An Algorithmic Method to Symbollically Compute Conservation Laws of Nonlinear PDEs in (N+1) Dimensions

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Outline

- Overview of conservation laws
- Preliminary definitions and tools for computing conservation laws
- Conservation laws for the Zakharov-Kuznetsov equation
- Other examples
  - The Camassa-Holm equation
  - The Khoklov-Zabolotskaya equation

Methods for Computing Conservation Laws

- Use Noether's Theorem (Lagrangian formulation)
- Direct methods (Anderson, Bluman, Anco, Wolf, etc.) based on solving ODEs (or PDEs)
- Proposed strategy:
  - Density is linear combination of scaling invariant terms with undetermined coefficients
  - Use variational derivative (Euler operator) to compute the undetermined coefficients
  - Use the homotopy operator to compute the flux (invert D_x or Div)
  - Work with linearly independent pieces in finite dimensional spaces

Notation

Independent variables: $\mathbf{x} = (x^1, x^2, x^3) = (x, y, z)$

Dependent variables: $\mathbf{u} = (u^1, u^2, \ldots, u^j, \ldots, u^N)$
In examples, $(u^1, u^2) = (u, v)$

Partial Derivatives: $\frac{\partial^2 u}{\partial x^2} = u_{2x}$, $\frac{\partial^5 u}{\partial x^3 \partial y^2} = u_{3x2y}$, $\frac{\partial^k u}{\partial x^k} = u_{kx}$

$\mathbf{u}^{(M)}$ represents all components of $\mathbf{u}$ and all partial derivatives of components of $\mathbf{u}$ up to order M

$f = f\big(\mathbf{x}, \mathbf{u}^{(M)}(\mathbf{x})\big)$ is a differential function where operations take place on the jet space

For $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$, the jet space for second order derivatives has the coordinate
$(\mathbf{x}, \mathbf{u}^{(2)}(\mathbf{x})) = (x, y, u, v, u_x, u_y, v_x, v_y, u_{2x}, u_{xy}, u_{2y}, v_{2x}, v_{xy}, v_{2y})$

Tool: The Total Derivative Operator

$$\mathsf{D}_x f = \frac{\partial f}{\partial x} + \sum_{j=1}^{N} \sum_{k_1=0}^{M_1^j} \sum_{k_2=0}^{M_2^j} u^j_{(k_1+1)x k_2 y} \frac{\partial f}{\partial u^j_{k_1 x k_2 y}}$$

Example: Let $f = x^2 u_y + y u_x^2 - u^2 u_{2x}$

$$\begin{aligned}
\mathsf{D}_x &= \frac{\partial f}{\partial x} + \sum_{k_1=0}^{2} \sum_{k_2=0}^{1} u_{(k_1+1)x k_2 y} \frac{\partial f}{\partial u_{k_1 x k_2 y}} \\
&= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{2x} \frac{\partial f}{\partial u_x} + u_{3x} \frac{\partial f}{\partial u_{2x}} + u_{xy} \frac{\partial f}{\partial u_y} \\
&= 2x u_y - u_x (2u u_{2x}) + u_{2x}(2y u_x) - u_{3x}(u^2) + u_{xy}(x^2) \\
&= 2x u_y + x^2 u_{xy} + 2y u_x u_{2x} - 2u u_x u_{2x} - u^2 u_{3x}
\end{aligned}$$

Conservation Law

$$\mathsf{D}_t\rho + \mathrm{Div}\,\mathbf{J} = 0 \quad \text{on PDE}$$

ρ is the "conserved density" $\mathbf{J}$ is the "flux"

The continuity equation is satisfied for all solutions of the PDE.

In 1-D, $\mathbf{J} = J$ and $\mathrm{Div}\,\mathbf{J} = \mathsf{D}_x J$
In 2-D, $\mathbf{J} = (J^1, J^2)$ and $\mathrm{Div}\,\mathbf{J} = \mathsf{D}_x J^1 + \mathsf{D}_y J^2$
In 3-D, $\mathbf{J} = (J^1, J^2, J^3)$ and $\mathrm{Div}\,\mathbf{J} = \mathsf{D}_x J^1 + \mathsf{D}_y J^2 + \mathsf{D}_z J^3$

Tool: Euler Operator (Variational Derivtive)

