Symbolic Computation of Conserved Densities of Nonlinear Evolution and Differential-Difference Equations

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- **Purpose**

  Design and implement an algorithm to compute polynomial conservation laws for nonlinear systems of evolution equations and differential-difference equations

- **Motivation**

  - Conservation laws describe the conservation of fundamental physical quantities such as linear momentum and energy. Compare with constants of motion (first integrals) in mechanics

  - For nonlinear PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws assures complete integrability

  - Conservation laws provide a simple and efficient method to study both quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures

  - Conservation laws can be used to test numerical integrators

PART I: Evolution Equations

• Conservation Laws for PDEs

Consider a single nonlinear evolution equation

$$u_t = \mathrm{F}(u, u_x, u_{2x}, ..., u_{nx})$$

or a system of N nonlinear evolution equations

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, ..., \mathbf{u}_{nx})$$

where $\mathbf{u} = [u_1, \ldots, u_N]^T$ and

$$u_t \stackrel{\text{def}}{=} \frac{\partial u}{\partial t}, \qquad u^{(n)} = u_{nx} \stackrel{\text{def}}{=} \frac{\partial^n u}{\partial x^n}$$

All components of $\mathbf{u}$ depend on x and t

Conservation law:

$$D_t \rho + D_x J = 0$$

ρ is the density, J is the flux

Both are polynomial in $u, u_x, u_{2x}, u_{3x}, ...$

Consequently

$$P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}$$

if J vanishes at infinity

- **The Euler Operator (calculus of variations)**

Useful tool to verify if an expression is a total derivative

Theorem:

If

$$f = f(x, y_1, \ldots, y_1^{(n)}, \ldots, y_N, \ldots, y_N^{(n)})$$

then

$$\mathcal{L}_{\mathbf{y}}(f) \equiv \mathbf{0}$$

if and only if

$$f = D_x g$$

where

$$g = g(x, y_1, \ldots, y_1^{(n-1)}, \ldots, y_N, \ldots, y_N^{(n-1)})$$

Notations:

$$\mathbf{y} = [y_1, \ldots, y_N]^T$$

$$\mathcal{L}_{\mathbf{y}}(f) = [\mathcal{L}_{y_1}(f), \ldots, \mathcal{L}_{y_N}(f)]^T$$

$$\mathbf{0} = [0, \ldots, 0]^T$$

(T for transpose)

and **Euler Operator**:

$$\mathcal{L}_{y_i} = \frac{\partial}{\partial y_i} - \frac{d}{dx}(\frac{\partial}{\partial {y_i}'}) + \frac{d^2}{dx^2}(\frac{\partial}{\partial {y_i}''}) + \cdots + (-1)^n \frac{d^n}{dx^n}(\frac{\partial}{\partial {y_i}^{(n)}})$$

- **Example: Korteweg-de Vries (KdV) equation**

$$u_t + uu_x + u_{3x} = 0$$

Conserved densities:

$$\rho_1 = u, \qquad (u)_t + (\frac{u^2}{2} + u_{2x})_x = 0$$

$$\rho_2 = u^2, \qquad (u^2)_t + (\frac{2u^3}{3} + 2uu_{2x} - {u_x}^2)_x = 0$$

$$\rho_3 = u^3 - 3{u_x}^2,$$

$$\left(u^3 - 3{u_x}^2\right)_t + \left(\frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2u_{2x} + 3{u_{2x}}^2 - 6u_xu_{3x}\right)_x = 0$$

$$\vdots$$

$$\rho_6 = u^6 - 60u^3{u_x}^2 - 30{u_x}^4 + 108u^2{u_{2x}}^2$$

$$+\frac{720}{7}{u_{2x}}^3 - \frac{648}{7}u{u_{3x}}^2 + \frac{216}{7}{u_{4x}}^2, \qquad \text{...... long ......}$$

$$\vdots$$

Note: KdV equation and conservation laws are invariant under dilation (scaling) symmetry

$$(x, t, u) \rightarrow (\lambda x, \lambda^3 t, \lambda^{-2} u)$$

u and t carry the weights of 2 and 3 derivatives with respect to x

$$u \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$$

- **Key Steps of the Algorithm**

  1. Determine weights (scaling properties) of variables & parameters

  2. Construct the form of the density (building blocks)

  3. Determine the unknown constant coefficients

- **Example: KdV equation**

$$u_t + uu_x + u_{3x} = 0$$

Compute the density of rank 6

(i) Compute the weights by solving a linear system

$$w(u) + w(\frac{\partial}{\partial t}) = 2w(u) + w(x) = w(u) + 3w(x)$$

With $w(x) = 1$, $w(\frac{\partial}{\partial t}) = 3, \ \ w(u) = 2.$

Thus, $\qquad (x, t, u) \rightarrow (\lambda x, \lambda^3 t, \lambda^{-2} u)$

(ii) Take all the variables, except $(\frac{\partial}{\partial t})$, with positive weight and list all possible powers of u, up to rank 6 : $\quad [u, u^2, u^3]$

Introduce x derivatives to 'complete' the rank

u has weight 2, introduce $\frac{\partial^4}{\partial x^4}$

u^2 has weight 4, introduce $\frac{\partial^2}{\partial x^2}$

u^3 has weight 6, no derivatives needed

Apply the derivatives and remove terms that are total derivatives with respect to x or total derivative up to terms kept earlier in the list

$$
\begin{aligned}
[u_{4x}] &\rightarrow [\,] \text{ empty list} \\
[{u_x}^2, uu_{2x}] &\rightarrow [{u_x}^2] \text{ since } uu_{2x} = (uu_x)_x - {u_x}^2 \\
[u^3] &\rightarrow [u^3]
\end{aligned}
$$

Combine the building blocks: $\quad \rho = c_1 u^3 + c_2 {u_x}^2$

(iii) Determine the coefficients c_1 and c_2

1. Compute $D_t\rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt}$
2. Replace u_t by $-(uu_x + u_{3x})$ and u_{xt} by $-(uu_x + u_{3x})_x$
3. Apply the Euler operator or integrate by parts

$$
\begin{aligned}
D_t\rho &= -[\frac{3}{4}c_1 u^4 - (3c_1 - c_2)uu_x^2 + 3c_1 u^2 u_{2x} - c_2 {u_{2x}}^2 + 2c_2 u_x u_{3x}]_x \\
&\quad - (3c_1 + c_2){u_x}^3
\end{aligned}
$$

