

Continuous and Discrete Homotopy Operators with Applications in Integrability Testing

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OUTLINE

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Exactness or ‘Total Difference’ Criterion: Discrete Euler Operator

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Application of Discrete Homotopy Operator

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Problem Statement

For continuous case:

Given, for example,

$$f = 3 u' v^2 \sin(u) - u'^3 \sin(u) - 6 v v' \cos(u) + 2 u' u'' \cos(u) + 8 v' v''$$

Find F so that $f = D_x F$ or $F = \int f \, dx$.

Result:

$$F = 4 v'^2 + u'^2 \cos(u) - 3 v^2 \cos(u)$$

Can this be done without integration by parts?

Can the problem be reduced to a single integral in one variable?

For discrete case:

Given, for example,

$$f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

Find F_n so that $f_n = \Delta F_n = F_{n+1} - F_n$ or $F_n = \Delta^{-1} f_n$.

Result:

$$F_n = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}.$$

How can this be done algorithmically?

Can this be done in the same way as the continuous case?

Part I: Continuous Case

Integration by Parts on the Jet Space

- Given f involving $u(x)$ and $v(x)$ and their derivatives

$$f = 3 u' v^2 \sin(u) - u'^3 \sin(u) - 6 v v' \cos(u) + 2 u' u'' \cos(u) + 8 v' v''$$

- Find F so that $f = D_x F$ or $F = \int f dx$.

Integrate by parts (compute F by hand)

f	F
$8 v' v''$	$\longrightarrow 4 v'^2$
$2 u' u'' \cos(u)$	$\longrightarrow u'^2 \cos(u)$
$-u'^3 \sin(u)$	
$-6 v v' \cos(u)$	$\longrightarrow -3 v^2 \cos(u)$
$3 u' v^2 \sin(u)$	

- Integral:

$$F = 4 v'^2 + u'^2 \cos(u) - 3 v^2 \cos(u)$$

Remark: For simplicity we denote $f(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \dots, \mathbf{u}^{(m)})$ as $f(\mathbf{u})$.

- **Exactness Criterion:**

Continuous Euler Operator (variational derivative)

Definition (exactness):

A function $f(\mathbf{u})$ is exact, i.e. can be integrated fully, if there exists a function $F(\mathbf{u})$, such that $f(\mathbf{u}) = D_x F(\mathbf{u})$ or equivalently $F(\mathbf{u}) = D_x^{-1} f(\mathbf{u}) = \int_x f(\mathbf{u}) dx$.

D_x is the (total) derivative with respect to x .

Theorem (exactness or integrability test):

A necessary and sufficient condition for a function f to be exact, i.e. the derivative of another function, is that $\mathcal{L}_{\mathbf{u}}^{(0)}(f) \equiv 0$ where $\mathcal{L}_{\mathbf{u}}^{(0)}$ is the continuous Euler operator (variational derivative) defined by

$$\begin{aligned} \mathcal{L}_{\mathbf{u}}^{(0)} &= \sum_{k=0}^{m_0} (-D_x)^k \frac{\partial}{\partial \mathbf{u}^{(k)}} \\ &= \frac{\partial}{\partial \mathbf{u}} - D_x \frac{\partial}{\partial \mathbf{u}'} + D_x^2 \frac{\partial}{\partial \mathbf{u}''} + \cdots + (-1)^{m_0} D_x^{m_0} \frac{\partial}{\partial \mathbf{u}^{(m_0)}} \end{aligned}$$

where m_0 is the order (of f).

Proof:

See calculus of variations (derivation of Euler-Lagrange equations — the forgotten case!).

Example: Apply the continuous Euler operator to

$$f(\mathbf{u}) = 3u' v^2 \sin(u) - u'^3 \sin(u) - 6v v' \cos(u) + 2u' u'' \cos(u) + 8v' v''$$

Here $\mathbf{u} = (u, v)$.

For component u (order 2):

$$\begin{aligned} \mathcal{L}_u^{(0)}(f) &= \frac{\partial}{\partial u}(f) - D_x \frac{\partial}{\partial u'}(f) + D_x^2 \frac{\partial}{\partial u''}(f) \\ &= 3u' v^2 \cos(u) - u'^3 \cos(u) + 6v v' \sin(u) - 2u' u'' \sin(u) \\ &\quad - D_x[3v^2 \sin(u) - 3u'^2 \sin(u) + 2u'' \cos(u)] + D_x^2[2u' \cos(u)] \\ &= 3u' v^2 \cos(u) - u'^3 \cos(u) + 6v v' \sin(u) - 2u' u'' \sin(u) \\ &\quad - [3u' v^2 \cos(u) + 6v v' \sin(u) - 3u'^3 \cos(u) - 6u u'' \sin(u) \\ &\quad - 2u' u'' \sin(u) + 2u''' \cos(u)] \\ &\quad + [-2u''' \cos(u) - 6u' u'' \sin(u) + 2u''' \cos(u)] \\ &\equiv 0 \end{aligned}$$

