Continuous and Discrete Homotopy Operators: A Theoretical Approach Made Concrete and Applicable

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Outline

Part I: Continuous Case

- Calculus-based formulas for the continuous homotopy operator (in multi-dimensions)
- Symbolic integration by parts and inversion of the total divergence operator
- Application: symbolic computation of conservation laws of nonlinear PDEs in multiple space dimensions

Part II: Discrete Case

- Simple formula for the homotopy operator
- Symbolic summation by parts and inversion of the forward difference operator
- Analogy: continuous and discrete formulas
- Application: symbolic computation of conservation laws of nonlinear DDEs
- Demonstration of *ConservationLawsMD.m*
- Demonstration of *DDEDensityFlux.m*
- Conclusions

Motivation of the Research

Conservation Laws for Nonlinear PDEs

- System of evolution equations of order M

$$\boxed{\mathbf{u}_t = \mathbf{F}(\mathbf{u}^{(M)}(\mathbf{x}))}$$

with $\mathbf{u} = (u, v, w, \ldots)$ and $\mathbf{x} = (x, y, z)$.

- Conservation law in (1+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \mathrm{D}_x J = 0}$$

evaluated on the PDE.

Conserved density ρ and flux J.

- Conservation law in (2+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \nabla \cdot \mathbf{J} = \mathrm{D}_t \rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 = 0}$$

evaluated on the PDE.

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$.

- Conservation law in (3+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \nabla \cdot \mathbf{J} = \mathrm{D}_t \rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 + \mathrm{D}_z J_3 = 0}$$

evaluated on the PDE.

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2, J_3)$.

Notation – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$

- Dependent variables $\mathbf{u} = (u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(N)})$
  In examples: $\mathbf{u} = (u, v, \theta, h, \ldots)$

- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kx\,ly} = \frac{\partial^{k+l} u}{\partial x^k y^l}$, etc.
  Examples: $u_{xxxxx} = u_{5x} = \frac{\partial^5 u}{\partial x^5}$
  $u_{xx\,yyyy} = u_{2x\,4y} = \frac{\partial^6 u}{\partial x^2 y^4}$

- *Differential functions*
  Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

- *Total derivatives*: $\mathrm{D}_t,\ \mathrm{D}_x, \mathrm{D}_y, \ldots$

Example: Let $f = uvv_x + x^2u_x^3v_x + u_xv_{xx}$. Then,

$$
\begin{aligned}
\mathrm{D}_x f &= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{xx} \frac{\partial f}{\partial u_x} \\
&\quad + v_x \frac{\partial f}{\partial v} + v_{xx} \frac{\partial f}{\partial v_x} + v_{xxx} \frac{\partial f}{\partial v_{xx}} \\
&= 2xu_x^3v_x + u_x(vv_x) + u_{xx}(3x^2u_x^2v_x + v_{xx}) \\
&\quad + v_x(uv_x) + v_{xx}(uv + x^2u_x^3) + v_{xxx}(u_x) \\
&= 2xu_x^3v_x + vu_xv_x + 3x^2u_x^2v_xu_{xx} + u_{xx}v_{xx} \\
&\quad + uv_x^2 + uvv_{xx} + x^2u_x^3v_{xx} + u_xv_{xxx}
\end{aligned}
$$

An Example in (2+1) Dimensions

- Example: Shallow water wave (SWW) equations

[P. Dellar, Phys. Fluids **15** (2003) 292-297]

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\,\mathbf{\Omega}\times\mathbf{u} + \nabla(\theta h) - \tfrac{1}{2}h\nabla\theta = \mathbf{0}$$

$$\theta_t + \mathbf{u}\cdot(\nabla\theta) = 0$$

$$h_t + \nabla\cdot(\mathbf{u}h) = 0$$

with constant $\mathbf{\Omega}$

In components:

$$u_t + uu_x + vu_y - 2\,\Omega v + \tfrac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\,\Omega u + \tfrac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

where $\mathbf{u}(x,y,t), \theta(x,y,t)$ and $h(x,y,t)$.

- First few densities-flux pairs of SWW system:

$$\rho_{(1)} = h \qquad \mathbf{J}^{(1)} = h \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(2)} = h\,\theta \qquad \mathbf{J}^{(2)} = h\,\theta \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(3)} = h\,\theta^2 \qquad \mathbf{J}^{(3)} = h\,\theta^2 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(4)} = h\,(u^2 + v^2 + h\theta) \qquad \mathbf{J}^{(4)} = h \begin{pmatrix} u\,(u^2 + v^2 + 2h\theta) \\ v\,(v^2 + u^2 + 2h\theta) \end{pmatrix}$$

$$\rho_{(5)} = \theta\,(2\Omega + v_x - u_y)$$

$$\mathbf{J}^{(5)} = \frac{1}{2}\,\theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}$$

Generalizations:

$$\mathrm{D}_t\left(f(\theta)h\right)+\mathrm{D}_x\left(f(\theta)hu\right)+\mathrm{D}_y\left(f(\theta)hv\right)=0$$

$$\begin{aligned}&\mathrm{D}_t\left(g(\theta)(2\Omega+v_x-u_x)\right)\\&+\mathrm{D}_x\left(\tfrac{1}{2}g(\theta)(4\Omega u-2uu_y+2uv_x-h\theta_y)\right)\\&+\mathrm{D}_y\left(\tfrac{1}{2}g(\theta)(4\Omega v-2u_yv+2vv_x+h\theta_x)\right)=0\end{aligned}$$

for any functions $f(\theta)$ and $g(\theta)$

Conservation Laws for Nonlinear Differential-Difference Equations (DDEs)

