{\sqrt{l^2 + x^2}}\right) - \frac{\alpha}{x} \quad (31)$$

and $\mu_n = n$. Proceeding as in §3, we obtain $\tilde{f}(z) = \zeta(z)\tilde{u}(z)$, where $\tilde{u}(z)$ is analytic for $1 < \sigma < 3$. We must find the singularities of $\tilde{u}(z)$ in $0 \leq \sigma \leq 1$.

For $1 < \sigma < 3$, we integrate by parts to give

$$\tilde{f}(z) = \frac{\alpha}{z}\zeta(z)\tilde{u}_1(z), \quad (32)$$

where

$$\tilde{u}_1(z) = \int_0^\infty x^2 \left\{\frac{x}{\sqrt{x^2 + l^2(x^2 + k^2)}} - \frac{1}{x^2}\right\} dx. \quad (33)$$

is analytic for $1 < \sigma < 3$. It turns out that $\tilde{u}_1$ can be continued analytically into $-2 < \sigma < 3$, apart from a simple pole at $z = 1$; near $z = 1$, we find that $\tilde{u}_1(z) \simeq -(z - 1)^{-1} + Q$, where

$$Q = \log (2/l) + (k/\alpha) \log [(k - \alpha)/l]. \quad (34)$$

Hence, (32), shows that $\tilde{f}(z)$ has a double pole at $z = 1$, a simple pole at $z = 0$, and is otherwise analytic in $-2 < \sigma < 3; \text{ near } z = 1,$

$$\tilde{f}(z) \simeq \alpha(1 + w)^{-1}(w^{-1} + \gamma)(-w^{-1} + Q) \simeq \alpha\{-w^{-2} + w^{-1}(Q - \gamma + 1)\},$$

where $w = z - 1$, whereas near $z = 0$,

$$\tilde{f}(z) \simeq (\alpha/z)\zeta(0)\tilde{u}_1(0) = -\frac{1}{2}z^{-1} \tan^{-1}(\alpha/l).$$

We can then move the inversion contour to the left of $z = 0$, giving

$$f(x) = (\alpha/x)\log x + (\alpha/x)(Q - \gamma + 1) - \frac{1}{2} \tan^{-1}(\alpha/l) + o(1)$$

as $x \to 0$. Replacing $x$ by $\pi/c$, and expanding for large $h$ gives

$$T_1 = (\alpha h/\pi)\{-\log h + \log \pi + Q - \gamma + 1\} - \frac{1}{2} \tan^{-1}(\alpha/l)$$

$$+ (\alpha d/\pi)\{\log (h/\pi) - Q - \gamma\} + o(1) \quad \text{as } h \to \infty. \quad (35)$$
Deep-water behaviour of $T_2$. As in Example 2, we expect the leading behaviour of the non-separable series $T_2$ to be given by (29) with $k_n h$ replaced by $\mu_n = (n - \frac{1}{2}) \pi$. So, consider

$$T_\infty(1/h) = \sum_{n=1}^{\infty} \left\{ \tan^{-1} \left( \frac{\alpha}{\sqrt{l^2 + \mu_n^2 h^2}} \right) - \frac{\alpha h}{n \pi} \right\}. \quad (36)$$

We have

$$T_\infty(x) = \sum_{n=1}^{\infty} \left\{ \tan^{-1} \left( \frac{\alpha}{\sqrt{l^2 + \mu_n^2 x^2}} \right) - \frac{\alpha}{\mu_n x} \right\} + \frac{\alpha}{x} \sum_{n=1}^{\infty} \left( \frac{1}{\mu_n} - \frac{1}{n \pi} \right);$$

the second sum is $(2/\pi) \log 2$. Hence, $T_\infty(x) = (2\alpha/(\pi x)) \log 2 + f(x)$, where $f(x)$ is the separable series (1), with $c_n = 1$, $\mu_n = (n - \frac{1}{2}) \pi$ and $u(x)$ is again given by (31). We obtain

$$\tilde{f}(z) = \pi^{-z}(2^z - 1)\zeta(z)\tilde{u}(z) = (\alpha/z)\pi^{-z}(2^z - 1)\zeta(z)\tilde{u}_1(z),$$

where $\tilde{u}_1(z)$ is defined by (33). Note that, unlike the function defined by (32), here, $\tilde{f}(z)$ does not have a pole at $z = 0$. However, it does have a double pole at $z = 1$; near $z = 1$,

$$\tilde{f}(z) \approx (\alpha/\pi)\{-z^{-2} + (z - 1)^{-1}(Q - \gamma + 1 + \log \pi - 2 \log 2)\}.$$ 

Hence $T_\infty(x) = (\alpha/\pi)\{x^{-1} \log x + x^{-1}(Q - \gamma + 1 + \log \pi)\} + o(1)$ as $x \to 0$, and so, as $h \to \infty$, we obtain

$$T_\infty(1/h) = -(\alpha/\pi) h \log h + (\alpha/\pi) h(Q - \gamma + 1 + \log \pi) + o(1). \quad (37)$$

We now examine the difference between $T_2$ and $T_\infty(1/h)$. Let

$$T_4 = T_2 - T_\infty(1/h) = \sum_{n=1}^{\infty} t_n, \quad \text{where} \quad (38)$$

$$t_n = \tan^{-1} \left( \frac{\alpha}{\sqrt{l^2 + k_n^2}} \right) - \tan^{-1} \left( \frac{\alpha}{\sqrt{l^2 + \mu_n^2 / h^2}} \right).$$