For component v (order 2):

$$\begin{aligned} \mathcal{L}_v^{(0)}(f) &= \frac{\partial}{\partial v}(f) - D_x \frac{\partial}{\partial v'}(f) + D_x^2 \frac{\partial}{\partial v''}(f) \\ &= 6u' v \sin(u) - 6v' \cos(u) - D_x[-6v \cos(u) + 8v''] + D_x^2[8v'] \\ &= 6u' v \sin(u) - 6v' \cos(u) - [6u' v \sin(u) - 6v' \cos(u) + 8v'''] + 8v''' \\ &\equiv 0 \end{aligned}$$

- **Computation of the integral F**

Definition (higher Euler operators):

The continuous higher Euler operators are defined by

$$\mathcal{L}_{\mathbf{u}}^{(i)} = \sum_{k=i}^{m_i} \binom{k}{i} (-D_x)^{k-i} \frac{\partial}{\partial \mathbf{u}^{(k)}}$$

These Euler operators all terminate at some maximal order m_i .

Examples (for component u) :

$$\mathcal{L}_u^{(0)} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u'} + D_x^2 \frac{\partial}{\partial u''} - D_x^3 \frac{\partial}{\partial u'''} + \cdots + (-1)^{m_0} D_x^{m_0} \frac{\partial}{\partial u^{(m_0)}}$$

$$\mathcal{L}_u^{(1)} = \frac{\partial}{\partial u'} - 2D_x \frac{\partial}{\partial u''} + 3D_x^2 \frac{\partial}{\partial u'''} - 4D_x^3 \frac{\partial}{\partial u^{(4)}} + \cdots - (-1)^{m_1} m_1 D_x^{m_1-1} \frac{\partial}{\partial u^{(m_1)}}$$

$$\mathcal{L}_u^{(2)} = \frac{\partial}{\partial u''} - 3D_x \frac{\partial}{\partial u'''} + 6D_x^2 \frac{\partial}{\partial u^{(4)}} - 10D_x^3 \frac{\partial}{\partial u^{(5)}} + \cdots + (-1)^{m_2} \binom{m_2}{2} D_x^{m_2-2} \frac{\partial}{\partial u^{(m_2)}}$$

$$\mathcal{L}_u^{(3)} = \frac{\partial}{\partial u'''} - 4D_x \frac{\partial}{\partial u^{(4)}} + 10D_x^2 \frac{\partial}{\partial u^{(5)}} - 20D_x^3 \frac{\partial}{\partial u^{(6)}} + \cdots - (-1)^{m_3} \binom{m_3}{3} D_x^{m_3-3} \frac{\partial}{\partial u^{(m_3)}}$$

Similar formulae for component $\mathcal{L}_v^{(i)}$

Definition (homotopy operator):

The continuous homotopy operator is defined by

$$\mathcal{H}_{\mathbf{u}} = \int_0^1 \sum_{r=1}^N f_r(\mathbf{u})[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where

$$f_r(\mathbf{u}) = \sum_{i=0}^{p_r} D_x^i [u_r \mathcal{L}_{u_r}^{(i+1)}]$$

p_r is the maximum order of u_r in f

N is the number of dependent variables

$f_r(\mathbf{u})[\lambda \mathbf{u}]$ means that in $f_r(\mathbf{u})$ one replaces $\mathbf{u} \rightarrow \lambda \mathbf{u}$, $\mathbf{u}' \rightarrow \lambda \mathbf{u}'$, etc.

Example:

For a two-component system ($N = 2$) where $\mathbf{u} = (u, v)$:

$$\mathcal{H}_{\mathbf{u}} = \int_0^1 \{f_1(\mathbf{u})[\lambda \mathbf{u}] + f_2(\mathbf{u})[\lambda \mathbf{u}]\} \frac{d\lambda}{\lambda}$$

with

$$f_1(\mathbf{u}) = \sum_{i=0}^{p_1} D_x^i [u \mathcal{L}_u^{(i+1)}]$$

and

$$f_2(\mathbf{u}) = \sum_{i=0}^{p_2} D_x^i [v \mathcal{L}_v^{(i+1)}]$$

Theorem (integration via homotopy operator):

Given an integrable function f

$$F = D_x^{-1} f = \int f dx = \mathcal{H}_{\mathbf{u}}(f)$$

Proof: Olver's book 'Applications of Lie Groups to Differential Equations', p. 372. Proof is given in terms of differential forms.

Work of Henri Poincaré (1854-1912), George de Rham (1950), and Ian Anderson & Tom Duchamp (1980).