4. The non-integrable term must vanish. Thus, $c_1 = -\frac{1}{3}c_2$.
   Set $c_2 = -3$, hence, $c_1 = 1$

   Result:

$$
\rho = u^3 - 3{u_x}^2
$$

   Expression [...] yields

$$
J = \frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x}
$$

- **Example: Boussinesq equation**

$$u_{tt} - u_{2x} + 3uu_{2x} + 3{u_x}^2 + \alpha u_{4x} = 0$$

with nonzero parameter α. Can be written as

$$\begin{aligned} u_t + v_x &= 0 \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0 \end{aligned}$$

The terms u_x and $\alpha\ u_{3x}$ are not uniform in rank

Introduce auxiliary parameter β with weight.
Replace the system by

$$\begin{aligned} u_t + v_x &= 0 \\ v_t + \beta u_x - 3uu_x - \alpha u_{3x} &= 0 \end{aligned}$$

The system is invariant under the scaling symmetry

$$(x, t, u, v, \beta) \rightarrow (\lambda x, \lambda^2 t, \lambda^{-2} u, \lambda^{-3} v, \lambda^{-2} \beta)$$

Hence

$$w(u) = 2,\ w(\beta) = 2,\ w(v) = 3 \ \text{ and } \ w(\frac{\partial}{\partial t}) = 2$$

or

$$u \sim \beta \sim \frac{\partial^2}{\partial x^2}, \qquad v \sim \frac{\partial^3}{\partial x^3}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2}$$

Form ρ of rank 6

$$\rho = c_1\ \beta^2 u + c_2\ \beta u^2 + c_3\ u^3 + c_4\ v^2 + c_5\ u_x v + c_6\ u_x^2$$

Compute the c_i. At the end set $\beta = 1$

$$\rho = u^2 - u^3 + v^2 + \alpha u_x^2$$

which is no longer uniform in rank!

- **Application: A Class of Fifth-Order Evolution Equations**

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

where α, β, γ are nonzero parameters, and $u \sim \frac{\partial^2}{\partial x^2}$

Special cases:

$$\begin{array}{llll} \alpha = 30 & \beta = 20 & \gamma = 10 & \text{Lax} \\ \alpha = 5 & \beta = 5 & \gamma = 5 & \text{Sawada} - \text{Kotera} \\ \alpha = 20 & \beta = 25 & \gamma = 10 & \text{Kaup}-\text{Kupershmidt} \\ \alpha = 2 & \beta = 6 & \gamma = 3 & \text{Ito} \end{array}$$

Under what conditions for the parameters α, β and γ does this equation admit a density of fixed rank?

- **Rank 2:**
  No condition

$$\rho = u$$

- **Rank 4:**
  Condition: $\beta = 2\gamma$ (Lax and Ito cases)

$$\rho = u^2$$

– **Rank 6:**

Condition:

$$10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2$$

(Lax, SK, and KK cases)

$$\rho = u^3 + \frac{15}{(-2\beta + \gamma)} {u_x}^2$$

– **Rank 8:**

1. $\beta = 2\gamma$ (Lax and Ito cases)

$$\rho = u^4 - \frac{6\gamma}{\alpha} u {u_x}^2 + \frac{6}{\alpha} {u_{2x}}^2$$

2. $\alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45}$ (SK, KK and Ito cases)

$$\rho = u^4 - \frac{135}{2\beta + \gamma} u {u_x}^2 + \frac{675}{(2\beta + \gamma)^2} {u_{2x}}^2$$

– **Rank 10:**

Condition:

$$\beta = 2\gamma$$

and

$$10\alpha = 3\gamma^2$$

(Lax case)

$$\rho = u^5 - \frac{50}{\gamma} u^2 {u_x}^2 + \frac{100}{\gamma^2} u {u_{2x}}^2 - \frac{500}{7\gamma^3} {u_{3x}}^2$$

What are the necessary conditions for the parameters α, β and γ for this equation to admit infinitely many polynomial conservation laws?

- If $\alpha = \frac{3}{10}\gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case)

- If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case)

- If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \frac{5}{2}\gamma$ then there is a sequence (with gaps!) of conserved densities (KK case)

- If

  $$\alpha = -\frac{2\beta^2 - 7\beta\gamma + 4\gamma^2}{45}$$

  or

  $$\beta = 2\gamma$$

  then there is a conserved density of rank 8

  Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case)

PART II: Differential-difference Equations

- **Conservation Laws for DDEs**

Consider a system of DDEs, continuous in time, discretized in space

$$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

$\mathbf{u}_n$ and $\mathbf{F}$ are vector dynamical variables

Conservation law:

$$\dot{\rho}_n = J_n - J_{n+1}$$

ρ_n is the density, J_n is the flux

Both are polynomials in $\mathbf{u}_n$ and its shifts

$$\frac{\mathrm{d}}{\mathrm{dt}}(\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1})$$

If J_n is bounded for all n, with suitable boundary or periodicity conditions

$$\sum_n \rho_n = \text{constant}$$

- **Definitions**

Define: D *shift-down* operator, U *shift-up* operator

$$Dm = m|_{n \to n-1} \qquad Um = m|_{n \to n+1}$$

For example,

$$Du_{n+2}v_n = u_{n+1}v_{n-1} \qquad Uu_{n-2}v_{n-1} = u_{n-1}v_n$$

Compositions of D and U define an *equivalence relation*
All shifted monomials are *equivalent*, e.g.

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}$$

Use *equivalence criterion*:

If two monomials, m_1 and m_2, are equivalent, $m_1 \equiv m_2$, then

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial M_n

For example, $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$$u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}]$$

with $M_n = u_{n-2}u_n$

Main representative of an equivalence class; the monomial with label n on u (or v)

For example, u_nu_{n+2} is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3}$, etc.

Use lexicographical ordering to resolve conflicts

For example, u_nv_{n+2} (not $u_{n-2}v_n$) is the main representative of the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4}$, etc.