- System of DDEs

$$\boxed{\dot{\mathbf{u}}_n = \mathbf{F}(\cdots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \cdots)}$$

- Conservation law in $(1+1)$ dimensions

$$\boxed{\mathrm{D}_t \rho_n + \Delta J_n = 0} \text{ (on DDE)}$$

conserved density ρ_n and flux J_n

- Example: Toda lattice

$$\begin{aligned} \dot{u}_n &= v_{n-1} - v_n \\ \dot{v}_n &= v_n(u_n - u_{n+1}) \end{aligned}$$

- First few densities-flux pairs for Toda lattice:

$$\rho_n^{(0)} = \ln(v_n) \qquad J_n^{(0)} = u_n$$

$$\rho_n^{(1)} = u_n \qquad J_n^{(1)} = v_{n-1}$$

$$\rho_n^{(2)} = \tfrac{1}{2}u_n^2 + v_n \qquad J_n^{(2)} = u_n v_{n-1}$$

$$\rho_n^{(3)} = \tfrac{1}{3}u_n^3 + u_n(v_{n-1} + v_n) \qquad J_n^{(3)} = u_{n-1}u_n v_{n-1} + v_{n-1}^2$$

PART I: CONTINUOUS CASE

Problem Statement

Continuous case in 1D:

Example: For $u(x)$ and $v(x)$

$$f = 8 v_x v_{xx} - u_x^3 \sin u + 2 u_x u_{xx} \cos u - 6 v v_x \cos u + 3 u_x v^2 \sin u$$

Question: Can the expression be integrated?

If yes, find $F = \int f \, dx$ (so, $f = D_x F$)

Result (by hand): $F = 4\, v_x^2 + u_x^2 \cos u - 3\, v^2 \cos u$

Continuous case in 2D or 3D:

Example: For $u(x,y)$ and $v(x,y)$

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

Question: Is there an $\mathbf{F}$ so that $f = \mathrm{Div}\,\mathbf{F}$?

If yes, find $\mathbf{F}$.

Result (by hand): $\mathbf{F} = (u v_y - u_x v_y, -u v_x + u_x v_x)$

Can this be done without integration by parts?

Can the computation be reduced to a single integral in one variable?

Tools from the Calculus of Variations

- Definition:

  A differential function f is a exact iff $f = \mathrm{Div}\,\mathbf{F}$.

  Special case (1D): $f = \mathrm{D}_x\,F$.

- Question: How can one test that $f = \mathrm{Div}\,\mathbf{F}$?

- Theorem (exactness test):

  $f = \mathrm{Div}\,\mathbf{F}$ iff $\mathcal{L}_{u^{(j)}(\mathbf{x})}\,f \equiv 0, \quad j = 1, 2, \ldots, N.$

  N is the number of dependent variables.

  The Euler operator annihilates divergences

- Euler operator in 1D (variable $u(x)$):

$$\mathcal{L}_{u(x)} = \sum_{k=0}^{M} (-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}}$$
$$= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots$$

- Euler operator in 2D (variable $u(x, y)$):

$$\mathcal{L}_{u(x,y)} = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-\mathrm{D}_x)^k (-\mathrm{D}_y)^\ell \frac{\partial}{\partial u_{kx\,\ell y}}$$
$$= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} - \mathrm{D}_y \frac{\partial}{\partial u_y}$$
$$+ \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} + \mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u_{xy}} + \mathrm{D}_y^2 \frac{\partial}{\partial u_{yy}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{xxx}} \cdots$$

Application: Testing Exactness – Continous Case

Example:

$$f = 8 v_x v_{xx} - u_x^3 \sin u + 2 u_x u_{xx} \cos u - 6 v v_x \cos u + 3 u_x v^2 \sin u$$

where $u(x)$ and $v(x)$

- f is exact
- After integration by parts (by hand):

$$F = \int f \, dx = 4 \, v_x^2 + u_x^2 \, \cos u - 3 \, v^2 \, \cos u$$

- Exactness test with Euler operator:

$$f = 8v_x v_{xx} - u_x^3 \sin u + 2u_x u_{xx} \cos u - 6vv_x \cos u + 3u_x v^2 \sin u$$

$$\mathcal{L}_{u(x)} f = \frac{\partial f}{\partial u} - \mathrm{D}_x \frac{\partial f}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial f}{\partial u_{xx}} \equiv 0$$

$$\mathcal{L}_{v(x)} f = \frac{\partial f}{\partial v} - \mathrm{D}_x \frac{\partial f}{\partial v_x} + \mathrm{D}_x^2 \frac{\partial f}{\partial v_{xx}} \equiv 0$$

- Question: How can one compute $\mathbf{F} = \mathrm{Div}^{-1} f$?
- Theorem (integration by parts):
  - In 1D: If f is exact then

$$\boxed{F = \mathrm{D}_x^{-1} f = \int f\,dx = \mathcal{H}_{\mathbf{u}(x)} f}$$

  - In 2D: If f is a divergence then

$$\boxed{\mathbf{F} = \mathrm{Div}^{-1} f = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f,\, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f)}$$

The homotopy operator inverts total derivatives and divergences!