Proofs based on calculus: Deconinck and Hereman.

Example: Apply the continuous homotopy operator to integrate

$$f(\mathbf{u}) = 3u'v^2 \sin(u) - u'^3 \sin(u) - 6vv' \cos(u) + 2u'u'' \cos(u) + 8v'v''$$

For component u (order 2):

i	$\mathcal{L}_u^{(i+1)}(f)$	$D_x^i \left(u \mathcal{L}_u^{(i+1)}(f) \right)$
0	$\frac{\partial}{\partial u'} f - 2D_x \left(\frac{\partial}{\partial u''} f \right)$ $= 3v^2 \sin u - 3u'^2 \sin u + 2u'' \cos u$ $- 2D_x(2u' \cos u)$ $= 3v^2 \sin u + u'^2 \sin u - 2u'' \cos u$	$3uv^2 \sin u + uu'^2 \sin u - 2uu'' \cos u$
1	$\frac{\partial}{\partial u''} f = 2u' \cos u$	$D_x[2uu' \cos u]$ $= 2u'^2 \cos u + 2uu'' \cos u - 2uu'^2 \sin u$

Hence, $f_1(\mathbf{u})(f) = 3uv^2 \sin(u) - uu'^2 \sin(u) + 2u'^2 \cos(u)$

For component v (order 2):

i	$\mathcal{L}_v^{(i+1)}(f)$	$D_x^i [v \mathcal{L}_v^{(i+1)}(f)]$
0	$-6v \cos(u) + 8v'' - 2D_x[8v']$ $= -6v \cos(u) - 8v''$	$-6v^2 \cos(u) - 8vv''$
1	$8v'$	$D_x[8vv'] = 8v'^2 + 8vv''$

Hence, $f_2(\mathbf{u})(f) = -6v^2 \cos(u) + 8v'^2$

The homotopy operator leads to an integral for (auxiliary) variable λ .
(Use standard integration by parts to work the integral).

$$\begin{aligned}
F(\mathbf{u}) &= \mathcal{H}_{\mathbf{u}}(f) = \int_0^1 \{ f_1(\mathbf{u})(f)[\lambda \mathbf{u}] + f_2(\mathbf{u})(f)[\lambda \mathbf{u}] \} \frac{d\lambda}{\lambda} \\
&= \int_0^1 [3\lambda^2 uv^2 \sin(\lambda u) - \lambda^2 uu'^2 \sin(\lambda u) + 2\lambda u'^2 \cos(\lambda u) \\
&\quad - 6\lambda v^2 \cos(\lambda u) + 8\lambda v'^2] d\lambda \\
&= 4v'^2 + u'^2 \cos(u) - 3v^2 \cos(u)
\end{aligned}$$

- **Application: Conserved densities and fluxes for PDEs with transcendental nonlinearities**

Definition (conservation law):

$$D_t \rho + D_x J = 0 \quad (\text{on PDE})$$

conserved density ρ and flux J .

Example: Sine-Gordon system (type $\mathbf{u}_t = \mathbf{F}$)

$$\begin{aligned} u_t &= v \\ v_t &= u_{xx} + \alpha \sin(u) \end{aligned}$$

has scaling symmetry

$$(t, x, u, v, \alpha) \rightarrow (\lambda^{-1}t, \lambda^{-1}x, \lambda^0 u, \lambda v, \lambda^2 \alpha)$$

In terms of weights:

$$w(D_x) = 1, \quad w(D_t) = 1, \quad w(u) = 0, \quad w(v) = 1, \quad w(\alpha) = 2$$

Conserved densities and fluxes

$$\begin{aligned} \rho_{(1)} &= 2\alpha \cos(u) + v^2 + u_x^2 & J_{(1)} &= -2u_x v \\ \rho_{(2)} &= u_x v & J_{(2)} &= -\left[\frac{1}{2}v^2 + \frac{1}{2}u_x^2 - \alpha \cos(u)\right] \end{aligned}$$

$$\rho_{(3)} = 12 \cos(u) v u_x + 2v^3 u_x + 2v u_x^3 - 16v_x u_{2x}$$

$$\begin{aligned} \rho_{(4)} &= 2 \cos^2(u) - 2 \sin^2(u) + v^4 + 6v^2 u_x^2 + u_x^4 + 4 \cos(u) v^2 \\ &\quad + 20 \cos(u) u_x^2 - 16v_x^2 - 16u_{2x}^2. \end{aligned}$$

are all scaling invariant!

Remark: $J_{(3)}$ and $J_{(4)}$ are not shown (too long).