- **Algorithm: Toda Lattice**

$$m\ddot{y}_n = a[e^{(y_{n-1}-y_n)} - e^{(y_n-y_{n+1})}]$$

Take $m = a = 1$ (scale on t), and set $u_n = \dot{y}_n,\ \ v_n = e^{(y_n-y_{n+1})}$

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1})$$

Simplest conservation law (by hand):

$$\dot{u}_n = \dot{\rho}_n = v_{n-1} - v_n = J_n - J_{n+1} \quad \text{with} \quad J_n = v_{n-1}$$

First pair:

$$\rho_n^{(1)} = u_n, \qquad J_n^{(1)} = v_{n-1}$$

Second pair:

$$\rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n, \qquad J_n^{(2)} = u_n v_{n-1}$$

Key observation: The DDE and the two conservation laws, $\dot{\rho}_n = J_n - J_{n+1}$, with

$$\rho_n^{(1)} = u_n, \qquad J_n^{(1)} = v_{n-1}$$

$$\rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n, \qquad J_n^{(2)} = u_n v_{n-1}$$

are invariant under the scaling symmetry

$$(t, u_n, v_n) \rightarrow (\lambda t, \lambda^{-1}u_n, \lambda^{-2}v_n)$$

Dimensional analysis:
u_n corresponds to one derivative with respect to t

For short, $u_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$, and similarly, $v_n \sim \frac{\mathrm{d}^2}{\mathrm{dt}^2}$

Our algorithm exploits this symmetry to find conserved densities:

1. Determining the weights
2. Constructing the form of density
3. Determining the unknown coefficients

- **Step 1: Determine the weights**

The *weight*, w, of a variable is equal to the number of derivatives with respect to t the variable carries.

Weights are positive, rational, and independent of n.

Requiring uniformity in rank for each equation

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1})$$

allows one to compute the weights of the dependent variables.

Solve the linear system

$$\begin{aligned} w(u_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) &= w(v_n) \\ w(v_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) &= w(v_n) + w(u_n) \end{aligned}$$

Set $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$, then $w(u_n) = 1$, and $w(v_n) = 2$

which is consistent with the scaling symmetry

$$(t, u_n, v_n) \rightarrow (\lambda t, \lambda^{-1} u_n, \lambda^{-2} v_n)$$

- **Step 2: Construct the form of the density**

The *rank* of a monomial is the total weight of the monomial.

For example, compute the form of the density of rank 3

List all monomials in u_n and v_n of rank 3 or less:

$$\mathcal{G} = \{{u_n}^3, {u_n}^2, u_n v_n, u_n, v_n\}$$

Next, for each monomial in $\mathcal{G}$, introduce enough t-derivatives, so that each term exactly has weight 3. Use the DDE to remove $\dot{u}_n$ and $\dot{v}_n$

$$\frac{\mathrm{d}^0}{\mathrm{dt}^0}({u_n}^3) = {u_n}^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,$$

$$\frac{\mathrm{d}}{\mathrm{dt}}({u_n}^2) = 2u_n v_{n-1} - 2u_n v_n, \qquad \frac{\mathrm{d}}{\mathrm{dt}}(v_n) = u_n v_n - u_{n+1} v_n,$$

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n$$

Gather the resulting terms in a set

$$\mathcal{H} = \{{u_n}^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n\}$$

Identify members of the same equivalence classes and replace them by the main representatives.

For example, since $u_n v_{n-1} \equiv u_{n+1} v_n$ both are replaced by $u_n v_{n-1}$.
$\mathcal{H}$ is replaced by

$$\mathcal{I} = \{{u_n}^3, u_n v_{n-1}, u_n v_n\}$$

containing the building blocks of the density.

Form a linear combination of the monomials in $\mathcal{I}$

$$\rho_n = c_1\, {u_n}^3 + c_2\, u_n v_{n-1} + c_3\, u_n v_n$$

with constant coefficients c_i

- **Step 3: Determine the unknown coefficients**

  Require that the conservation law, $\dot{\rho}_n = J_n - J_{n+1}$, holds

  Compute $\dot{\rho}_n$ and use the equations to remove $\dot{u}_n$ and $\dot{v}_n$.

  Group the terms

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2){u_n}^2 v_{n-1} + (c_3 - 3c_1){u_n}^2 v_n + (c_3 - c_2) v_{n-1} v_n \\ &\quad + c_2 u_{n-1} u_n v_{n-1} + c_2 {v_{n-1}}^2 - c_3 u_n u_{n+1} v_n - c_3 {v_n}^2\end{aligned}$$

  Use the equivalence criterion to modify $\dot{\rho}_n$

  Replace $u_{n-1} u_n v_{n-1}$ by $u_n u_{n+1} v_n + [u_{n-1} u_n v_{n-1} - u_n u_{n+1} v_n]$.

  The goal is to introduce the main representatives. Therefore,

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2){u_n}^2 v_{n-1} + (c_3 - 3c_1){u_n}^2 v_n \\ &\quad + (c_3 - c_2) v_n v_{n+1} + [(c_3 - c_2) v_{n-1} v_n - (c_3 - c_2) v_n v_{n+1}] \\ &\quad + c_2 u_n u_{n+1} v_n + [c_2 u_{n-1} u_n v_{n-1} - c_2 u_n u_{n+1} v_n] \\ &\quad + c_2 {v_n}^2 + [c_2 {v_{n-1}}^2 - c_2 {v_n}^2] - c_3 u_n u_{n+1} v_n - c_3 {v_n}^2\end{aligned}$$

  Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom. Rearrange the latter terms so that they match the pattern $[J_n - J_{n+1}]$. Hence,

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2){u_n}^2 v_{n-1} + (c_3 - 3c_1){u_n}^2 v_n \\ &\quad + (c_3 - c_2) v_n v_{n+1} + (c_2 - c_3) u_n u_{n+1} v_n + (c_2 - c_3){v_n}^2 \\ &\quad + [\{(c_3 - c_2) v_{n-1} v_n + c_2 u_{n-1} u_n v_{n-1} + c_2 {v_{n-1}}^2\} \\ &\quad - \{(c_3 - c_2) v_n v_{n+1} + c_2 u_n u_{n+1} v_n + c_2 {v_n}^2\}]\end{aligned}$$

The terms inside the square brackets determine:

$$J_n = (c_3 - c_2)v_{n-1}v_n + c_2 u_{n-1}u_n v_{n-1} + c_2 {v_{n-1}}^2$$

The terms outside the square brackets must vanish, thus

$$\mathcal{S} = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}$$

The solution is $3c_1 = c_2 = c_3$. Choose $c_1 = \frac{1}{3}$, thus $c_2 = c_3 = 1$

$$\rho_n = \tfrac{1}{3}\,{u_n}^3 + u_n(v_{n-1} + v_n), \qquad J_n = u_{n-1}u_n v_{n-1} + {v_{n-1}}^2$$

Analogously, conserved densities of rank ≤ 5:

$$\rho_n^{(1)} = u_n, \qquad \rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n$$

$$\rho_n^{(3)} = \tfrac{1}{3}{u_n}^3 + u_n(v_{n-1} + v_n)$$

$$\rho_n^{(4)} = \tfrac{1}{4}{u_n}^4 + {u_n}^2(v_{n-1} + v_n) + u_n u_{n+1} v_n + \tfrac{1}{2}{v_n}^2 + v_n v_{n+1}$$

$$\begin{aligned}\rho_n^{(5)} &= \tfrac{1}{5}{u_n}^5 + {u_n}^3(v_{n-1} + v_n) + u_n u_{n+1} v_n(u_n + u_{n+1}) \\ &\quad + u_n v_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_n v_n(v_{n-1} + v_n + v_{n+1})\end{aligned}$$

- **Application: A parameterized Toda lattice**

$$\dot{u}_n = \alpha\, v_{n-1} - v_n, \quad \dot{v}_n = v_n\,(\beta\, u_n - u_{n+1})$$

α and β are *nonzero* parameters. The system is integrable if $\alpha = \beta = 1$

Compute the *compatibility conditions* for α and β, so that there is a conserved densities of, say, rank 3.