- Homotopy Operator in 1D (variable x):

$$\boxed{\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^{N} (I_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}}$$

with integrand

$$I_{u^{(j)}} f = \sum_{k=1}^{M_x^{(j)}} \left(\sum_{i=0}^{k-1} u_{ix}^{(j)} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}^{(j)}}$$

$(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u} \to \lambda \mathbf{u}, \ \mathbf{u}_x \to \lambda \mathbf{u}_x,$ *etc.*

More general: $\mathbf{u} \to \lambda(\mathbf{u} - \mathbf{u}_0) + \mathbf{u}_0$

$\mathbf{u}_x \to \lambda(\mathbf{u}_x - \mathbf{u}_{x0}) + \mathbf{u}_{x0}$ etc.

- Homotopy Operator in 2D (variables x and y):

$$\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(x)}_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$\mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(y)}_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where for dependent variable $u(x,y)$

$$\mathcal{I}^{(x)}_u f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix\,jy} \frac{\binom{i+j}{i}\binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} (-\mathrm{D}_x)^{k-i-1} (-\mathrm{D}_y)^{\ell-j} \right) \frac{\partial f}{\partial u_{kx\,\ell y}}$$

Application of Homotopy Operator in 1D

Example:

$$f = 8 v_x v_{xx} - u_x^3 \sin u + 2 u_x u_{xx} \cos u - 6 v v_x \cos u + 3 u_x v^2 \sin u$$

- Compute

$$\begin{aligned} I_u\, f &= u \frac{\partial f}{\partial u_x} + \left(u_x \mathrm{I} - u \mathrm{D}_x \right) \frac{\partial f}{\partial u_{xx}} \\ &= -u u_x^2 \sin u + 3 u v^2 \sin u + 2 u_x^2 \cos u \end{aligned}$$

- Similarly,

$$\begin{aligned} I_v\, f &= v \frac{\partial f}{\partial v_x} + (v_x \mathrm{I} - v \mathrm{D}_x) \frac{\partial f}{\partial v_{xx}} \\ &= -6v^2 \cos u + 8v_x^2 \end{aligned}$$

- Finally,

$$\begin{aligned} F = \mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 (I_u\, f + I_v\, f)\,[\lambda \mathbf{u}]\,\frac{d\lambda}{\lambda} \\ = \int_0^1 \Big(3\lambda^2 u v^2 \sin(\lambda u) - \lambda^2 u u_x^2 \sin(\lambda u) \\ +2\lambda u_x^2 \cos(\lambda u) - 6\lambda v^2 \cos(\lambda u) + 8\lambda v_x^2\Big)\, d\lambda \\ = 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u \end{aligned}$$

Application of Homotopy Operator in 2D

- Example: $f = u_x v_y - u_{xx} v_y - u_y v_x + u_{xy} v_x$
- By hand: $\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$
- Compute

$$
\begin{aligned}
I_u^{(x)} f &= u \frac{\partial f}{\partial u_x} + \left(u_x \mathrm{I} - u \mathrm{D}_x \right) \frac{\partial f}{\partial u_{xx}} \\
&\quad + \left(\tfrac{1}{2} u_y \mathrm{I} - \tfrac{1}{2} u \mathrm{D}_y \right) \frac{\partial f}{\partial u_{xy}} \\
&= u v_y + \tfrac{1}{2} u_y v_x - u_x v_y + \tfrac{1}{2} u v_{xy}
\end{aligned}
$$

- Similarly,

$$I_v^{(x)} f = v \frac{\partial f}{\partial v_x} = -u_y v + u_{xy} v$$

- Hence,

$$\begin{aligned} F_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \left(I_u^{(x)} f + I_v^{(x)} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \lambda \left(u v_y + \tfrac{1}{2} u_y v_x - u_x v_y + \tfrac{1}{2} u v_{xy} - u_y v + u_{xy} v \right) d\lambda \\ &= \tfrac{1}{2} u v_y + \tfrac{1}{4} u_y v_x - \tfrac{1}{2} u_x v_y + \tfrac{1}{4} u v_{xy} - \tfrac{1}{2} u_y v + \tfrac{1}{2} u_{xy} v \end{aligned}$$

- Analogously,

$$F_2 = \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f = \int_0^1 \left(I^{(y)}_u f + I^{(y)}_v f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$= \int_0^1 \left(\lambda \left(-u v_x - \tfrac{1}{2} u v_{xx} + \tfrac{1}{2} u_x v_x \right) + \lambda \left(u_x v - u_{xx} v \right) \right) d\lambda$$

$$= -\tfrac{1}{2} u v_x - \tfrac{1}{4} u v_{xx} + \tfrac{1}{4} u_x v_x + \tfrac{1}{2} u_x v - \tfrac{1}{2} u_{xx} v$$

- So,

$$\mathbf{F} = \frac{1}{4} \begin{pmatrix} 2u v_y + u_y v_x - 2u_x v_y + u v_{xy} - 2u_y v + 2u_{xy} v \\ -2u v_x - u v_{xx} + u_x v_x + 2u_x v - 2u_{xx} v \end{pmatrix}$$

Let $\mathbf{K} = \tilde{\mathbf{F}} - \mathbf{F}$ then

$$\mathbf{K} = \frac{1}{4} \begin{pmatrix} 2uv_y - u_y v_x - 2u_x v_y - uv_{xy} + 2u_y v - 2u_{xy} v \\ -2uv_x + uv_{xx} + 3u_x v_x - 2u_x v + 2u_{xx} v \end{pmatrix}$$

then $\mathrm{Div}\,\mathbf{K} = 0$

- Also, $\mathbf{K} = (\mathrm{D}_y \phi, -\mathrm{D}_x \phi)$ with $\phi = \frac{1}{4}(2uv - uv_x - 2u_x v)$

  ($curl$ in 2D)

  After removing the curl term $\mathbf{K}$:

$$\tilde{\mathbf{F}} = \mathbf{F} + \mathbf{K} = (uv_y - u_x v_y, -uv_x + u_x v_x)$$

Avoid curl terms algorithmically!