- **Algorithm for Conserved Densities and Fluxes**

Example: Density and flux of rank 2 for sine-Gordon system

Step 1: Construct the form of the density

$$\rho = \alpha h_1(u) + h_2(u)v^2 + h_3(u)u_x^2 + h_4(u)u_x v$$

where $h_i(u)$ are unknown functions.

Step 2: Determine the functions h_i

Compute

$$\begin{aligned} E = D_t \rho &= \frac{\partial \rho}{\partial t} + \rho'(\mathbf{u})[\mathbf{F}] \quad (\text{on PDE}) \\ &= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{m_1} \frac{\partial \rho}{\partial u_{kx}} D_x^k u_t + \sum_{k=0}^{m_2} \frac{\partial \rho}{\partial v_{kx}} D_x^k v_t \end{aligned}$$

Since $E = D_t \rho = -D_x J$, the expression E must be integrable.

Require that $\mathcal{L}_u^{(0)}(E) \equiv 0$ and $\mathcal{L}_v^{(0)}(E) \equiv 0$.

Solve the system of linear mixed system (algebraic eqs. and ODEs):

$$\begin{aligned} h_2(u) - h_3(u) &= 0 \\ h_2'(u) &= 0 \\ h_3'(u) &= 0 \\ h_4'(u) &= 0 \\ h_2''(u) &= 0 \\ h_4''(u) &= 0 \\ 2h_2'(u) - h_3'(u) &= 0 \\ 2h_2''(u) - h_3''(u) &= 0 \\ h_1'(u) + 2\sin(u)h_2(u) &= 0 \\ h_1''(u) + 2\sin(u)h_2'(u) + 2\cos(u)h_2(u) &= 0 \end{aligned}$$

Solution:

$$\begin{aligned}h_1(u) &= 2c_1 \cos(u) + c_3 \\h_2(u) &= h_3(u) = c_1 \\h_4(u) &= c_2\end{aligned}$$

(with arbitrary constants c_i).

Substitute in ρ

$$\rho = c_1(2\alpha \cos(u) + v^2 + u_x^2) + c_2(u_x v) + c_3\alpha$$

Step 3: Compute the flux J

First, compute

$$\begin{aligned}E = D_t \rho &= c_1(-2\alpha u_t \sin u + 2vv_t + 2u_x u_{xt}) + c_2(u_{xt}v + u_x v_t) \\&= c_1(-2\alpha v \sin u + 2v(u_{2x} + \alpha \sin u) + 2u_x v_x) \\&\quad + c_2(v_x v + u_x(u_{2x} + \alpha \sin u)) \\&= c_1(2u_{2x}v + 2u_x v_x) + c_2(vv_x + u_x u_{2x} + \alpha u_x \sin u)\end{aligned}$$

Since $E = D_t \rho = -D_x J$, one must integrate $f = -E$.

Apply the homotopy operator for each component of $\mathbf{u} = (u, v)$.

For component u (order 2):

i	$\mathcal{L}_u^{(i+1)}(f)$	$D_x^i(u\mathcal{L}_u^{(i+1)}(f))$
0	$2c_1 v_x + c_2(u_{2x} - \alpha \sin u)$	$2c_1 u v_x + c_2(u u_{2x} - \alpha u \sin u)$
1	$-2c_1 v - c_2 u_x$	$-2c_1(u_x v + u v_x) - c_2(u_x^2 + u u_{2x})$

Hence, $f_1(\mathbf{u})(f) = -2c_1 u_x v - c_2(u_x^2 + \alpha u \sin u)$

For component v (order 1):

i	$\mathcal{L}_v^{(i+1)}(f)$	$D_x^i \left(v \mathcal{L}_v^{(i+1)}(f) \right)$
0	$-2c_1 u_x - c_2 v$	$-2c_1 u_x v - c_2 v^2$

Hence, $f_2(\mathbf{u})(f) = -2c_1 u_x v - c_2 v^2$

The homotopy operator leads to an integral for (one) variable λ :

$$\begin{aligned}
J(\mathbf{u}) &= \mathcal{H}_{\mathbf{u}}(f) = \int_0^1 (f_1(\mathbf{u})(f)[\lambda \mathbf{u}] + f_2(\mathbf{u})(f)[\lambda \mathbf{u}]) \frac{d\lambda}{\lambda} \\
&= - \int_0^1 \left(4c_1 \lambda u_x v + c_2 (\lambda u_x^2 + \alpha u \sin(\lambda u) + \lambda v^2) \right) d\lambda \\
&= -2c_1 u_x v - c_2 \left(\frac{1}{2} v^2 + \frac{1}{2} u_x^2 - \alpha \cos u \right)
\end{aligned}$$

Split the density and flux in independent pieces (for c_1 and c_2):

$$\begin{aligned}
\rho_{(1)} &= 2\alpha \cos u + v^2 + u_x^2 & J_{(1)} &= -2u_x v \\
\rho_{(2)} &= u_x v & J_{(2)} &= -\frac{1}{2} v^2 - \frac{1}{2} u_x^2 + \alpha \cos u
\end{aligned}$$

Remark: Computation of $J_{(3)}$ and $J_{(4)}$ requires integration with the homotopy operator!

