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SYMBOLIC SOFTWARE FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS: SYMMETRIES, INTEGRABILITY AND EXACT SOLUTIONS

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Calculation of Lie-point and Generalized Symmetries

- SPDE by Schwarz (Reduce, Scratchpad, 1986)
- Symmetries via exterior differential forms by Kersten and Gragert (Reduce, 1987)
- Lie-Bäcklund symmetries by Fedorova, Kornyak and Fushchich (Reduce, 1987)
- Crackstar by Wolf (Formac, 1987)
- CRACK by Wolf (Reduce, 1990)
- Lie-point symmetries by Schwarzmeier and Rosenau (Macsyma, 1988)
- Special symmetries by Mikhailov (Pascal, 1988)

- LIE by Head (muMath, 1990)
- Lie program by Nucci (Reduce, 1990)
- PDELIE by Vafeades (Macsyma, 1990)
- DEliA by Bocharov (Pascal, 1990)
- SYM_DE by Steinberg (Macsyma, 1990)
- SYMCAL by Reid (Macsyma, 1990)
- SYMMGRP.MAX by Champagne, Hereman and Winternitz (Macsyma, 1990)
- Lie symmetries by Herod, Berube, Wilcox (Mathematica, in development 1992)

Example 1 - Macsyma
Lie-point Symmetries

- System of m differential equations of order k

$$\Delta^i(x, u^{(k)}) = 0, \quad i = 1, 2, ..., m$$

with p independent and q dependent variables

$$\begin{aligned} x &= (x_1, x_2, ..., x_p) \in \mathbb{R}^p \\ u &= (u^1, u^2, ..., u^q) \in \mathbb{R}^q \end{aligned}$$

- The group transformations have the form

$$\tilde{x} = \Lambda_{group}(x, u), \quad \tilde{u} = \Omega_{group}(x, u)$$

where the functions Λ_{group} and Ω_{group} are to be determined

- Look for the Lie algebra $\mathcal{L}$ realized by the vector field

$$\alpha = \sum_{i=1}^{p} \eta^i(x, u)\frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x, u)\frac{\partial}{\partial u^l}$$

Procedure for finding the coefficients

- Construct the k^{th} prolongation $\mathrm{pr}^{(k)}\alpha$ of the vector field α
- Apply it to the system of equations
- Request that the resulting expression vanishes on the solution set of the given system

$$\mathrm{pr}^{(k)}\alpha\Delta^i\,|_{\Delta^j=0} \qquad i,j=1,...,m$$

- This results in a system of linear homogeneous PDEs for η^i and φ_l, with independent variables x and u (*determining equations*)
- Procedure thus consists of two major steps:

deriving the determining equations
solving the determining equations

Procedure for Computing the Determining Equations

- Use multi-index notation $J = (j_1, j_2, ..., j_p) \in \mathbb{N}^p$,
  to denote partial derivatives of u^l

$$u_J^l \equiv \frac{\partial^{|J|} u^l}{\partial {x_1}^{j_1} \partial {x_2}^{j_2} ... \partial {x_p}^{j_p}} \,,$$

  where $|J| = j_1 + j_2 + ... + j_p$

- $u^{(k)}$ denotes a vector whose components are all the partial derivatives of order 0 up to k of all the u^l

- Steps:

  (1) Construct the k^{th} prolongation of the vector field

$$\mathrm{pr}^{(k)}\alpha = \alpha + \sum_{l=1}^{q} \sum_{J} \psi_l^J(x, u^{(k)}) \frac{\partial}{\partial u_J^l}, \quad 1 \le |J| \le k$$

  The coefficients ψ_l^J of the first prolongation are:

$$\psi_l^{J_i} = D_i \varphi_l(x, u) - \sum_{j=1}^{p} u_{J_j}^l D_i \eta^j(x, u),$$

  where J_i is a $p-$tuple with 1 on the i^{th} position and zeros elsewhere

D_i is the total derivative operator

$$D_i = \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \sum_J u^l_{J+J_i} \frac{\partial}{\partial u^l_J}, \quad 0 \leq |J| \leq k$$