For the the 2-D case,

$$\mathcal{L}_{\mathbf{u}(x,y)} f = \left(\mathcal{L}_{u^1(x,y)} f, \mathcal{L}_{u^2(x,y)} f, \ldots, \mathcal{L}_{u^j(x,y)} f, \ldots, \mathcal{L}_{u^N(x,y)} f \right)$$

$$\mathcal{L}_{u^j(x,y)} f = \sum_{k_1=0}^{M_1^j} \sum_{k_2=0}^{M_2^j} (-\mathsf{D}_x)^{k_1} (-\mathsf{D}_y)^{k_2} \frac{\partial f}{\partial u^j_{k_1 x\, k_2 y}}, \qquad j = 1, \ldots, N$$

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Definition: In 1-D, f is exact when there is F such that $f = \mathsf{D}_x F$.
In multi-D, f is exact when there is $\mathbf{F}$ such that $f = \mathrm{Div}\, \mathbf{F}$.

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Key Theorem: f is exact if and only if $\mathcal{L}_{\mathbf{u}(\mathbf{x})} f \equiv \mathbf{0}$.

Example: Let $f = u_x^2 v_{xy} + 2u_x u_{2x} v_y + 3v_x^2 v_{xy}$

$$\begin{aligned}
\mathcal{L}_{u(x,y)} f &= (-\mathsf{D}_x)\frac{\partial f}{\partial u_x} + (-\mathsf{D}_x)^2 \frac{\partial f}{\partial u_{2x}} \\
&= -\mathsf{D}_x(2u_x v_{xy} + 2u_{2x} v_y) + \mathsf{D}_x^2(2u_x v_y) \\
&= -2u_x v_{2xy} - 4u_{2x} v_{xy} - 2u_{3x} v_x + 2u_x v_{2xy} \\
&\qquad + 4u_{2x} v_{xy} + 2u_{3x} v_x \equiv 0 \\
\mathcal{L}_{v(x,y)} f &= (-\mathsf{D}_x)\frac{\partial f}{\partial v_x} + (-\mathsf{D}_y)\frac{\partial f}{\partial v_y} + (-\mathsf{D}_x)(-\mathsf{D}_y)\frac{\partial f}{\partial v_{xy}} \\
&= -\mathsf{D}_x(3v_{xy}) - \mathsf{D}_y(2u_x u_{2x}) + \mathsf{D}_x \mathsf{D}_y(u_x^2 + 3v_x) \\
&= -6v_{2x} v_{xy} - 6v_x v_{2xy} - 2u_{xy} u_{2x} - 2u_x u_{2xy} + 6v_{2x} v_{xy} \\
&\qquad + 6v_x v_{2xy} + 2u_{xy} u_{2x} + 2u_x u_{2xy} \equiv 0
\end{aligned}$$

Example: Let $f = u_x^2 v_{xy} + 2u_x u_{2x} v_y + 3v_x^2 v_{xy}$

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$$= -\mathsf{D}_x(2u_x v_{xy} + 2u_{2x} v_y) + \mathsf{D}_x^2(2u_x v_y)$$

$$= -2u_x v_{2xy} - 4u_{2x} v_{xy} - 2u_{3x} v_x + 2u_x v_{2xy} + 4u_{2x} v_{xy} + 2u_{3x} v_x \equiv 0$$

$$\mathcal{L}_{v(x,y)} f = (-\mathsf{D}_x)\frac{\partial f}{\partial v_x} + (-\mathsf{D}_y)\frac{\partial f}{\partial v_y} + (-\mathsf{D}_x)(-\mathsf{D}_y)\frac{\partial f}{\partial v_{xy}}$$

$$= -\mathsf{D}_x(3v_{xy}) - \mathsf{D}_y(2u_x u_{2x}) + \mathsf{D}_x \mathsf{D}_y(u_x^2 + 3v_x)$$

$$= -6v_{2x} v_{xy} - 6v_x v_{2xy} - 2u_{xy} u_{2x} - 2u_x u_{2xy} + 6v_{2x} v_{xy} + 6v_x v_{2xy} + 2u_{xy} u_{2x} + 2u_x u_{2xy} \equiv 0$$