Computer Demos

(1) Use continuous homotopy operator to integrate

$$f = 3 u' v^2 \sin(u) - u'^3 \sin(u) - 6 v v' \cos(u) + 2 u' u'' \cos(u) + 8 v' v''$$

(2) Compute densities of rank 8 and fluxes for 5th-order Korteweg-de Vries equation with three parameters:

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

(α, β, γ are nonzero constant parameters).

(3) Compute density of rank 4 and flux for sine-Gordon system:

$$\begin{aligned} u_t &= v \\ v_t &= u_{xx} + \alpha \sin(u) \end{aligned}$$

Analogy PDEs and DDEs

	Continuous Case (PDEs)	Semi-discrete Case (DDEs)
System	$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \dots)$	$\dot{\mathbf{u}}_n = \mathbf{F}(\dots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \dots)$
Conservation Law	$D_t \rho + D_x J = 0$	$\dot{\rho}_n + J_{n+1} - J_n = 0$
Symmetry	$D_t \mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}]$ $= \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u} + \epsilon \mathbf{G}) _{\epsilon=0}$	$D_t \mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}]$ $= \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u}_n + \epsilon \mathbf{G}) _{\epsilon=0}$
Recursion Operator	$D_t \mathcal{R} + [\mathcal{R}, \mathbf{F}'(u)] = 0$	$D_t \mathcal{R} + [\mathcal{R}, \mathbf{F}'(\mathbf{u}_n)] = 0$

Table 1: Conservation Laws and Symmetries

	KdV Equation	Volterra Lattice
Equation	$u_t = 6uu_x + u_{3x}$	$\dot{u}_n = u_n(u_{n+1} - u_{n-1})$
Densities	$\rho = u, \quad \rho = u^2$ $\rho = u^3 - \frac{1}{2}u_x^2$	$\rho_n = u_n, \quad \rho_n = u_n(\frac{1}{2}u_n + u_{n+1})$ $\rho_n = \frac{1}{3}u_n^3 + u_n u_{n+1}(u_n + u_{n+1} + u_{n+2})$
Symmetries	$G = u_x, \quad G = 6uu_x + u_{3x}$ $G = 30u^2u_x + 20u_xu_{2x}$ $+10uu_{3x} + u_{5x}$	$G = u_n u_{n+1}(u_n + u_{n+1} + u_{n+2})$ $-u_{n-1}u_n(u_{n-2} + u_{n-1} + u_n)$
Recursion Operator	$\mathcal{R} = D_x^2 + 4u + 2u_x D_x^{-1}$	$\mathcal{R} = u_n(I + D)(u_n D - D^{-1}u_n)$ $(D - I)^{-1} \frac{1}{u_n}$

Table 2: Prototypical Examples

Part II: Discrete Case

Definitions (shift and total difference operators):

D is the **up-shift** (forward or right-shift) operator if for F_n

$$DF_n = F_{n+1} = F_n|_{n \rightarrow n+1}$$

D^{-1} the **down-shift** (backward or left-shift) operator if

$$D^{-1}F_n = F_{n-1} = F_n|_{n \rightarrow n-1}$$

$\Delta = D - I$ is the total **difference operator**

$$\Delta F_n = (D - I)F_n = F_{n+1} - F_n$$

D (up-shift operator) corresponds the differential operator D_x

$$D_x F(x) \rightarrow \frac{F_{n+1} - F_n}{\Delta x} = \frac{\Delta F_n}{\Delta x} \quad (\text{set } \Delta x = 1)$$

For $k > 0$, $D^k = D \circ D \circ \dots \circ D$ (k times).

Similarly, $D^{-k} = D^{-1} \circ D^{-1} \circ \dots \circ D^{-1}$.

Problem to be solved:

Continuous case:

Given f . Find F so that $f = D_x F$ or $F = D_x^{-1} f = \int f dx$.

Discrete case:

Given f_n . Find F_n so that $f_n = \Delta F_n = F_{n+1} - F_n$ or $F_n = \Delta^{-1} f_n$.

Inverting the Δ Operator

- Given f_n involving u_n and v_n and their shifts:

$$f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

- Find F_n so that $f_n = \Delta F_n = F_{n+1} - F_n$ or $F_n = \Delta^{-1} f_n$.