Why does this work?

Sketch of Derivation and Proof

(in 1D with variable x, and for one component u)

Definition: Degree operator $\mathcal{M}$

$$\mathcal{M}f = \sum_{i=0}^{M} u_{ix} \frac{\partial f}{\partial u_{ix}} = u \frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} + u_{2x} \frac{\partial f}{\partial u_{2x}} + \cdots + u_{Mx} \frac{\partial f}{\partial u_{Mx}}$$

f is of order M in x

Example: $f = u^p u_x^q u_{3x}^r$ (p, q, r non-negative integers)

$$g = \mathcal{M}f = \sum_{i=0}^{3} u_{ix} \frac{\partial f}{\partial u_{ix}} = (p + q + r)\, u^p u_x^q u_{3x}^r$$

Application of $\mathcal{M}$ computes the total *degree*

Theorem (inverse operator) $\mathcal{M}^{-1}g(u) = \int_0^1 g[\lambda u]\, \frac{d\lambda}{\lambda}$

Proof:

$$\frac{d}{d\lambda}g[\lambda u] = \sum_{i=0}^{M} \frac{\partial g[\lambda u]}{\partial \lambda u_{ix}} \frac{d\lambda u_{ix}}{d\lambda} = \frac{1}{\lambda}\sum_{i=0}^{M} u_{ix}\frac{\partial g[\lambda u]}{\partial u_{ix}} = \frac{1}{\lambda}\mathcal{M}g[\lambda u]$$

Integrate both sides with respect to λ

$$\begin{aligned}\int_0^1 \frac{d}{d\lambda}g[\lambda u]\, d\lambda &= g[\lambda u]\Big|_{\lambda=0}^{\lambda=1} = g(u) - g(0) \\ &= \int_0^1 \mathcal{M}g[\lambda u]\, \frac{d\lambda}{\lambda} = \mathcal{M}\int_0^1 g[\lambda u]\, \frac{d\lambda}{\lambda}\end{aligned}$$

Assuming $g(0) = 0$,

$$\mathcal{M}^{-1}g(u) = \int_0^1 g[\lambda u]\, \frac{d\lambda}{\lambda}$$

Example:

If $g(u) = (p+q+r)\, u^p u_x^q u_{3x}^r$, then

$$g[\lambda u] = (p+q+r) \lambda^{p+q+r}\, u^p u_x^q u_{3x}^r$$

Hence,

$$\begin{aligned} \mathcal{M}^{-1} g &= \int_0^1 (p+q+r)\, \lambda^{p+q+r-1}\, u^p u_x^q u_{3x}^r \, d\lambda \\ &= u^p u_x^q u_{3x}^r \;\; \lambda^{p+q+r} \Big|_{\lambda=0}^{\lambda=1} = u^p u_x^q u_{3x}^r \end{aligned}$$

Theorem: If f is an exact differential function, then

$$F = \mathcal{D}_x^{-1} f = \int f \, dx = \mathcal{H}_{u(x)} f$$

Proof: Multiply

$$\mathcal{L}_{u(x)} \, f = \sum_{k=0}^{M} (-\mathcal{D}_x)^k \frac{\partial f}{\partial u_{kx}}$$

by u to restore the degree.

Split off $u \frac{\partial f}{\partial u}$. Integrate by parts.

Split off $u_x \frac{\partial f}{\partial u_x}$. Repeat the process.

Lastly, split off $u_{Mx} \frac{\partial f}{\partial u_{Mx}}$.

$$
\begin{aligned}
u\mathcal{L}_{u(x)} f &= u \sum_{k=0}^{M} (-\mathcal{D}_x)^k \frac{\partial f}{\partial u_{kx}} \\
&= u\frac{\partial f}{\partial u} - \mathcal{D}_x \left(u \sum_{k=1}^{M} (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \right) + u_x \sum_{k=1}^{M} (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \\
&= u\frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} - \mathcal{D}_x \left(u \sum_{k=1}^{M} (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \right. \\
&\qquad \left. + u_x \sum_{k=2}^{M} (-\mathcal{D}_x)^{k-2} \frac{\partial f}{\partial u_{kx}} \right) + u_{2x} \sum_{k=2}^{M} (-\mathcal{D}_x)^{k-2} \frac{\partial f}{\partial u_{kx}} \\
&= \ldots
\end{aligned}
$$

