Symbolic Computation of Conservation Laws of Nonlinear PDEs

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Outline

- Overview of conservation laws
- Conservation laws for the Zakharov-Kuznetsov equation
- Presentation of ConservationLawsMD.m, a *Mathematica* program to compute conservation laws
- Conservation laws for the Manakov-Santini System

Overview of Conservation Laws

- The computation of conservation laws was an integral part of an in-depth study to find solutions to the Korteweg-de Vries (KdV) equation
- An infinite set of conservations laws is a factor determining that the PDE is completely integrable
- In a system described by a PDE, conservation laws
  - show which physical quantities are conserved
  - lead to discoveries toward finding solutions of PDEs such as the Inverse Scattering Transform
  - aid in the study of qualitative properties of a PDE
  - aid in the design of numerical solvers

Conservation Law

$$\mathrm{D}_t\rho + \mathrm{Div}\,\mathbf{J} = 0 \quad \text{on PDE}$$

ρ is the "conserved density" $\mathbf{J}$ is the "flux"

The continuity equation is satisfied for all solutions of the PDE.

In 1-D, $\mathbf{J} = J$ and $\mathrm{Div}\,\mathbf{J} = \mathrm{D}_x J$
In 2-D, $\mathbf{J} = (J^1, J^2)$ and $\mathrm{Div}\,\mathbf{J} = \mathrm{D}_x J^1 + \mathrm{D}_y J^2$
In 3-D, $\mathbf{J} = (J^1, J^2, J^3)$ and $\mathrm{Div}\,\mathbf{J} = \mathrm{D}_x J^1 + \mathrm{D}_y J^2 + \mathrm{D}_z J^3$

The Zakharov-Kuznetsov (ZK) Equation and Conservation Laws

The ZK equation models ion-sound solitons in a low pressure uniform magnetized plasma

The (2+1)-dimensional ZK equation:

$$u_t + \alpha u u_x + \beta (u_{2x} + u_{2y})_x = 0$$

Conservation Laws:

$$\mathrm{D}_t \Big(u \Big) + \mathrm{D}_x \Big(\tfrac{\alpha}{2} u^2 + \beta u_{2x} \Big) + \mathrm{D}_y \Big(\beta u_{xy} \Big) = 0$$

$$\mathrm{D}_t \Big(u^2 \Big) + \mathrm{D}_x \Big(\tfrac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u (u_{2x} + u_{2y}) \Big) - \mathrm{D}_y \Big(2\beta u_x u_y \Big) = 0$$

More Conservation Laws:

$$\mathsf{D}_t\Big(u^3 - \tfrac{3\beta}{\alpha}(u_x^2 + u_y^2)\Big) + \mathsf{D}_x\Big(\tfrac{3\alpha}{4}u^4 + 3\beta u^2 u_{2x} - 6\beta u(u_x^2 + u_y^2)$$

$$+ \tfrac{3\beta^2}{\alpha}(u_{2x}^2 - u_{2y}^2) - \tfrac{6\beta^2}{\alpha}(u_x(u_{3x} + u_{x2y}) + u_y(u_{2xy} + u_{3y}))\Big)$$

$$+ \mathsf{D}_y\Big(3\beta u^2 u_{xy} + \tfrac{6\beta^2}{\alpha}u_{xy}(u_{2x} + u_{2y})\Big) = 0$$

$$\mathsf{D}_t\Big(tu^2 - \tfrac{2}{\alpha}xu\Big) + \mathsf{D}_x\Big(t(\tfrac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{2x} + u_{2y}))$$

$$- x(u^2 + \tfrac{2\beta}{\alpha}u_{2x}) + \tfrac{2\beta}{\alpha}u_x\Big) - \mathsf{D}_y\Big(2\beta(tu_xu_y + \tfrac{1}{\alpha}xu_{xy})\Big) = 0$$