Invert the Δ operator (compute F_n by hand)

f_n	$\longrightarrow$	F_n
$-v_n^2$	$\longrightarrow$	v_n^2
v_{n+1}^2		
$-u_n u_{n+1} v_n$	$\longrightarrow$	$u_n u_{n+1} v_n$
$u_{n+1} u_{n+2} v_{n+1}$		
$-u_{n+1} v_n$	$\longrightarrow$	$u_{n+1} v_n$
$+u_{n+2} v_{n+1}$		
$-u_{n+2} v_{n+1}$	$\longrightarrow$	$u_{n+2} v_{n+1}$
$u_{n+3} v_{n+2}$		

- Result:

$$F_n = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}.$$

Remarks: We denote $f(\mathbf{u}_n, \mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \dots, \mathbf{u}_{n+p})$ as $f(\mathbf{u}_n)$.

Assume that all negative shifts have been removed via up-shifting

Replace $f_n = u_{n-2} v_n v_{n+3}$ by $\tilde{f}_n = D^2 f_n = u_n v_{n+2} v_{n+5}$.

- **‘Total Difference’ Criterion:**

Discrete Euler Operator (variational derivative)

Definition (exactness):

A function $f_n(\mathbf{u}_n)$ is exact, i.e. a total difference, if there exists a function $F_n(\mathbf{u}_n)$, such that $f_n = \Delta F_n$ or equivalently $F_n = \Delta^{-1} f_n$.

D is the up-shift operator.

Theorem (exactness or total difference test):

A necessary and sufficient condition for a function f_n to be exact, i.e. a total difference, is that $\mathcal{L}_{\mathbf{u}_n}^{(0)}(f_n) \equiv 0$, where $\mathcal{L}_{\mathbf{u}_n}^{(0)}$ is the discrete Euler operator (variational derivative) defined by

$$\begin{aligned}
\mathcal{L}_{\mathbf{u}_n}^{(0)} &= \sum_{k=0}^{m_0} D^{-k} \frac{\partial}{\partial \mathbf{u}_{n+k}} \\
&= \frac{\partial}{\partial \mathbf{u}_n} + D^{-1} \left(\frac{\partial}{\partial \mathbf{u}_{n+1}} \right) + D^{-2} \left(\frac{\partial}{\partial \mathbf{u}_{n+2}} \right) + \cdots + D^{-m_0} \left(\frac{\partial}{\partial \mathbf{u}_{n+m_0}} \right) \\
&= \frac{\partial}{\partial \mathbf{u}_n} \left(\sum_{k=0}^{m_0} D^{-k} \right) \\
\mathcal{L}_{\mathbf{u}_n}^{(0)} &= \frac{\partial}{\partial \mathbf{u}_n} (I + D^{-1} + D^{-2} + \cdots + D^{-m_0})
\end{aligned}$$

where m_0 is the highest forward shift (in f_n).

Example: Apply the discrete Euler operator to

$$f_n(\mathbf{u}_n) = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

Here $\mathbf{u}_n = (u_n, v_n)$.

For component u_n (highest shift 3):

$$\begin{aligned} \mathcal{L}_{u_n}^{(0)}(f_n) &= \frac{\partial}{\partial u_n} [I + D^{-1} + D^{-2} + D^{-3}](f_n) \\ &= [-u_{n+1} v_n] + [-u_{n-1} v_{n-1} + u_{n+1} v_n - v_{n-1}] + [u_{n-1} v_{n-1}] + [v_{n-1}] \\ &\equiv 0 \end{aligned}$$

For component v_n (highest shift 2):

$$\begin{aligned} \mathcal{L}_{v_n}^{(0)}(f_n) &= \frac{\partial}{\partial v_n} [I + D^{-1} + D^{-2}](f_n) \\ &= [-u_n u_{n+1} - 2v_n - u_{n+1}] + [u_n u_{n+1} + 2v_n] + [u_{n+1}] \\ &\equiv 0 \end{aligned}$$

- **Computation of F_n**

Definition (higher Euler operators):

The discrete higher Euler operators are defined by

$$\mathcal{L}_{\mathbf{u}_n}^{(i)} = \frac{\partial}{\partial \mathbf{u}_n} \left(\sum_{k=i}^{m_i} \binom{k}{i} D^{-k} \right)$$

These Euler operators all terminate at some maximal shifts m_i .

Examples (for component u_n):

$$\mathcal{L}_{u_n}^{(0)} = \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2} + D^{-3} + \dots + D^{-m_0})$$

$$\mathcal{L}_{u_n}^{(1)} = \frac{\partial}{\partial u_n} (D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \dots + m_1 D^{-m_1})$$

$$\mathcal{L}_{u_n}^{(2)} = \frac{\partial}{\partial u_n} (D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \dots + \frac{1}{2}m_2(m_2 - 1) D^{-m_2})$$

$$\mathcal{L}_{u_n}^{(3)} = \frac{\partial}{\partial u_n} (D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \dots + \binom{m_3}{3} D^{-m_3})$$

Similar formulae for $\mathcal{L}_{v_n}^{(i)}$.