$$
\begin{aligned}
&= u\frac{\partial f}{\partial u} + u_x\frac{\partial f}{\partial u_x} + \ldots + u_{Mx}\frac{\partial f}{\partial u_{Mx}} \\
&\quad -\mathcal{D}_x\left(u\sum_{k=1}^{M}(-\mathcal{D}_x)^{k-1}\frac{\partial f}{\partial u_{kx}} + u_x\sum_{k=2}^{M}(-\mathcal{D}_x)^{k-2}\frac{\partial f}{\partial u_{kx}}\right. \\
&\quad \left. + \ldots + u_{(M-1)x}\sum_{k=M}^{M}(-\mathcal{D}_x)^{k-M}\frac{\partial f}{\partial u_{kx}}\right) \\
&= \sum_{i=0}^{M} u_{ix}\frac{\partial f}{\partial u_{ix}} - \mathcal{D}_x\left(\sum_{i=0}^{M-1} u_{ix}\sum_{k=i+1}^{M}(-\mathcal{D}_x)^{k-(i+1)}\frac{\partial f}{\partial u_{kx}}\right) \\
&= \mathcal{M}f - \mathcal{D}_x\left(\sum_{i=0}^{M-1} u_{ix}\sum_{k=i+1}^{M}(-\mathcal{D}_x)^{k-(i+1)}\frac{\partial f}{\partial u_{kx}}\right) \\
&= 0
\end{aligned}
$$

$$\mathcal{M}f = \mathcal{D}_x \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right)$$

Apply $\mathcal{M}^{-1}$ and use $\mathcal{M}^{-1}\mathcal{D}_x = \mathcal{D}_x \mathcal{M}^{-1}$.

$$f = \mathcal{D}_x \left(\mathcal{M}^{-1} \sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right)$$

Apply $\mathcal{D}_x^{-1}$ and use the formula for $\mathcal{M}^{-1}$.

$$\begin{aligned} F &= \mathcal{D}_x^{-1} f = \int_0^1 \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right) [\lambda u] \, \frac{d\lambda}{\lambda} \\ &= \int_0^1 \left(\sum_{k=1}^{M} \left(\sum_{i=0}^{k-1} u_{ix} (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}} \right) [\lambda u] \, \frac{d\lambda}{\lambda} \\ &= \mathcal{H}_{u(x)} f \end{aligned}$$

Computation of Conservation Laws for SWW

Quick Recapitulation

- Conservation law in (2+1) dimensions

$$\boxed{\mathrm{D}_t\rho + \nabla\cdot\mathbf{J} = \mathrm{D}_t\rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 = 0} \quad \text{(on PDE)}$$

  conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$

- Example: Shallow water wave (SWW) equations

$$u_t + uu_x + vu_y - 2\,\Omega v + \tfrac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\,\Omega u + \tfrac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- Typical density-flux pair:

$$
\begin{aligned}
\rho_{(5)} &= \theta\left(v_x - u_y + 2\Omega\right) \\
\mathbf{J}^{(5)} &= \frac{1}{2}\,\theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}
\end{aligned}
$$

Algorithm for PDEs in (2+1)-dimensions

- **Step 1: Construct the form of the density**

The SWW equations are invariant under the scaling symmetries

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda\theta, \lambda h, \lambda^2\Omega)$$

and

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda^2\theta, \lambda^0 h, \lambda^2\Omega)$$

Construct a candidate density, for example,

$$\rho = c_1\Omega\theta + c_2 u_y\theta + c_3 v_y\theta + c_4 u_x\theta + c_5 v_x\theta$$

which is scaling invariant under *both* symmetries.

- **Step 2: Determine the constants c_i**

  Compute $E = -\mathrm{D}_t \rho$ and remove time derivatives

$$
\begin{aligned}
E &= -\left(\frac{\partial \rho}{\partial u_x} u_{tx} + \frac{\partial \rho}{\partial u_y} u_{ty} + \frac{\partial \rho}{\partial v_x} v_{tx} + \frac{\partial \rho}{\partial v_y} v_{ty} + \frac{\partial \rho}{\partial \theta} \theta_t\right) \\
&= c_4 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_x \\
&\quad + c_2 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_y \\
&\quad + c_5 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_x \\
&\quad + c_3 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_y \\
&\quad + (c_1 \Omega + c_2 u_y + c_3 v_y + c_4 u_x + c_5 v_x)(u \theta_x + v \theta_y)
\end{aligned}
$$

  Require that

$$
\mathcal{L}_{u(x,y)} E = \mathcal{L}_{v(x,y)} E = \mathcal{L}_{\theta(x,y)} E = \mathcal{L}_{h(x,y)} E \equiv 0.
$$

- Solution: $c_1 = 2,\ c_2 = -1,\ c_3 = c_4 = 0,\ c_5 = 1$ gives

$$\boxed{\rho = \theta\,(2\Omega - u_y + v_x)}$$

- **Step 3: Compute the flux $\mathbf{J}$**

$$\begin{aligned} E \;=\; & \theta(u_x v_x + u v_{xx} + v_x v_y + v v_{xy} + 2\Omega u_x \\ & + \tfrac{1}{2}\theta_x h_y - u_x u_y - u u_{xy} - u_y v_y - u_{yy} v \\ & + 2\Omega v_y - \tfrac{1}{2}\theta_y h_x) + 2\Omega u \theta_x + 2\Omega v \theta_y \\ & - u u_y \theta_x - u_y v \theta_y + u v_x \theta_x + v v_x \theta_y \end{aligned}$$

Apply the 2D homotopy operator:

$$\mathbf{J} = (J_1, J_2) = \mathrm{Div}^{-1} E = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E)$$