In this case, we have $\mathcal{S}$:

$\{3\alpha c_1 - c_2 = 0, \beta c_3 - 3c_1 = 0, \alpha c_3 - c_2 = 0, \beta c_2 - c_3 = 0, \alpha c_2 - c_3 = 0\}$

A non-trivial solution $3c_1 = c_2 = c_3$ will exist *iff* $\alpha = \beta = 1$

Analogously, the parameterized Toda lattice has density

$\rho_n^{(1)} = u_n$ of rank 1 if $\alpha = 1$

and density

$\rho_n^{(2)} = \frac{\beta}{2} {u_n}^2 + v_n$ of rank 2 if $\alpha\,\beta = 1$

Only when $\alpha = \beta = 1$ will the parameterized system have conserved densities of rank ≥ 3

- **Example: Nonlinear Schrödinger (NLS) equation**

Ablowitz and Ladik discretization of the NLS equation:

$$i\,\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n^* u_n (u_{n+1} + u_{n-1})$$

where u_n^* is the complex conjugate of u_n.

Treat u_n and $v_n = u_n^*$ as independent variables, add the complex conjugate equation, and absorb i in the scale on t

$$\begin{aligned} \dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}) \\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}) \end{aligned}$$

Since $v_n = u_n^*, \;\; w(v_n) = w(u_n)$.

No uniformity in rank! Circumvent this problem by introducing an auxiliary parameter α with weight,

$$\begin{aligned} \dot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}) \\ \dot{v}_n &= -\alpha(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

Uniformity in rank requires that

$$\begin{aligned} w(u_n) + 1 &= w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n) \\ w(v_n) + 1 &= w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n) \end{aligned}$$

which yields

$$w(u_n) = w(v_n) = \frac{1}{2}, \quad w(\alpha) = 1$$

Uniformity in rank is essential for the first two steps of the algorithm. After Step 2, you can already set $\alpha = 1$.

The computations now proceed as in the previous examples

Conserved densities:

$$
\begin{aligned}
\rho_n^{(1)} &= c_1 u_n v_{n-1} + c_2 u_n v_{n+1} \\
\rho_n^{(2)} &= c_1 \big(\tfrac{1}{2} {u_n}^2 {v_{n-1}}^2 + u_n u_{n+1} v_{n-1} v_n + u_n v_{n-2}\big) \\
&+ c_2 \big(\tfrac{1}{2} {u_n}^2 {v_{n+1}}^2 + u_n u_{n+1} v_{n+1} v_{n+2} + u_n v_{n+2}\big) \\
\rho_n^{(3)} &= c_1 \big[\tfrac{1}{3} {u_n}^3 {v_{n-1}}^3 \\
&\quad + u_n u_{n+1} v_{n-1} v_n \big(u_n v_{n-1} + u_{n+1} v_n + u_{n+2} v_{n+1}\big) \\
&\quad + u_n v_{n-1} \big(u_n v_{n-2} + u_{n+1} v_{n-1}\big) \\
&\quad + u_n v_n \big(u_{n+1} v_{n-2} + u_{n+2} v_{n-1}\big) + u_n v_{n-3}\big] \\
&+ c_2 \big[\tfrac{1}{3} {u_n}^3 {v_{n+1}}^3 \\
&\quad + u_n u_{n+1} v_{n+1} v_{n+2} \big(u_n v_{n+1} + u_{n+1} v_{n+2} + u_{n+2} v_{n+3}\big) \\
&\quad + u_n v_{n+2} \big(u_n v_{n+1} + u_{n+1} v_{n+2}\big) \\
&\quad + u_n v_{n+3} \big(u_{n+1} v_{n+1} + u_{n+2} v_{n+2}\big) + u_n v_{n+3}\big]
\end{aligned}
$$

PART III: Symmetries of PDEs and DDEs

– Symmetries of PDEs

Consider the system of PDEs

$$\mathbf{u}_t = \mathbf{F}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ..., \mathbf{u}_{mx})$$

space variable x, time variable t

dynamical variables $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{F} = (F_1, F_2, ..., F_n)$

Definition of Symmetry

Vector function $\mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$ is a *symmetry* if and only if the PDE is invariant for the replacement

$$\mathbf{u} \to \mathbf{u} + \epsilon \mathbf{G}$$

within order ϵ. Hence

$$\frac{\partial}{\partial t}(\mathbf{u} + \epsilon \mathbf{G}) = \mathbf{F}(\mathbf{u} + \epsilon \mathbf{G})$$

must hold up to order ϵ, or

$$\frac{\partial \mathbf{G}}{\partial t} = \mathbf{F}'(\mathbf{u})[\mathbf{G}]$$

where **F**' is the Gateaux derivative of **F**

$$\mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u} + \epsilon \mathbf{G})|_{\epsilon=0}$$

Equivalently, **G** is a symmetry if the compatibility condition

$$\frac{\partial}{\partial \tau} \mathbf{F}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ..., \mathbf{u}_{nx}) = \frac{\partial}{\partial t} \mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$$

is satisfied, where τ is the new time variable such that

$$\frac{\partial \mathbf{u}}{\partial \tau} = \mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$$