COMPUTER DEMONSTRATION

The Generalized Zakharov-Kuznetsov Equation and Conservation Laws

$$u_t + \alpha u^n u_x + \beta(u_{2x} + u_{2y})_x = 0,$$

where n is rational, $n \neq 0$

Conservation Laws:

$$\mathsf{D}_t\Big(u\Big) + \mathsf{D}_x\Big(\tfrac{\alpha}{2}u^2 + \beta u_{2x}\Big) + \mathsf{D}_y\Big(\beta u_{xy}\Big) = 0$$

$$\mathsf{D}_t\Big(u^2\Big) + \mathsf{D}_x\Big(\tfrac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{2x} + u_{2y})\Big)$$
$$- \mathsf{D}_y\Big(2\beta u_x u_y\Big) = 0$$

The third conservation law:

$$
\begin{aligned}
&\mathsf{D}_t\Big(u^{n+2} - \tfrac{(n+1)(n+2)\beta}{2\alpha}(u_x^2 + u_y^2)\Big) \\
&\quad + \mathsf{D}_x\Big(\tfrac{(n+2)\alpha}{2(n+1)}u^{2(n+1)} + (n+2)\beta u^{n+1} u_{2x} \\
&\quad - (n+1)(n+2)\beta u^n (u_x^2 + u_y^2) + \tfrac{(n+1)(n+2)\beta^2}{2\alpha}(u_{2x}^2 - u_{2y}^2) \\
&\quad - \tfrac{(n+1)(n+2)\beta^2}{\alpha}(u_x(u_{3x} + u_{x2y}) + u_y(u_{2xy} + u_{3y}))\Big) \\
&\quad + \mathsf{D}_y\Big((n+2)\beta u^{n+1} u_{xy} + \tfrac{(n+1)(n+2)\beta^2}{\alpha} u_{xy}(u_{2x} + u_{2y})\Big) = 0.
\end{aligned}
$$

The Manakov-Santini System and Conservation Laws

$$u_{tx} + u_{2y} + (uu_x)_x + v_x u_{xy} - u_{2x} v_y = 0$$

$$v_{tx} + v_{2y} + uv_{2x} + v_x v_{xy} - v_y v_{2x} = 0$$

Conservation Laws:

$$\mathsf{D}_t\Big(f u_x v_x\Big) + \mathsf{D}_x\Big(f(uu_x v_x - u_x v_x v_y - u_y v_y)$$

$$- f' y(u_t + uu_x - u_x v_y)\Big) + \mathsf{D}_y\Big(f(u_x v_y + u_y v_x + u_x v_x^2)$$

$$+ f'(u - yu_y - yu_x v_x)\Big) = 0,$$

where $f = f(t)$ is arbitrary

$$
\begin{aligned}
&\mathsf{D}_t\Big(f(2u + v_x^2 - y u_x v_x)\Big) + \mathsf{D}_x\Big(f(u^2 + u v_x^2 + u_y v \\
&\quad - v_y^2 - v_x^2 v_y - y(u u_x v_x - u_x v_x v_y - u_y v_y)) \\
&\quad - f' y(v_t + u v_x - v_x v_y) - (2 f x - f') y^2 (u_t + u u_x - u_x v_y)\Big) \\
&\quad - \mathsf{D}_y\Big(f(u_x v - 2 v_x v_y - v_x^3 + y(u_x v_x^2 + u_x v_y + u_y v_x)) \\
&\quad - f'(v - y(2u + v_y + v_x^2)) + (2 f x - f' y^2)(u_x v_x + u_y)\Big) = 0,
\end{aligned}
$$

where $f = f(t)$ is arbitrary

plus 3 others

Conclusions and Future Work

- The conservation laws program is fast and has computed conservation laws for a variety of PDEs. Improvements to the code will allow for a broader class of PDEs.
- Future research will include a study of integrability of DDEs, IDEs and delay DEs, leading to techniques, algorithms, and software to compute conservation laws, symmetries, and recursion operators.
- Software can be found at http://inside.mines.edu/~whereman