Compute

$$I_u^{(x)}E = u\frac{\partial E}{\partial u_x} + \left(\tfrac{1}{2}u_y\mathrm{I} - \tfrac{1}{2}u\mathrm{D}_y\right)\frac{\partial E}{\partial u_{xy}}$$
$$= uv_x\theta + 2\Omega u\theta + \tfrac{1}{2}u^2\theta_y - uu_y\theta$$

Similarly, compute

$$\begin{aligned} I_v^{(x)}E &= vv_y\theta + \tfrac{1}{2}v^2\theta_y + uv_x\theta \\ I_\theta^{(x)}E &= \tfrac{1}{2}\theta^2 h_y + 2\Omega u\theta - uu_y\theta + uv_x\theta \\ I_h^{(x)}E &= -\tfrac{1}{2}\theta\theta_y h \end{aligned}$$

Next,

$$
\begin{aligned}
J_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E \\
&= \int_0^1 \left(I^{(x)}_u E + I^{(x)}_v E + I^{(x)}_\theta E + I^{(x)}_h E \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
&= \int_0^1 \Bigg(4\lambda \Omega u \theta + \lambda^2 \Big(3 u v_x \theta + \tfrac{1}{2} u^2 \theta_y - 2 u u_y \theta + v v_y \theta \\
&\qquad + \tfrac{1}{2} v^2 \theta_y + \tfrac{1}{2} \theta^2 h_y - \tfrac{1}{2} \theta \theta_y h \Big) \Bigg) d\lambda \\
&= 2\Omega u \theta - \tfrac{2}{3} u u_y \theta + u v_x \theta + \tfrac{1}{3} v v_y \theta + \tfrac{1}{6} u^2 \theta_y \\
&\quad + \tfrac{1}{6} v^2 \theta_y - \tfrac{1}{6} h \theta \theta_y + \tfrac{1}{6} h_y \theta^2
\end{aligned}
$$

Analogously,

$$\begin{aligned} J_2 &= \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E \\ &= 2\Omega v\theta + \tfrac{2}{3} v v_x \theta - v u_y \theta - \tfrac{1}{3} u u_x \theta - \tfrac{1}{6} u^2 \theta_x - \tfrac{1}{6} v^2 \theta_x \\ &\qquad + \tfrac{1}{6} h\theta\theta_x - \tfrac{1}{6} h_x \theta^2 \end{aligned}$$

Hence,

$$\mathbf{J} = \frac{1}{6} \begin{pmatrix} 12\Omega u\theta - 4u u_y \theta + 6u v_x \theta + 2v v_y \theta + (u^2 + v^2)\theta_y - h\theta\theta_y + h_y \theta^2 \\ 12\Omega v\theta + 4v v_x \theta - 6v u_y \theta - 2u u_x \theta - (u^2 + v^2)\theta_x + h\theta\theta_x - h_x \theta^2 \end{pmatrix}$$

There are curl terms in $\mathbf{J}$

Indeed, subtract $\mathbf{K}$ where $\mathrm{Div}\,\mathbf{K} = 0$

Here

$$\mathbf{K} = \frac{1}{6}\begin{pmatrix} -(2uu_y\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y + 2h\theta\theta_y + h_y\theta^2) \\ 2vv_x\theta + 2uu_x\theta + u^2\theta_x + v^2\theta_x + 2h\theta\theta_x + h_x\theta^2 \end{pmatrix}$$

Note that $\mathbf{K} = (\mathrm{D}_y\phi, -\mathrm{D}_x\phi)$ with $\phi = -(h\theta^2 + u^2\theta + v^2\theta)$

($curl$ in 2D).

After removing the curl term $\mathbf{K}$,

$$\tilde{\mathbf{J}}^{(5)} = \frac{1}{2}\,\theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}$$

PART II: DISCRETE CASE

Problem Statement

Discrete case in 1D:

Example:

$$f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

Question: Can the expression be summed by parts?

If yes, find $F_n = \Delta^{-1} f_n$ (so, $f_n = \Delta F_n = F_{n+1} - F_n$)

Result (by hand): $F_n = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}$

How can this be done algorithmically?

Can this be done as in the continuous case?

Tools from the Discrete Calculus of Variations

- Definitions:

D is the up-shift (forward or right-shift) operator

$$\mathrm{D} F_n = F_{n+1} = F_{n}|_{n \to n+1}$$

D^{-1} the down-shift (backward or left-shift) operator

$$\mathrm{D}^{-1} F_n = F_{n-1} = F_{n}|_{n \to n-1}$$

$\Delta = \mathrm{D} - \mathrm{I}$ is the forward difference operator

$$\Delta F_n = (\mathrm{D} - \mathrm{I}) F_n = F_{n+1} - F_n$$

- Problem to be solved: Given f_n.