Higher order prolongations are defined recursively:

$$\psi_l^{J+J_i} = D_i \psi_l^J - \sum_{j=1}^{p} u^l_{J+J_j} D_i \eta^j(x,u), \quad |J| \geq 1$$

(2) Apply the prolonged operator $\mathrm{pr}^{(k)}\alpha$ to each equation $\Delta^i(x, u^{(k)}) = 0$

Require that $\mathrm{pr}^{(k)}\alpha$ vanishes on the solution set of the system

$$\mathrm{pr}^{(k)}\alpha \; \Delta^i \,|_{\Delta^j=0} \; = 0 \quad i, j = 1, ..., m$$

(3) Choose m components of the vector $u^{(k)}$, say $v^1, ..., v^m$, such that:

(a) Each v^i is equal to a derivative of a u^l $\;(l = 1, ..., q)$ with respect to at least one variable x_i $\;(i = 1, ..., p)$.

(b) None of the v^i is the derivative of another one in the set.

(c) The system can be solved algebraically for the v^i in terms of the remaining components of $u^{(k)}$, which we denoted by w:

$$v^i = S^i(x, w), \quad i = 1, ..., m.$$

(d) The derivatives of v^i,

$$v^i_J = D_J S^i(x, w),$$

where $D_J \equiv D_1^{j_1} D_2^{j_2} ... D_p^{j_p}$, can all be expressed in terms of the components of w and their derivatives, without ever reintroducing the v^i or their derivatives.

For instance, for a system of evolution equations

$$u^i_t(x_1, ..., x_{p-1}, t) = F^i(x_1, ..., x_{p-1}, t, u^{(k)}), \quad i = 1, ..., m,$$

where $u^{(k)}$ involves derivatives with respect to the variables x_i but not t, choose $v^i = u^i_t$.

(4) Eliminate all v^i and their derivatives from the expression prolonged vector field, so that all the remaining variables are independent

(5) Obtain the determining equations for $\eta^i(x, u)$ and $\varphi_l(x, u)$ by equating to zero the coefficients of the remaining independent derivatives u^l_J.

Example 2 – Macsyma
Painlevé Integrability Test

Integrability of a PDE requires that the only **movable singularities** in its solution are **poles**

Definition: A simple equation or system has the *Painlevé Property* if its solution in the complex plane has no worse singularities than movable poles

Aim: Verify if the PDE satisfies the **necessary criteria** to have the *Painlevé Property*

The solution f expressed as a Laurent series,

$$f = g^{\alpha} \sum_{k=0}^{\infty} u_k g^k$$

should only have movable poles.

$u_0(t,x) \neq 0$, α is a negative integer
$u_k(t,x)$ are analytic functions in a neighborhood of the singular, non-characteristic manifold $g(t,x) = 0$, with $g_x(t,x) \neq 0$

Steps of the Painlevé Test

- **Step 1**:
  1. Substitute the **leading order** term,
  $$f \propto u_0 \, g^{\alpha}$$
  into the given equation
  2. Determine the integer $\alpha < 0$ by balancing the most singular terms in g
  3. Calculate u_0
- **Step 2**:
  1. Substitute the generic terms
  $$f \propto u_0 \, g^{\alpha} + u_r \, g^{\alpha+r}$$
  into the equation, retaining its most singular terms
  2. Require that u_r is arbitrary
  3. Calculate the corresponding values of $r > 0$, called **resonances**

- **Step 3**:
  1. Substitute the truncated expansion
  $$f = g^{\alpha} \sum_{k=0}^{R} u_k \, g^k, \tag{4}$$
  where R represents the largest resonance, into the complete equation
  2. Determine u_k unambiguously at the non-resonance levels
  3. Check whether or not the **compatibility condition** is satisfied at resonance levels