By hand, $\mathbf{F} = \mathrm{Div}^{-1} f = (u_x^2 v_y, v_x^3)$

The Zakharov-Kuznetsov (ZK) Equation and Conservation Laws

The (2+1)-dimensional equation models ion-sound solitons in a low pressure uniform magnetized plasma

$$u_t + \alpha u u_x + \beta (u_{2x} + u_{2y})_x = 0$$

Conservation Laws:

$$\mathsf{D}_t \Big(u \Big) + \mathsf{D}_x \Big(\tfrac{\alpha}{2} u^2 + \beta u_{2x} \Big) + \mathsf{D}_y \Big(\beta u_{xy} \Big) = 0$$

$$\mathsf{D}_t \Big(u^2 \Big) + \mathsf{D}_x \Big(\tfrac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u (u_{2x} + u_{2y}) \Big)$$
$$+ \mathsf{D}_y \Big(- 2\beta u_x u_y \Big) = 0$$

More Conservation Laws:

$$\mathsf{D}_t\Big(u^3 - \tfrac{3\beta}{\alpha}(u_x^2 + u_y^2)\Big) + \mathsf{D}_x\Big(\tfrac{3\alpha}{4}u^4 + 3\beta u^2 u_{2x} - 6\beta u(u_x^2 + u_y^2)$$

$$+ \tfrac{3\beta^2}{\alpha}(u_{2x}^2 - u_{2y}^2) - \tfrac{6\beta^2}{\alpha}(u_x(u_{3x} + u_{x2y}) + u_y(u_{2xy} + u_{3y}))\Big)$$

$$+ \mathsf{D}_y\Big(3\beta u^2 u_{xy} + \tfrac{6\beta^2}{\alpha}u_{xy}(u_{2x} + u_{2y})\Big) = 0$$

$$\mathsf{D}_t\Big(tu^2 - \tfrac{2}{\alpha}xu\Big) + \mathsf{D}_x\Big(t(\tfrac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{2x} + u_{2y}))$$

$$- x(u^2 + \tfrac{2\beta}{\alpha}u_{2x}) + \tfrac{2\beta}{\alpha}u_x\Big) - \mathsf{D}_y\Big(2\beta(tu_x u_y + \tfrac{1}{\alpha}xu_{xy})\Big) = 0$$

How Conservation Laws Are Computed

1. Determine a scaling symmetry for the PDE

   Example: The ZK equation is invariant under the scaling symmetry

$$(t, x, y, u) \rightarrow (\frac{t}{\lambda^3}, \frac{x}{\lambda}, \frac{y}{\lambda}, \lambda^2 u)$$

   Note: $\frac{x}{\lambda} \Rightarrow \lambda \frac{\partial}{\partial x}$ $\quad \frac{y}{\lambda} \Rightarrow \lambda \frac{\partial}{\partial y}$ $\quad \frac{t}{\lambda^3} \Rightarrow \lambda^3 \frac{\partial}{\partial t}$

   Using the scaling symmetry, assign weights to each variable

$$W(u) = 2 \qquad W(\frac{\partial}{\partial t}) = 3$$

$$W(\frac{\partial}{\partial x}) = 1 \qquad W(\frac{\partial}{\partial y}) = 1.$$

2. Construct a candidate density

Definition: The "rank" of a monomial is the sum of the weights of the variables

Example: The rank of the monomial $\alpha u u_x$ from the ZK equation is $W(u) + W(u) + W(\frac{\partial}{\partial x}) = 5$

2. Construct a candidate density

Definition: The "rank" of a monomial is the sum of the weights of the variables

Example: The rank of the monomial $\alpha u u_x$ from the ZK equation is $W(u) + W(u) + W(\frac{\partial}{\partial x}) = 5$

The conservation law is invariant under the scaling symmetry of the PDE

Choose rank 6 and find all terms with rank 6 using the scaling symmetry

$$\{u^3, u_x^2, uu_{2x}, u_y^2, uu_{2y}, u_x u_y, uu_{xy}, u_{4x}, u_{3xy}, u_{2x2y}, u_{x3y}, u_{4y}\}$$

Remove divergences and divergence-equivalent terms. The candidate density is a linear combination of the remaining terms

$$\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y$$

3. Calculate the undetermined coefficients in the candidate density

Find $\mathrm{D}_t\rho$.

$$\mathrm{D}_t\rho = 3c_1u^2u_t + 2c_2u_xu_{tx} + 2c_3u_yu_{ty} + c_4(u_{tx}u_y + u_xu_{ty})$$