Find $F_n = \Delta^{-1} f_n$ (so $f_n = \Delta F_n = F_{n+1} - F_n$)

Analogy Continuous & Discrete Cases

Euler Operators

Continuous Case	Discrete Case
$\mathcal{L}_{\mathbf{u}(x)} = \sum_{k=0}^{M} (-\mathrm{D}_x)^k \frac{\partial}{\partial \mathbf{u}_{k\,x}}$	$\mathcal{L}_{\mathbf{u}_n} = \sum_{k=0}^{M} \mathrm{D}^{-k} \frac{\partial}{\partial \mathbf{u}_{n+k}} = \frac{\partial}{\partial \mathbf{u}_n} \sum_{k=0}^{M} \mathrm{D}^{-k}$

Analogy Continuous & Discrete Cases

Homotopy Operators & Integrands

Continuous Case	Discrete Case
$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^{N} (I_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$	$\mathcal{H}_{\mathbf{u}_n} f_n = \int_0^1 \sum_{j=1}^{N} (I_{u_n^{(j)}} f_n)[\lambda \mathbf{u}_n] \frac{d\lambda}{\lambda}$
$I_{u^{(j)}} f = \sum_{k=1}^{M^{(j)}} \left(\sum_{i=0}^{k-1} u_{ix}^{(j)} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}^{(j)}}$	$I_{u_n^{(j)}} f_n = \sum_{i=0}^{M^{(j)}-1} u_{n+i}^{(j)} \frac{\partial}{\partial u_{n+i}^{(j)}} \sum_{k=i+1}^{M^{(j)}} \mathrm{D}^{-(k-i)} f_n$

Euler Operators Side by Side

Continuous Case (for component u)

$$\mathcal{L}_u = \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{2x}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{3x}} + \cdots$$

Discrete Case (for component u_n)

$$\begin{aligned} \mathcal{L}_{u_n} &= \frac{\partial}{\partial u_n} + \mathrm{D}^{-1} \frac{\partial}{\partial u_{n+1}} + \mathrm{D}^{-2} \frac{\partial}{\partial u_{n+2}} + \mathrm{D}^{-3} \frac{\partial}{\partial u_{n+3}} + \cdots \\ &= \frac{\partial}{\partial u_n} (\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \mathrm{D}^{-3} + \cdots) \end{aligned}$$

Homotopy Operators Side by Side

Continuous Case (for components u and v)

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \left(I_u\, f + I_v\, f \right) [\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

with

$$I_u\, f = \sum_{k=1}^{M^{(1)}} \left(\sum_{i=0}^{k-1} u_{ix} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}}$$

and

$$I_v\, f = \sum_{k=1}^{M^{(2)}} \left(\sum_{i=0}^{k-1} v_{ix} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial v_{kx}}$$

Discrete Case (for components u_n and v_n)

$$\mathcal{H}_{\mathbf{u}_n} f_n = \int_0^1 \left(I_{u_n} f_n + I_{v_n} f_n \right) [\lambda \mathbf{u}_n] \frac{d\lambda}{\lambda}$$

with

$$I_{u_n} f_n = \sum_{i=0}^{M^{(1)}-1} u_{n+i} \frac{\partial}{\partial u_{n+i}} \sum_{k=i+1}^{M^{(1)}} \mathrm{D}^{-(k-i)} f_n$$

and

$$I_{v_n} f_n = \sum_{i=0}^{M^{(2)}-1} v_{n+i} \frac{\partial}{\partial v_{n+i}} \sum_{k=i+1}^{M^{(2)}} \mathrm{D}^{-(k-i)} f_n$$

Analogy of Definitions & Theorems

Continuous Case (PDE)	Semi-discrete Case (DDE)
$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \cdots)$	$\dot{\mathbf{u}}_n = \mathbf{F}(\cdots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \cdots)$
$\mathrm{D}_t \rho + \mathrm{D}_x J = 0$	$\mathrm{D}_t \rho_n + \Delta\, J_n = 0$

- Definition: f_n is exact iff $f_n = \Delta F_n = F_{n+1} - F_n$
- Theorem (exactness test): $f_n = \Delta F_n$ iff $\mathcal{L}_{\mathbf{u}_n} f_n \equiv 0$
- Theorem (summation with homotopy operator):

If f_n is exact then $F_n = \Delta^{-1} f_n = \mathcal{H}_{\mathbf{u}_n}(f_n)$

Testing Exactness – Discrete Case

For example,

$$f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

- f_n is exact
- After summation by parts (done by hand):

$$F_n = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}$$

- Exactness test with Euler operator:

For component u_n (highest shift 3):

$$
\begin{aligned}
\mathcal{L}_{u_n} f_n &= \frac{\partial}{\partial u_n}\Big(\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \mathrm{D}^{-3}\Big) f_n \\
&= -u_{n+1}v_n - u_{n-1}v_{n-1} + u_{n+1}v_n - v_{n-1} \\
&\quad + u_{n-1}v_{n-1} + v_{n-1} \\
&\equiv 0
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{L}_{v_n} f_n &= \frac{\partial}{\partial v_n}\Big(\mathrm{I} + \mathrm{D}^{-1}\Big) f_n \\
&= u_n u_{n+1} + 2v_n - u_n u_{n+1} - 2v_n \\
&\equiv 0
\end{aligned}
$$

Application of Discrete Homotopy Operator

Example:

$$f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

Here, $M^{(1)} = 3$ and $M^{(2)} = 2$.