• **Definition** (homotopy operator):

The discrete homotopy operator is defined by

$$\mathcal{H}_{\mathbf{u}_n} = \int_0^1 \sum_{r=1}^N f_{r,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] \frac{d\lambda}{\lambda}$$

where

$$f_{r,n}(\mathbf{u}_n) = \sum_{i=0}^{p_r} (D - I)^i [u_{r,n} \mathcal{L}_{u_{r,n}}^{(i+1)}]$$

p_r is the maximum shift of $u_{r,n}$ in f_n

N is the number of dependent variables

$f_{r,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n]$ means that in $f_{r,n}(\mathbf{u}_n)$ one replaces $\mathbf{u}_n \rightarrow \lambda \mathbf{u}_n$, $\mathbf{u}_{n+1} \rightarrow \lambda \mathbf{u}_{n+1}$, etc.

Example:

For a two-component system ($N = 2$) where $\mathbf{u}_n = (u_n, v_n)$:

$$\mathcal{H}_{\mathbf{u}_n} = \int_0^1 \{f_{1,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] + f_{2,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n]\} \frac{d\lambda}{\lambda}$$

with

$$f_{1,n}(\mathbf{u}_n) = \sum_{i=0}^{p_1} (D - I)^i [u_n \mathcal{L}_{u_n}^{(i+1)}]$$

and

$$f_{2,n}(\mathbf{u}_n) = \sum_{i=0}^{p_2} (D - I)^i [v_n \mathcal{L}_{v_n}^{(i+1)}]$$

Theorem (total difference via homotopy operator):

Given a function f_n which is a total difference, then

$$F_n = \Delta^{-1} f_n = \mathcal{H}_{\mathbf{u}_n}(f_n)$$

Proof: Recent work by Mansfield and Hydon on discrete variational bi-complexes. Proof is given in terms of differential forms.

Proof based on calculus: Deconinck and Hereman.

Higher Euler Operators Side by Side

Continuous Case (for component u)

$$\begin{aligned}
 \mathcal{L}_u^{(0)} &= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{2x}} - D_x^3 \frac{\partial}{\partial u_{3x}} + \dots \\
 \mathcal{L}_u^{(1)} &= \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} - 4D_x^3 \frac{\partial}{\partial u_{4x}} + \dots \\
 \mathcal{L}_u^{(2)} &= \frac{\partial}{\partial u_{2x}} - 3D_x \frac{\partial}{\partial u_{3x}} + 6D_x^2 \frac{\partial}{\partial u_{4x}} - 10D_x^3 \frac{\partial}{\partial u_{5x}} + \dots \\
 \mathcal{L}_u^{(3)} &= \frac{\partial}{\partial u_{3x}} - 4D_x \frac{\partial}{\partial u_{4x}} + 10D_x^2 \frac{\partial}{\partial u_{5x}} - 20D_x^3 \frac{\partial}{\partial u_{6x}} + \dots
 \end{aligned}$$

Discrete Case (for component u_n)

$$\begin{aligned}
 \mathcal{L}_{u_n}^{(0)} &= \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2} + D^{-3} + \dots) \\
 \mathcal{L}_{u_n}^{(1)} &= \frac{\partial}{\partial u_n} (D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \dots) \\
 \mathcal{L}_{u_n}^{(2)} &= \frac{\partial}{\partial u_n} (D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \dots) \\
 \mathcal{L}_{u_n}^{(3)} &= \frac{\partial}{\partial u_n} (D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \dots)
 \end{aligned}$$

Homotopy Operators Side by Side

Continuous Case (for components u and v)

$$\mathcal{H}_{\mathbf{u}} = \int_0^1 \{f_1(\mathbf{u})[\lambda \mathbf{u}] + f_2(\mathbf{u})[\lambda \mathbf{u}]\} \frac{d\lambda}{\lambda}$$

with

$$f_1(\mathbf{u}) = \sum_{i=0}^{p_1} D_x^i [u \mathcal{L}_u^{(i+1)}]$$

and

$$f_2(\mathbf{u}) = \sum_{i=0}^{p_2} D_x^i [v \mathcal{L}_v^{(i+1)}]$$

Discrete Case (for components u_n and v_n)

$$\mathcal{H}_{\mathbf{u}_n} = \int_0^1 \{f_{1,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] + f_{2,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n]\} \frac{d\lambda}{\lambda}$$

with

$$f_{1,n}(\mathbf{u}_n) = \sum_{i=0}^{p_1} (D - I)^i [u_n \mathcal{L}_{u_n}^{(i+1)}]$$

and

$$f_{2,n}(\mathbf{u}_n) = \sum_{i=0}^{p_2} (D - I)^i [v_n \mathcal{L}_{v_n}^{(i+1)}]$$