Compute $E = -\mathrm{D}_t\rho$ by using the ZK equation to replace all u_t

$$\begin{aligned} E = {} & 3c_1u^2(\alpha uu_x + \beta(u_{2x} + u_{2y})_x) \\ & + 2c_2u_x(\alpha uu_x + \beta(u_{2x} + u_{2y})_x)_x \\ & + 2c_3u_y(\alpha uu_x + \beta(u_{2x} + u_{2y})_x)_y \\ & + c_4(u_y(\alpha uu_x + \beta(u_{2x} + u_{2y})_x)_x \\ & + u_x(\alpha uu_x + \beta(u_{2x} + u_{2y})_x)_y) \end{aligned}$$

Require $E = -\mathsf{D}_t\rho$ to be a divergence since

$$-\mathsf{D}_t\rho = \mathrm{Div}\,\mathbf{J}$$

$$\begin{aligned} 0 &\equiv \mathcal{L}_{u(x,y)}E \\ &= -2\Big((3c_1\beta + c_3\alpha)u_x u_{2y} + 2(3c_1\beta + c_3\alpha)u_y u_{xy} \\ &\qquad + 2c_4\alpha u_x u_{xy} + c_4\alpha u_y u_{2x} + 3(3c_1\beta + c_2\alpha)u_x u_{2x}\Big) \end{aligned}$$

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$$
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&\qquad + 2c_4\alpha u_x u_{xy} + c_4\alpha u_y u_{2x} + 3(3c_1\beta + c_2\alpha) u_x u_{2x}\Big)
\end{aligned}
$$

Solve the linear system of coefficients.

$$
3c_1\beta + c_3\alpha = 0, \qquad c_4\alpha = 0, \qquad 3c_1\beta + c_2\alpha = 0
$$

$$
c_1 = 1, \quad c_2 = -\tfrac{3\beta}{\alpha}, \quad c_3 = -\tfrac{3\beta}{\alpha}, \quad c_4 = 0
$$

The density is $\rho = u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2)$.

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3c_1\beta + c_3\alpha = 0, \qquad c_4\alpha = 0, \qquad 3c_1\beta + c_2\alpha = 0
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The density is $\rho = u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2)$.

4. Compute the flux

Computer Demonstration

The program **ConservationLawsMD.m** will compute the conservation laws for the ZK equation

Tool: The Homotopy Operator

- integrates an exact 1-D differential expression by parts
- inverts the divergence of an exact multi-D differential expression

Let $f = u_x v_{2x} \cos u + v_{3x} \sin u - v_{4x}$

$\left(\mathcal{L}_{u(x)}, \mathcal{L}_{v(x)}\right) = (0, 0)$, so f is exact

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Mathematica's `Integrate` returns

$$\int (u_x v_{2x} \cos u + v_{3x} \sin u - v_{4x})\, dx$$

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The homotopy operator returns $v_{2x} \sin u - v_{3x}$

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Mathematica's `Integrate` returns

$$\int (u_x v_{2x} \cos u + v_{3x} \sin u - v_{4x}) \, dx$$

The homotopy operator returns $v_{2x} \sin u - v_{3x}$

Mathematica cannot invert a divergence!!!

The 2-D Homotopy Operator

f is an exact differential function, i. e., $f = \mathrm{Div}\,\mathbf{F}$

$$\mathbf{F} = \left(\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f\right)$$

$$= \left(\int_{\lambda_0}^{1} \sum_{j=1}^{N} \left(\mathcal{I}^{(x)}_{u^j(x,y)} f\right)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda},\ \int_{\lambda_0}^{1} \sum_{j=1}^{N} \left(\mathcal{I}^{(y)}_{u^j(x,y)} f\right)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \right)$$

$$\mathcal{I}^{(x)}_{u^j(x,y)} f = \sum_{k_1=1}^{M_1^j} \sum_{k_2=0}^{M_2^j} \left(\sum_{i_1=0}^{k_1-1} \sum_{i_y=0}^{k_2} \frac{\binom{i_1+i_2}{i_1}\binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}}{\binom{k_1+k_2}{k_1}} u_{i_1 x\, i_2 y} \left(-\mathsf{D}_x\right)^{k_1-i_1-1} \left(-\mathsf{D}_y\right)^{k_2-i_2} \right) \frac{\partial f}{\partial u^j_{k_1 x\, k_2 y}}$$