– **Example: The KdV Equation**

$$u_t = 6uu_x + u_{3x}$$

has infinitely many symmetries:

$$\begin{aligned}
G^{(1)} &= u_x \qquad G^{(2)} = 6uu_x + u_{3x} \\
G^{(3)} &= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x} \\
G^{(4)} &= 140u^3u_x + 70{u_x}^3 + 280uu_xu_{2x} + 70u^2u_{3x} + 70u_{2x}u_{3x} + 42u_xu_{4x} \\
&\quad +14uu_{5x} + u_{7x} \\
G^{(5)} &= 630u^4u_x + 1260u{u_x}^3 + 2520u^2u_xu_{2x} + 1302u_x{u_{2x}}^2 + 420u^3u_{3x} \\
&\quad +966{u_x}^2u_{3x} + 1260uu_{2x}u_{3x} + 756uu_xu_{4x} + 252u_{3x}u_{4x} \\
&\quad +126u^2u_{5x} + 168u_{2x}u_{5x} + 72u_xu_{6x} + 18uu_{7x} + u_{9x}
\end{aligned}$$

The recursion operator connecting them is:

$$R = D^2 + 4u + 2u_xD^{-1}$$

– **Algorithm (KdV equation)**

Use the dilation symmetry $(t, x, u) \to (\lambda^{-3}t, \lambda^{-1}x, \lambda^2u)$
λ is arbitrary parameter. Hence, $u \sim \frac{\partial^2}{\partial x^2}$ and $\frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$

Step 1: Determine the weights of variables

We choose $w(x) = -1$, then $w(u) = 2$ and $w(t) = -3$

Step 2: Construct the form of the symmetry

Compute the form of the symmetry with rank 7

List all monomials in u of rank 7 or less

$$\mathcal{L} = \{1, u, u^2, u^3\}$$

Introduce x-derivatives so that each term has weight 7

$$\frac{\partial}{\partial x}(u^3) = 3u^2u_x, \quad \frac{\partial^3}{\partial x^3}(u^2) = 6u_xu_{2x}+2uu_{3x}, \quad \frac{\partial^5}{\partial x^5}(u) = u_{5x}, \quad \frac{\partial^7}{\partial x^7}(1) = 0$$

Gather the non-zero resulting terms in a set

$$\mathcal{R} = \{u^2u_x, u_xu_{2x}, uu_{3x}, u_{5x}\}$$

which contains the building blocks of the symmetry

Linear combination of the monomials in $\mathcal{R}$ determines the symmetry

$$G = c_1\, u^2 u_x + c_2\, u_x u_{2x} + c_3\, u u_{3x} + c_4\, u_{5x}$$

Step 3: Determine the unknown coefficients in the symmetry

Requiring that

$$\frac{\partial}{\partial \tau}\mathbf{F}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ..., \mathbf{u}_{nx}) = \frac{\partial}{\partial t}\mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$$

holds. Compute G_t and F_τ

Use the PDE,

$$\mathbf{u}_t = \mathbf{F}$$

to replace $u_t, u_{tx}.u_{txx}, ...$

Use

$$\mathbf{u}_\tau = \mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$$

to replace $u_\tau, u_{\tau x}, u_{\tau xx}, ...$

After grouping the terms

$$\begin{aligned} F_\tau - G_t \;&=\; (12c_1 - 18c_2)u_x^2 u_{2x} + (6c_1 - 18c_3)u u_{2x}^2 + (6c_1 - 18c_3)u u_x u_{3x} + \\ &\quad (3c_2 - 60c_4)u_{3x}^2 + (3c_2 + 3c_3 - 90c_4)u_{2x}u_{4x} + (3c_3 - 30c_4)u_x u_{5x} \\ &\equiv 0 \end{aligned}$$

This yields

$$\begin{aligned} \mathcal{S} \;&=\; \{12c_1 - 18c_2 = 0, 6c_1 - 18c_3 = 0, 3c_2 - 60c_4 = 0, \\ &\quad 3c_2 + 3c_3 - 90c_4 = 0, 3c_3 - 30c_4 = 0\} \end{aligned}$$

Choosing $c_4 = 1$, the solution is $c_1 = 30, c_2 = 20, c_3 = 10$

Hence

$$G = 30u^2 u_x + 20u_x u_{2x} + 10u u_{3x} + u_{5x}$$

which leads to Lax equation (in the KdV hierarchy)

$$u_t + 30u^2 u_x + 20u_x u_{2x} + 10u u_{3x} + u_{5x}$$

– x-t Dependent Symmetries

Algorithm can be used provided the **degree** in x or t is given

Compute the symmetry of the KdV equation with rank 2 (**linear** in x or t)

Build list of monomials in u, x and t of rank 2 or less

$$\mathcal{L} = \{1, u, x, xu, t, tu, tu^2\}$$

Introduce the correct number of x-derivatives to make each term weight 2

$$\frac{\partial}{\partial x}(xu) = u + xu_x, \quad \frac{\partial}{\partial x}(tu^2) = 2tuu_x, \quad \frac{\partial^3}{\partial x^3}(tu) = tu_{3x},$$
$$\frac{\partial^2}{\partial x^2}(1) = \frac{\partial^3}{\partial x^3}(x) = \frac{\partial^5}{\partial x^5}(t) = 0$$

Gather the non-zero resulting terms

$$\mathcal{R} = \{u, xu_x, tuu_x, tu_{3x}\}$$

Linearly combine the monomials to obtain

$$G = c_1\, u + c_2\, xu_x + c_3\, tuu_x + c_4\, tu_{3x}$$

Determine the coefficients c_1 through c_4

Compute G_t and F_τ and remove all t and τ derivatives (as before)

Group the terms

$$\begin{aligned} F_\tau - G_t \;&=\; (6c_1 + 6c_2 - c_3)uu_x + (3c_3 - 18c_4)tu_{2x}^2 + (3c_2 - c_4)u_{3x} + \\ &\quad\; (3c_3 - 18c_4)tu_x u_{3x} \equiv 0 \end{aligned}$$

This yields

$$\mathcal{S} = \{6c_1 + 6c_2 - c_3 = 0, 3c_3 - 18c_4 = 0, 3c_2 - c_4 = 0\}$$

The solution is $c_1 = \frac{2}{3}, c_2 = \frac{1}{3}, c_3 = 6, c_4 = 1$

Hence

$$G = \frac{2}{3}u + \frac{1}{3}xu_x + 6tuu_x + tu_{3x}$$

These are two $x - t$ dependent symmetries (of rank 0 and 2)

$$G = 1 + 6tu_x \quad \text{and} \quad G = 2u + xu_x + t(6uu_x + u_{3x})$$

– Symmetries of DDEs

Consider a system of DDEs (continuous in time, discretized in space)

$$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

$\mathbf{u}_n$ and $\mathbf{F}$ have any number of components

Definition of Symmetry

A vector function $\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ is called a *symmetry* of the DDE if the infinitesimal transformation

$$\mathbf{u} \to \mathbf{u} + \epsilon \mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

leaves the DDE invariant within order ϵ

Equivalently

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...) = \frac{\mathrm{d}}{\mathrm{dt}}\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

where τ is the new time variable such that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{u} = \mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