Clearly, $t_n = o(1)$ as $h \to \infty$, for fixed $n$, so we have

$$T_4 = \sum_{n=M+1}^{\infty} t_n + o(1) \quad \text{as} \quad h \to \infty,$$

where $M$ is fixed (cf. (16)). Writing $k_n h = \mu_n + \nu_n$, as in (12), we have

$$l^2 + k_n^2 \simeq \Delta_n^2 + 2\nu_n \mu_n / h^2$$

as $|\nu_n/\mu_n|$ is small, where $\Delta_n^2 = l^2 + \mu_n^2 / h^2$. Hence, using the Taylor approximation (18), we find that

$$t_n \simeq \frac{-\alpha \nu_n \mu_n}{h^2 \Delta_n (\Delta_n^2 + \alpha^2)}. $$
Finally, we use the approximation (13), \( \nu_n \simeq \nu_n^{(1)} = \tan^{-1}(\mu_n/(Kh)) \), giving \( t_n \simeq -(\alpha/h)t_n^{(1)}(1/h) \), where

\[
t_n^{(1)}(y) = \frac{\mu_n y \tan^{-1}(\mu_n y/K)}{\sqrt{l^2 + \mu_n^2 y^2 (k^2 + \mu_n^2 y^2)}}, \tag{39}
\]

using \( k^2 = \alpha^2 + l^2 \). So, we have approximated \( T_4 \) by a separable series:

\[
T_4 = -(\alpha/h)T_\infty^M(1/h) + o(1) \quad \text{as } h \to \infty, \tag{40}
\]

and \( t_n^{(1)}(x) \) is defined by (39). As \( t_n^{(1)}(x) \sim \frac{1}{2}\pi(\mu_n x)^{-2} \) as \( x \to \infty \), we see that \( \tilde{T}_\infty^M(z) \) is analytic in a strip \( \beta < \sigma < 2 \), for some \( \beta \), so we can take the inversion contour just to the left of \( \sigma = 2 \). We have

\[
\tilde{T}_\infty^M(z) = \pi^{-1}\psi_M(z)\tilde{u}(z),
\]

where \( \psi_M(z) \) is defined by (23) and

\[
\tilde{u}(z) = \int_0^\infty \frac{y \tan^{-1}(y/K)}{\sqrt{y^2 + l^2 (y^2 + k^2)}} \, dy.
\tag{41}
\]

Hence, as the conditions of Theorem 1 are satisfied, we obtain

\[
T_M^\infty(x) = L/(\pi x) + o(1)
\]
as \( x \to 0 \), whence (40) gives \( T_4 = -(\alpha/\pi)L + o(1) \) as \( h \to \infty \). Finally, we combine this result with (37) and (38) to give

\[
T_2 = (\alpha/\pi)(-h \log h + h(Q - \gamma + 1 + \log \pi) - L) + o(1) \quad \text{as } h \to \infty. \tag{42}
\]

The integral defining \( L \), (41), is not elementary. However, it can be expressed in terms of dilogarithms [3]; we find that

\[
\alpha L = \frac{1}{4}\pi^2 - \frac{1}{2}A \log(k + K) + \frac{1}{2}A \log(k - K) - \delta \tan^{-1}(\psi/\alpha) - \mathcal{L} \tag{43}
\]

where \( \psi = \sqrt{l^2 - K^2} \), \( A = -\log((k - \alpha)/l) \), \( \delta = \tan^{-1}(\psi/K) \) and

\[
\mathcal{L} = \text{Li}_2(e^{-A}, \delta) - \text{Li}_2(e^{-A}, \pi + \delta). \tag{44}
\]
Here, the dilogarithm is defined by

$$\text{Li}_2(z) = -\int_0^z \log(1 - w) \frac{dw}{w}$$  \hspace{1cm} (45)$$

for complex $z$, and $\text{Li}_2(r, \theta) = \text{Re}\{\text{Li}_2(re^{i\theta})\}$.

**Synthesis.** From (35) and (42), we have

$$T_1 - T_2 = \left(\frac{\alpha}{\pi}\right) \{L + d(\log h - \log \pi - Q + \gamma)\} - \frac{1}{2} \tan^{-1}(\alpha/l) + o(1),$$  \hspace{1cm} (46)

as $h \to \infty$, so that the terms involving $h$ and $h \log h$ in (35) and (42) cancel. Moreover, when (46) is substituted into (31), we see that the terms in $\log h$ cancel, leaving only bounded terms as $h \to \infty$. Finally, substituting for $Q$ and $L$, we obtain

$$\begin{align*}
E_\infty &= \frac{\alpha d}{\pi} \left\{ \log \frac{ld}{2\pi} + \gamma - 1 \right\} - \frac{k d}{\pi} \log \frac{k - \alpha}{l} - \tan^{-1} \frac{\alpha}{\psi} - \frac{1}{2} \tan^{-1} \frac{l}{\alpha} \\
&\quad + T_3 - \frac{1}{\pi} L + \frac{1}{2\pi} \log \frac{k - \alpha}{l} \log \frac{k + K}{k - K} - \frac{1}{\pi} \tan^{-1} \frac{\psi}{K} \tan^{-1} \frac{\psi}{\alpha}.
\end{align*}$$  \hspace{1cm} (47)

This expression for $E_\infty$ bears little resemblance to $E(h)$; indeed, it is perhaps surprising to see terms involving products of logarithms and products of inverse tangents. Nevertheless, the result can be checked by solving the deep-water problem directly. This has been done by Parsons [7], using the Wiener-Hopf technique; the two approaches yield the same result.

**References**