$$\mathcal{I}^{(y)}_{u^j(x,y)} f = \sum_{k_1=0}^{M_1^j} \sum_{k_2=1}^{M_2^j} \left(\sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2-1} \frac{\binom{i_1+i_2}{i_2}\binom{k_1+k_2-i_1-i_2-1}{k_2-i_2-1}}{\binom{k_1+k_2}{k_2}} u_{i_1 x\, i_2 y} \left(-\mathsf{D}_x\right)^{k_1-i_1} \left(-\mathsf{D}_y\right)^{k_2-i_2-1} \right) \frac{\partial f}{\partial u^j_{k_1 x\, k_2 y}}$$

Example: Compute the flux corresponding to the density $\rho = u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2)$ for the ZK equation. From the continuity equation: $\mathbf{J} = \mathrm{Div}^{-1}(-\mathrm{D}_t \rho) = \mathrm{Div}^{-1} E$.

Let $A = \alpha u u_x + \beta(u_{3x} + u_{x2y})$ so that

$$E = 3u^2 A - \frac{6\beta}{\alpha} u_x A_x - \frac{6\beta}{\alpha} u_y A_y$$

$$\begin{aligned}
\mathbf{J} &= \left(\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E \right) \\
&= \big(\tfrac{3}{4}\alpha u^4 + \beta u^2 (3u_{2x} + 2u_{2y}) - \beta u (6u_x^2 + 2u_y^2) \\
&\quad + \tfrac{3\beta^2}{4\alpha} u (u_{2x2y} + u_{4y}) - \tfrac{\beta^2}{\alpha} u_x (\tfrac{7}{2} u_{x2y} + 6u_{3x}) \\
&\quad - \tfrac{\beta^2}{\alpha} u_y (4u_{2xy} + \tfrac{3}{2} u_{3y}) + \tfrac{\beta^2}{\alpha} (3u_{2x}^2 + \tfrac{5}{2} u_{xy}^2 + \tfrac{3}{4} u_{2y}^2) \\
&\quad + \tfrac{5\beta^2}{4\alpha} u_{2x} u_{2y}, \quad \beta u^2 u_{xy} - 4\beta u u_x u_y \\
&\quad - \tfrac{3\beta^2}{4\alpha} u (u_{x3y} + u_{3xy}) - \tfrac{\beta^2}{4\alpha} u_x (13u_{2xy} + 3u_{3y}) \\
&\quad - \tfrac{5\beta^2}{4\alpha} u_y (u_{3x} + 3u_{x2y}) + \tfrac{9\beta^2}{4\alpha} u_{xy} (u_{2x} + u_{2y}) \big)
\end{aligned}$$

$\text{Div}^{-1} f$ is not unique

$f = \text{Div}\,\mathbf{F}$ where $\mathbf{F} = (G^1 + \mathsf{D}_y\theta,\ G^2 - \mathsf{D}_x\theta)$, θ is any differential function, and $f = \text{Div}\,(G^1, G^2)$

The homotopy operator often returns $\mathbf{F}$ with a particular "curl" term. In 2-D, $\mathbf{K} = (\mathsf{D}_y\theta,\ -\mathsf{D}_x\theta)$.

For $\mathbf{J}$ from the ZK equation,

$$\theta = 2\beta u^2 u_y + \frac{3\beta^2}{4\alpha} u(u_{2xy} + u_{3y}) + \frac{5\beta^2}{2\alpha} u_x u_{xy} + \frac{15\beta^2}{4\alpha} u_y u_{2y} + \frac{5\beta^2}{4\alpha} u_{2x} u_y$$

The flux without the "curl" term, $\tilde{\mathbf{J}} = \mathbf{J} - \mathbf{K}$,

$$\tilde{\mathbf{J}} = \Big(\frac{3\alpha}{4} u^4 + 3\beta u^2 u_{2x} - 6\beta u(u_x^2 + u_y^2) + \frac{3\beta^2}{\alpha}(u_{2x}^2 - u_{2y}^2) - \frac{6\beta^2}{\alpha}(u_x(u_{3x} + u_{x2y}) + u_y(u_{2xy} + u_{3y})), \\ 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{2x} + u_{2y}) \Big)$$

Conservation Laws for the (2+1)-dimensional Camassa-Holm (CH) equation

$$(u_t + \kappa u_x - u_{t2x} + 3uu_x - 2u_x u_{2x} - uu_{3x})_x + u_{2y} = 0$$

Two problems:

- The CH equation is not an evolution equation
- The CH equation does not have scaling symmetry

Evolution equations are required to be able to compute values for the undetermined coefficients in the candidate density.