– Algorithm

Consider the one-dimensional Toda lattice

$$\ddot{y}_n = \exp\left(y_{n-1} - y_n\right) - \exp\left(y_n - y_{n+1}\right)$$

Change the variables

$$u_n = \dot{y}_n, \qquad v_n = \exp\left(y_n - y_{n+1}\right)$$

to write the lattice in algebraic form

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1})$$

This system is invariant under the scaling symmetry

$$(t, u_n, v_n) \to (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n)$$

λ is an arbitrary parameter. Hence, $u_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$ and $v_n \sim \frac{\mathrm{d}^2}{\mathrm{dt}^2}$

Step 1: Determine the weights of variables

Set $w(t) = -1$. Then $w(u_n) = 1$, and $w(v_n) = 2$

Step 2: Construct the form of the symmetry

Compute the form of the symmetry of ranks $\{3, 4\}$

List all monomials in u_n and v_n of rank 3 or less

$$\mathcal{L}_1 = \{{u_n}^3, {u_n}^2, u_n v_n, u_n, v_n\}$$

and of rank 4 or less

$$\mathcal{L}_2 = \{{u_n}^4, {u_n}^3, {u_n}^2 v_n, {u_n}^2, u_n v_n, u_n, {v_n}^2, v_n\}$$

For each monomial in both lists, introduce the adjusting number of t-derivatives so that each term exactly has weight 3 and 4, resp.

For the monomials in $\mathcal{L}_1$

$$\frac{\mathrm{d}^0}{\mathrm{dt}^0}({u_n}^3) = {u_n}^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,$$

$$\frac{\mathrm{d}}{\mathrm{dt}}({u_n}^2) = 2u_n\dot{u}_n = 2u_n v_{n-1} - 2u_n v_n, \qquad \frac{\mathrm{d}}{\mathrm{dt}}(v_n) = \dot{v}_n = u_n v_n - u_{n+1} v_n,$$

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = \frac{\mathrm{d}}{\mathrm{dt}}(\dot{u}_n) = \frac{\mathrm{d}}{\mathrm{dt}}(v_{n-1} - v_n) = u_{n-1}v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n$$

Gather the resulting terms in a set

$$\mathcal{R}_1 = \{{u_n}^3, u_{n-1}v_{n-1}, u_n v_{n-1}, u_n v_n, u_{n+1} v_n\}$$

Similarly

$$\begin{aligned} \mathcal{R}_2 \;=\; & \{{u_n}^4, {u_{n-1}}^2 v_{n-1}, u_{n-1}u_n v_{n-1}, {u_n}^2 v_{n-1}, v_{n-2}v_{n-1}, {v_{n-1}}^2, {u_n}^2 v_n, \\ & u_n u_{n+1} v_n, {u_{n+1}}^2 v_n, v_{n-1}v_n, {v_n}^2, v_n v_{n+1}\} \end{aligned}$$

Linear combination of the monomials in $\mathcal{R}_1$ and $\mathcal{R}_2$ determines

$$
\begin{aligned}
G_1 &= c_1\, u_n{}^3 + c_2\, u_{n-1}v_{n-1} + c_3\, u_n v_{n-1} + c_4\, u_n v_n + c_5\, u_{n+1}v_n \\
G_2 &= c_6\, u_n{}^4 + c_7\, u_{n-1}{}^2 v_{n-1} + c_8\, u_{n-1}u_n v_{n-1} + c_9\, u_n{}^2 v_{n-1} + c_{10}\, v_{n-2}v_{n-1} + \\
&\quad c_{11}\, v_{n-1}{}^2 + c_{12}\, u_n{}^2 v_n + c_{13}\, u_n u_{n+1} v_n + c_{14}\, u_{n+1}{}^2 v_n + c_{15}\, v_{n-1}v_n + \\
&\quad c_{16}\, v_n{}^2 + c_{17}\, v_n v_{n+1}
\end{aligned}
$$

Step 3: Determine the unknown coefficients in the symmetry

Requiring that $F_\tau = G_t$ holds

Compute $\frac{\mathrm{d}}{\mathrm{dt}}G_1$, $\frac{\mathrm{d}}{\mathrm{dt}}G_2$, $\frac{\mathrm{d}}{\mathrm{d}\tau}F_1$ and $\frac{\mathrm{d}}{\mathrm{d}\tau}F_2$ and remove all $\dot{u_n}, \dot{v_n}, \frac{\mathrm{d}}{\mathrm{d}\tau}u_n, \frac{\mathrm{d}}{\mathrm{d}\tau}v_n$

Require that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}F_1 - \frac{\mathrm{d}}{\mathrm{dt}}G_1 \equiv 0, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}F_2 - \frac{\mathrm{d}}{\mathrm{dt}}G_2 \equiv 0$$

which gives

$$
\begin{aligned}
c_1 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0, \\
-c_2 = -c_3 = c_4 = c_5 = -c_{12} = c_{14} = -c_{15} = c_{17}
\end{aligned}
$$

With $c_{17} = 1$ the symmetry is

$$
\begin{aligned}
G_1 &= u_n v_n - u_{n-1}v_{n-1} + u_{n+1}v_n - u_n v_{n-1} \\
G_2 &= u_{n+1}^2 v_n - u_n^2 v_n + v_n v_{n+1} - v_{n-1}v_n
\end{aligned}
$$

- **Scope and Limitations of Algorithm & Software**
  - Systems of PDEs or DDEs must be polynomial in dependent variables
  - Only one space variable (continuous x for PDEs, discrete n for DDEs) is allowed
  - No terms should *explicitly* depend on x and t for PDEs, or n for DDEs
  - Program only computes polynomial conserved densities;
    only polynomials in the dependent variables and their derivatives;
    no explicit dependencies on x and t for PDEs (or n for DDEs)
  - No limit on the number of PDEs or DDEs.
    In practice: time and memory constraints
  - Input systems may have (nonzero) parameters.
    Program computes the compatibility conditions for parameters such that densities (of a given rank) exist
  - Systems can also have parameters with (unknown) weight.
    Allows one to test PDEs or DDEs of non-uniform rank
  - For systems where one or more of the weights are free,
    the program prompts the user to enter values for the free weights
  - Negative weights are not allowed
  - Fractional weights and ranks are permitted
  - Form of ρ can be given in the data file (testing purposes)