Compute

$$\begin{aligned} I_{u_n} f_n &= u_n \frac{\partial}{\partial u_n} (\mathrm{D}^{-1} + \mathrm{D}^{-2} + \mathrm{D}^{-3}) f_n \\ &\quad + u_{n+1} \frac{\partial}{\partial u_{n+1}} (\mathrm{D}^{-1} + \mathrm{D}^{-2}) f_n \\ &\quad + u_{n+2} \frac{\partial}{\partial u_{n+2}} \mathrm{D}^{-1} f_n \\ &= 2 u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1} \end{aligned}$$

$$\begin{aligned} I_{v_n}\, f_n &= v_n \frac{\partial}{\partial v_n}(\mathrm{D}^{-1} + \mathrm{D}^{-2}) f_n + v_{n+1} \frac{\partial}{\partial v_{n+1}}(\mathrm{D}^{-1}) f_n \\ &= u_n u_{n+1} v_n + 2v_n^2 + u_{n+1} v_n + u_{n+2} v_{n+1} \end{aligned}$$

Finally,

$$\begin{aligned} F_n &= \int_0^1 \left(I_{u_n}\, f_n + I_{v_n}\, f_n\right) [\lambda \mathbf{u}_n] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \Big(2\lambda v_n^2 + 3\lambda^2 u_n u_{n+1} v_n + 2\lambda u_{n+1} v_n + 2\lambda u_{n+2} v_{n+1}\Big) d\lambda \\ &= v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1} \end{aligned}$$

Application: Computation of Conservation Laws

- System of DDEs

$$\dot{\mathbf{u}}_n = \mathbf{F}(\cdots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \cdots)$$

- Conservation law

$$\mathrm{D}_t \rho_n + \Delta J_n = 0 \quad \text{(on DDE)}$$

conserved density ρ_n and flux J_n

- Example: Toda lattice

$$\begin{aligned} \dot{u}_n &= v_{n-1} - v_n \\ \dot{v}_n &= v_n(u_n - u_{n+1}) \end{aligned}$$

- Typical density-flux pair:

$$\begin{aligned} \rho_n^{(3)} &= \tfrac{1}{3}u_n^3 + u_n(v_{n-1} + v_n) \\ J_n^{(3)} &= u_{n-1}u_n v_{n-1} + v_{n-1}^2 \end{aligned}$$

Computation Conservation Laws for Toda Lattice

Step 1: Construct the form of the density

The Toda lattice is invariant under scaling symmetry

$$(t, u_n, v_n) \rightarrow (\lambda^{-1} t, \lambda u_n, \lambda^2 v_n)$$

Construct a candidate density, for example,

$$\rho_n = c_1\, u_n^3 + c_2\, u_n v_{n-1} + c_3\, u_n v_n$$

which is scaling invariant under the symmetry

Step 2: Determine the constants c_i

Compute $E_n = \mathrm{D}_t \rho_n$ and remove time derivatives

$$
\begin{aligned}
E_n &= (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n + (c_3 - c_2)v_{n-1}v_n \\
&\quad + c_2 u_{n-1}u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2
\end{aligned}
$$

Compute $\tilde{E}_n = \mathrm{D} E_n$ to remove negative shift $n-1$

Require that $\mathcal{L}_{u_n} \tilde{E}_n = \mathcal{L}_{v_n} \tilde{E}_n \equiv 0$

Solution: $c_1 = \frac{1}{3}, c_2 = c_3 = 1$ gives

$$
\boxed{\rho_n = \frac{1}{3}\, u_n^3 + u_n(v_{n-1} + v_n)}
$$

Step 3: Compute the flux J_n

$$\tilde{E}_n = \mathrm{D}E_n = u_n u_{n+1} v_n + v_n^2 - u_{n+1} u_{n+2} v_{n+1} - v_{n+1}^2$$

Apply the homotopy operator

$$\tilde{J}_n = \mathrm{D}J_n = -\Delta^{-1}(\tilde{E}_n) = -\mathcal{H}_{\mathbf{u}_n}(\tilde{E}_n)$$

Compute

$$\begin{aligned} I_{u_n} \tilde{E}_n &= u_n \frac{\partial}{\partial u_n}(\mathrm{D}^{-1} + \mathrm{D}^{-2})\tilde{E}_n + u_{n+1} \frac{\partial}{\partial u_{n+1}}(\mathrm{D}^{-1})\tilde{E}_n \\ &= -(2u_n u_{n+1} v_n) \end{aligned}$$

Likewise,

$$I_{v_n} \tilde{E}_n = v_n \frac{\partial}{\partial v_n}(\mathrm{D}^{-1})\tilde{E}_n = -(u_n u_{n+1} v_n + 2v_n^2)$$

Next, compute

$$\begin{aligned}\tilde{J}_n &= -\int_0^1 \left(I_{u_n}\, \tilde{E}_n + I_{v_n}\, \tilde{E}_n \right) [\lambda \mathbf{u}_n] \, \frac{d\lambda}{\lambda} \\ &= \int_0^1 (3\lambda^2 u_n u_{n+1} v_n + 2\lambda v_n^2) \, d\lambda \\ &= u_n u_{n+1} v_n + v_n^2\end{aligned}$$

Finally, backward shift $J_n = \mathrm{D}^{-1}(\tilde{J}_n)$ given

$$\boxed{J_n = u_{n-1} u_n v_{n-1} + v_{n-1}^2}$$

Software Demonstration

Demonstrations of *ConservationLawsMD.m*

Demonstration of *DDEDensityFlux.m*

Software packages in *Mathematica*

Codes are available via the Internet:

URL: http://inside.mines.edu/∼whereman/

Conclusions

- Continuous Euler and homotopy operators:
  - Testing exactness
  - Integration by parts: D_x^{-1} and Div^{-1}
  - Application: conservation laws of PDEs
- Discrete Euler and homotopy operators:
  - Testing exactness (summability)
  - Summation by parts: Δ^{-1}
  - Application: conservation laws of DDEs and lattices
- Useful analogy between the formulas.

Thank You

Publications

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