The CH equation is transformed by interchanging t with y

$$(u_y + \kappa u_x - u_{2xy} + 3uu_x - 2u_x u_{2x} - uu_{3x})_x + u_{2t} = 0$$

Then set $v = u_t$. This forms a system of evolution equations

$$\begin{aligned} u_t &= v \\ v_t &= -u_{xy} - \kappa u_{2x} + u_{3xy} - 3u_x^2 - 3uu_{2x} + 2u_{2x}^2 \\ &\quad + 3u_x u_{3x} + uu_{4x} \end{aligned}$$

A scaling symmetry exists when parameters α and β are introduced

$$\begin{aligned} u_t &= v, \\ v_t &= -\alpha u_{xy} - \kappa u_{2x} + u_{3xy} - 3\beta u_x^2 - 3\beta u u_{2x} + 2u_{2x}^2 \\ &\quad + 3u_x u_{3x} + u u_{4x} \end{aligned}$$

Parameters α, β, and κ are included in the scaling symmetry

$$(u, v, \alpha, \beta, \kappa, t, x, y) \longrightarrow (\lambda u, \lambda^{7/2} v, \lambda^2 \alpha, \lambda^2 \beta, \lambda^3 \kappa, \frac{t}{\lambda^{5/2}}, \frac{x}{\lambda}, \frac{y}{\lambda^2})$$

Conservation Laws:

$$\mathsf{D}_t\Big(fu\Big) + \mathsf{D}_x\Big(\tfrac{1}{\alpha}f(\tfrac{3}{2}\beta u^2 + \kappa u - \tfrac{1}{2}u_x^2 - uu_{2x} - u_{tx})$$
$$- (\tfrac{1}{\alpha}fx - \tfrac{1}{2}f'y^2)(\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_x u_{2x} - uu_{3x}$$
$$- u_{t2x})) - \mathsf{D}_y\Big(u_y(\tfrac{1}{\alpha}fx - \tfrac{1}{2}f'y^2) + f'yu\Big) = 0$$

$$\mathsf{D}_t\Big(fyu\Big) + \mathsf{D}_x\Big(\tfrac{1}{\alpha}fy(\tfrac{3}{2}\beta u^2 + \kappa u - \tfrac{1}{2}u_x^2 - uu_{2x} - u_{tx})$$
$$- y(\tfrac{1}{\alpha}fx - \tfrac{1}{6}f'y^2)(\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_x u_{2x} - uu_{3x}$$
$$- u_{t2x})) - \mathsf{D}_y\Big(yu_y(\tfrac{1}{\alpha}fx - \tfrac{1}{6}f'y^2) - u(\tfrac{1}{\alpha}fx - \tfrac{1}{2}f'y^2)\Big) = 0$$

where $f = f(t)$ is an arbitrary function

Computer Demonstration

Results when **ConservationLawsMD.m** computes several conservation laws for the CH equation

Conservation Laws for the (3+1)-dimensional Khoklov-Zabolotskaya (KZ) equation

$$(u_t - u u_x)_x - u_{2y} - u_{2z} = 0$$

Conservation Law:

For $f = f(t,y,z)$ and $g = g(t,y,z)$,

$$\mathsf{D}_t\Big(fu\Big) - \mathsf{D}_x\Big(\tfrac{1}{2} f u^2 + (fx+g)(u_t - u u_x)\Big)$$
$$- \mathsf{D}_y\Big((f_y x + g_y)u - (fx+g)u_y\Big)$$
$$- \mathsf{D}_z\Big((f_z x + g_z)u - (fx+g)u_z\Big) = 0$$

under the constraints $\Delta f = 0$ and $\Delta g = f_t$

Conclusions

- The conservation laws program is fast and has computed conservation laws for a variety of PDEs. Improvements to the code will allow for a broader class of PDEs.

- The program ConservationLawsMD.m is available at http://inside.mines.edu/∼whereman under scientific software