- **Conserved Densities Software**

  - Conserved densities programs **CONSD** and **SYMCD** by Ito and Kako (Reduce, 1985, 1994 & 1996).
  - Conserved densities in **DELiA** by Bocharov (Pascal, 1990)
  - Conserved densities and formal symmetries **FS** by Gerdt and Zharkov (Reduce, 1993)
  - Formal symmetry approach by Mikhailov and Yamilov (MuMath, 1990)
  - Recursion operators and symmetries by Roelofs, Sanders and Wang (Reduce 1994, Maple 1995, Form 1995-present)
  - Conserved densities **condens.m** by Hereman and Göktaş (Mathematica, 1996)
  - Conservation laws, based on **CRACK** by Wolf (Reduce, 1995)
  - Conservation laws by Hickman (Maple, 1996)
  - Conserved densities by Ahner *et al.* (Mathematica, 1995). Project halted.
  - Conserved densities **diffdens.m** by Göktaş and Hereman (Mathematica, 1997)

- **Conclusions and Further Research**
  - Two *Mathematica* programs are available:
    condens.m for evolution equations (PDEs)
    diffdens.m for differential-difference equations (DDEs)
  - Usefulness
    * Testing models for integrability
    * Study of classes of nonlinear PDEs or DDEs
  - Comparison with other programs
    * Parameter analysis is possible
    * Not restricted to uniform rank equations
    * Not restricted to evolution equations provided that one can write the equation(s) as a system of evolution equations
  - Future work
    * Generalization towards broader classes of equations (e.g. u_{xt})
    * Generalization towards more space variables (e.g. KP equation)
    * Conservation laws with time and space dependent coefficients
    * Conservation laws with n dependent coefficients

* Exploit other symmetries in the hope to find conserved densities of non-polynomial form

* Constants of motion for dynamical systems
  (e.g. Lorenz and Hénon-Heiles systems)

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– In collaboration with Ünal Göktaş and Grant Erdmann

– Papers submitted to: J. Symb. Comp., Phys. Lett. A and Physica D

– Software: available via FTP, ftp site *mines.edu* in subdirectories

pub/papers/math_cs_dept/software/condens
pub/papers/math_cs_dept/software/diffdens

or via the Internet

URL: http://www.mines.edu/fs_home/whereman/

- **More Examples**

- **Nonlinear Schrödinger Equation**

$$iq_t - q_{2x} + 2|q|^2 q = 0$$

Program can not handle complex equations
Replace by

$$\begin{aligned} u_t - v_{2x} + 2v(u^2 + v^2) &= 0 \\ v_t + u_{2x} - 2u(u^2 + v^2) &= 0 \end{aligned}$$

where $q = u + iv$

Scaling properties

$$u \sim v \sim \frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2}$$

First seven conserved densities:

$$\rho_1 = u^2 + v^2$$

$$\rho_2 = vu_x$$

$$\rho_3 = u^4 + 2u^2v^2 + v^4 + {u_x}^2 + {v_x}^2$$

$$\rho_4 = u^2vu_x + \frac{1}{3}v^3u_x - \frac{1}{6}vu_{3x}$$

$$\rho_5 = -\frac{1}{2}u^6 - \frac{3}{2}u^4v^2 - \frac{3}{2}u^2v^4 - \frac{1}{2}v^6 - \frac{5}{2}u^2{u_x}^2 - \frac{1}{2}v^2{u_x}^2$$
$$-\frac{3}{2}u^2{v_x}^2 - \frac{5}{2}v^2{v_x}^2 + uv^2u_{2x} - \frac{1}{4}{u_{2x}}^2 - \frac{1}{4}{v_{2x}}^2$$

$$\rho_6 = -\frac{3}{4}u^4vu_x - \frac{1}{2}u^2v^3u_x - \frac{3}{20}v^5u_x + \frac{1}{4}v{u_x}^3 - \frac{1}{4}vu_x{v_x}^2$$
$$+uvu_xu_{2x} + \frac{1}{4}u^2vu_{3x} + \frac{1}{12}v^3u_{3x} - \frac{1}{40}vu_{5x}$$

$$\rho_7 = \frac{5}{4}u^8 + 5u^6v^2 + \frac{15}{2}u^4v^4 + 5u^2v^6 + \frac{5}{4}v^8 + \frac{35}{2}u^4{u_x}^2$$
$$-5u^2v^2{u_x}^2 + \frac{5}{2}v^4{u_x}^2 - \frac{7}{4}{u_x}^4 + \frac{15}{2}u^4{v_x}^2 + 25u^2v^2{v_x}^2$$
$$+\frac{35}{2}v^4{v_x}^2 - \frac{5}{2}{u_x}^2{v_x}^2 - \frac{7}{4}{v_x}^4 - 10u^3v^2u_{2x} - 5uv^4u_{2x}$$
$$-5u{v_x}^2u_{2x} + \frac{7}{2}u^2{u_{2x}}^2 + \frac{1}{2}v^2{u_{2x}}^2 + \frac{5}{2}u^2{v_{2x}}^2$$
$$+\frac{7}{2}v^2{v_{2x}}^2 - v^2u_xu_{3x} + \frac{1}{4}{u_{3x}}^2 + \frac{1}{4}{v_{3x}}^2 + uv^2u_{4x}$$

- **The Ito system**

$$\begin{aligned} u_t - u_{3x} - 6uu_x - 2vv_x &= 0 \\ v_t - 2u_x v - 2uv_x &= 0 \end{aligned}$$

$$u \sim \frac{\partial^2}{\partial x^2}, \qquad v \sim \frac{\partial^2}{\partial x^2}$$

$$\rho_1 = c_1 u + c_2 v$$

$$\rho_2 = u^2 + v^2$$

$$\rho_3 = 2u^3 + 2uv^2 - {u_x}^2$$

$$\rho_4 = 5u^4 + 6u^2v^2 + v^4 - 10u{u_x}^2 + 2v^2u_{2x} + {u_{2x}}^2$$

$$\begin{aligned} \rho_5 = {} & 14u^5 + 20u^3v^2 + 6uv^4 - 70u^2{u_x}^2 + 10v^2{u_x}^2 \\ & -4v^2{v_x}^2 + 20uv^2u_{2x} + 14u{u_{2x}}^2 - {u_{3x}}^2 + 2v^2u_{4x} \end{aligned}$$

and more conservation laws

- **The dispersiveless long-wave system**

$$\begin{aligned} u_t + vu_x + uv_x &= 0 \\ v_t + u_x + vv_x &= 0 \end{aligned}$$