- An equation or system has the **Painlevé Property** and is conjectured to be integrable if:
  1. Step 1 thru 3 can be carried out consistently with $\alpha < 0$ and with positive resonances
  2. The compatibility conditions are identically satisfied for all resonances

- For an equation to be integrable it is **necessary** but **not sufficient** that it passes the Painlevé test
- The above algorithm does not detect the existence of essential singularities

Demo Painlevé Test

Korteweg-de Vries equation

$$u_t + 6uu_x + u_{3x} = 0$$

Example 4 – Macsyma
Positioning of Equipment

A Trilateration Problem

Calculate the unknown 3D-position of a point,
given the distances from that point to a set of fixed points

x and y are the horizontal coordinates
z is the altitude of the unknown point

Questions:

- Is there a mathematical solution?
- What is the smallest number of beacons needed?
- Can the position of the bulldozer be determined "fairly accurately" if the distances are inaccurate?
- What are the optimal positions of the beacons?
- What is the 'best' optimal algorithm for the solution?
- Can the algorithm be translated into a fast C-program?
- What are the possible applications of the problem?

A Mathematical Solution

Linearization

(x_i, y_i, z_i) $(i = 1, 2, ..., n)$ are the known coordinates of the n beacons

(x, y, z) are the unknown coordinates of the bulldozer

r_i are *measured* approximate slope distances from bulldozer to beacons

Constraints

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = {r_i}^2$$

Use the j^{th} constraint as a *linearizing* tool

Add and subtract x_j, y_j and z_j

$$(x - x_j + x_j - x_i)^2 + (y - y_j + y_j - y_i)^2 + (z - z_j + z_j - z_i)^2 = {r_i}^2$$

Expand and regroup terms

$$\begin{aligned}
&(x - x_j)(x_i - x_j) + (y - y_j)(y_i - y_j) + (z - z_j)(z_i - z_j) \\
&= \frac{1}{2}[(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2 \\
&\quad -{r_i}^2 + (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]
\end{aligned}$$

$$= \frac{1}{2}[{r_j}^2 - {r_i}^2 + d_{ij}^2]$$

where

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

is the distance between beacons i and j

Select $j = 1$, thus for $i = 2, 3, ..., n$
linear system of $(n - 1)$ eqs. in 3 unknowns:

$$(x - x_1)(x_2 - x_1) + (y - y_1)(y_2 - y_1) + (z - z_1)(z_2 - z_1)$$
$$= \frac{1}{2}[{r_1}^2 - {r_2}^2 + d_{21}^2] = b_{21}$$

$$(x - x_1)(x_3 - x_1) + (y - y_1)(y_3 - y_1) + (z - z_1)(z_3 - z_1)$$
$$= \frac{1}{2}[{r_1}^2 - {r_3}^2 + d_{31}^2] = b_{31}$$

$$\vdots$$

$$(x - x_1)(x_n - x_1) + (y - y_1)(y_n - y_1) + (z - z_1)(z_n - z_1)$$
$$= \frac{1}{2}[{r_1}^2 - {r_n}^2 + d_{n1}^2] = b_{n1}$$

In matrix form $A\vec{x} = \vec{b}$, with

$$A = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ \vdots & \ddots & \vdots \\ x_n - x_1 & y_n - y_1 & z_n - z_1 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} b_{21} \\ b_{31} \\ \vdots \\ b_{n1} \end{pmatrix}$$

The Least Squares Method

The distances r_i are only approximate
Determine $\vec{x}$ such that $A\vec{x} \approx \vec{b}$

Minimize the sum of the squares of the residuals

$$S = \vec{r}^{\,T}\vec{r} = (\vec{b} - A\vec{x})^T(\vec{b} - A\vec{x})$$

Solve the *normal* equation

$$A^T A\vec{x} = A^T\vec{b}$$

If $A^T A$ is non-singular then

$$\vec{x} = (A^T A)^{-1} A^T\vec{b}$$

Determine the effect of adding errors to the radii

Demo with the actual data from the coal mine