$$u \sim 2v \quad w(v) \text{ is free}$$

$$\text{choose} \quad \text{u} \sim \frac{\partial}{\partial \text{x}} \quad \text{and} \quad 2\text{v} \sim \frac{\partial}{\partial \text{x}}$$

$$\begin{aligned} \rho_1 &= v \\ \rho_2 &= u \\ \rho_3 &= uv \\ \rho_4 &= u^2 + uv^2 \\ \rho_5 &= 3u^2v + uv^3 \\ \rho_6 &= \frac{1}{3}u^3 + u^2v^2 + \frac{1}{6}uv^4 \\ \rho_7 &= u^3v + u^2v^3 + \frac{1}{10}uv^5 \\ \rho_8 &= \frac{1}{3}u^4 + 2u^3v^2 + u^2v^4 + \frac{1}{15}uv^6 \end{aligned}$$

and more

Always the same set irrespective the choice of weights

- **A generalized Schamel equation**

$$n^2 u_t + (n+1)(n+2) u^{\frac{2}{n}} u_x + u_{3x} = 0$$

where n is a positive integer

$$\rho_1 = u, \qquad \rho_2 = u^2$$

$$\rho_3 = \frac{1}{2} {u_x}^2 - \frac{n^2}{2} u^{2+\frac{2}{n}}$$

For $n \neq 1, 2$ no further conservation laws

- **Three-Component Korteweg-de Vries Equation**

$$
\begin{aligned}
u_t - 6uu_x + 2vv_x + 2ww_x - u_{3x} &= 0 \\
v_t - 2vu_x - 2uv_x &= 0 \\
w_t - 2wu_x - 2uw_x &= 0
\end{aligned}
$$

Scaling properties

$$
u \sim v \sim w \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}
$$

First five densities:

$$
\begin{aligned}
\rho_1 &= c_1 u + c_2 v + c_3 w \\
\rho_2 &= u^2 - v^2 - w^2 \\
\rho_3 &= -2u^3 + 2uv^2 + 2uw^2 + {u_x}^2
\end{aligned}
$$

$$
\begin{aligned}
\rho_4 &= -\frac{5}{2}u^4 + 3u^2v^2 - \frac{1}{2}v^4 + 3u^2w^2 - v^2w^2 - \frac{1}{2}w^4 \\
&\quad + 5u{u_x}^2 + v^2u_{2x} + w^2u_{2x} - \frac{1}{2}{u_{2x}}^2
\end{aligned}
$$

$$
\begin{aligned}
\rho_5 &= -\frac{7}{10}u^5 + u^3v^2 - \frac{3}{10}uv^4 + u^3w^2 - \frac{3}{5}uv^2w^2 - \frac{3}{10}uw^4 \\
&\quad + \frac{7}{2}u^2{u_x}^2 + \frac{1}{2}v^2{u_x}^2 + \frac{1}{2}w^2{u_x}^2 + \frac{1}{5}v^2{v_x}^2 \\
&\quad - \frac{1}{5}w^2{v_x}^2 + \frac{1}{5}w^2{w_x}^2 + uv^2u_{2x} + uw^2u_{2x} - \frac{7}{10}u{u_{2x}}^2 \\
&\quad - \frac{1}{5}vw^2v_{2x} + \frac{1}{20}{u_{3x}}^2 + \frac{1}{10}v^2u_{4x} + \frac{1}{10}w^2u_{4x}
\end{aligned}
$$

- **The Deconinck-Meuris-Verheest equation**

Consider the modified vector derivative NLS equation:

$$\frac{\partial \mathbf{B}_\perp}{\partial t} + \frac{\partial}{\partial x}(B_\perp^2 \mathbf{B}_\perp) + \alpha \mathbf{B}_{\perp 0} \mathbf{B}_{\perp 0} \cdot \frac{\partial \mathbf{B}_\perp}{\partial x} + \mathbf{e}_x \times \frac{\partial^2 \mathbf{B}_\perp}{\partial x^2} = 0$$

Replace the vector equation by

$$\begin{aligned} u_t + \Big(u(u^2+v^2) + \beta u - v_x\Big)_x &= 0 \\ v_t + \Big(v(u^2+v^2) + u_x\Big)_x &= 0 \end{aligned}$$

u and v denote the components of $\mathbf{B}_\perp$ parallel and perpendicular to $\mathbf{B}_{\perp 0}$ and $\beta = \alpha B_{\perp 0}^2$

$$u^2 \sim \frac{\partial}{\partial x}, \qquad v^2 \sim \frac{\partial}{\partial x}, \qquad \beta \sim \frac{\partial}{\partial x}$$

First 6 conserved densities

$$\rho_1 = c_1 u + c_2 v$$

$$\rho_2 = u^2 + v^2$$

$$\rho_3 = \frac{1}{2}(u^2+v^2)^2 - u v_x + u_x v + \beta u^2$$

$$\rho_4 = \frac{1}{4}(u^2+v^2)^3 + \frac{1}{2}({u_x}^2 + {v_x}^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4)$$

$$
\begin{aligned}
\rho_5 &= \frac{1}{4}(u^2+v^2)^4 - \frac{2}{5}(u_x v_{2x} - u_{2x} v_x) + \frac{4}{5}(u u_x + v v_x)^2 \\
&\quad + \frac{6}{5}(u^2+v^2)({u_x}^2 + {v_x}^2) - (u^2+v^2)^2(u v_x - u_x v) \\
&\quad + \frac{\beta}{5}(2{u_x}^2 - 4u^3 v_x + 2u^6 + 3u^4 v^2 - v^6) + \frac{\beta^2}{5} u^4
\end{aligned}
$$

$$
\begin{aligned}
\rho_6 &= \frac{7}{16}(u^2+v^2)^5 + \frac{1}{2}(u_{2x}^2 + v_{2x}^2) \\
&- \frac{5}{2}(u^2+v^2)(u_x v_{2x} - u_{2x} v_x) + 5(u^2+v^2)(u u_x + v v_x)^2 \\
&+ \frac{15}{4}(u^2+v^2)^2({u_x}^2 + {v_x}^2) - \frac{35}{16}(u^2+v^2)^3(u v_x - u_x v) \\
&+ \frac{\beta}{8}(5u^8 + 10u^6 v^2 - 10u^2 v^6 - 5v^8 + 20u^2 {u_x}^2 \\
&- 12u^5 v_x + 60 u v^4 v_x - 20 v^2 {v_x}^2) \\
&+ \frac{\beta^2}{4}(u^6 + v^6)
\end{aligned}
$$