Example: Apply the discrete homotopy operator to

$$f_n(\mathbf{u}_n) = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

For component u_n (highest shift 3):

i	$\mathcal{L}_{u_n}^{(i+1)}(f_n)$	$(D - I)^i[u_n \mathcal{L}_{u_n}^{(i+1)}(f_n)]$
0	$u_{n-1}v_{n-1} + u_{n+1}v_n + 2v_{n-1}$	$u_{n-1}u_n v_{n-1} + u_n u_{n+1} v_n + 2u_n v_{n-1}$
1	$u_{n-1}v_{n-1} + 3v_{n-1}$	$u_n u_{n+1} v_n + 3u_{n+1} v_n - u_{n-1} u_n v_{n-1} - 3u_n v_{n-1}$
2	v_{n-1}	$u_{n+2} v_{n+1} - u_{n+1} v_n - u_{n+1} v_n + u_n v_{n-1}$

Hence, $f_{1,n}(\mathbf{u}_n)(f_n) = 2u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}$

For component v_n (highest shift 2):

i	$\mathcal{L}_{v_n}^{(i+1)}(f_n)$	$(D - I)^i[v_n \mathcal{L}_{v_n}^{(i+1)}(f_n)]$
0	$u_n u_{n+1} + 2v_n + 2u_{n+1}$	$u_n u_{n+1} v_n + 2v_n^2 + 2u_{n+1} v_n$
1	u_{n+1}	$u_{n+2} v_{n+1} - u_{n+1} v_n$

Hence, $f_{2,n}(\mathbf{u}_n)(f_n) = u_n u_{n+1} v_n + 2v_n^2 + u_{n+1} v_n + u_{n+2} v_{n+1}$

The homotopy operator leads to an integral for variable λ . (Use standard integration by parts to work the integral).

$$\begin{aligned}
F_n(\mathbf{u}_n) &= \mathcal{H}_{\mathbf{u}_n}(f_n) = \int_0^1 \{f_{1,n}(\mathbf{u}_n)(f_n)[\lambda \mathbf{u}_n] + f_{2,n}(\mathbf{u}_n)(f_n)[\lambda \mathbf{u}_n]\} \frac{d\lambda}{\lambda} \\
&= \int_0^1 [2\lambda v_n^2 + 3\lambda^2 u_n u_{n+1} v_n + 2\lambda u_{n+1} v_n + 2\lambda u_{n+2} v_{n+1}] d\lambda \\
&= v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}
\end{aligned}$$

- **Application: Conserved densities and fluxes for DDEs**

Definition (conservation law):

$$D_t \rho_n + \Delta J_n = D_t \rho_n + J_{n+1} - J_n = 0 \quad (\text{on DDE})$$

conserved density ρ_n and flux J_n .

Example The Toda lattice (type $\dot{\mathbf{u}}_n = \mathbf{F}$) :

$$\begin{aligned} \dot{u}_n &= v_{n-1} - v_n \\ \dot{v}_n &= v_n(u_n - u_{n+1}) \end{aligned}$$

has scaling symmetry

$$(t, u_n, v_n) \rightarrow (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n).$$

In terms of weights:

$$w\left(\frac{d}{dt}\right) = 1, \quad w(u_n) = w(u_{n+1}) = 1, \quad w(v_n) = w(v_{n-1}) = 2.$$

Conserved densities and fluxes

$$\rho_n^{(0)} = \ln(v_n)$$

$$J_n^{(0)} = u_n$$

$$\rho_n^{(1)} = u_n$$

$$J_n^{(1)} = v_{n-1}$$

$$\rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n$$

$$J_n^{(2)} = u_n v_{n-1}$$

$$\rho_n^{(3)} = \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n)$$

$$J_n^{(3)} = u_{n-1}u_n v_{n-1} + v_{n-1}^2$$

are all scaling invariant!

- **Algorithm for Conserved Densities and Fluxes**

Example: Density of rank 3 for Toda system

Step 1: Construct the form of the density.

$$\rho_n = c_1 u_n^3 + c_2 u_n v_{n-1} + c_3 u_n v_n$$

where c_i are unknown constants.

Step 2: Determine the constants c_i .

Compute

$$\begin{aligned} E_n = D_t \rho_n &= \frac{\partial \rho_n}{\partial t} + \rho'_n(\mathbf{u}_n)[\mathbf{F}] \quad (\text{on DDE}) \\ &= (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n + (c_3 - c_2)v_{n-1}v_n \\ &\quad + c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2 \end{aligned}$$

Compute $\tilde{E}_n = D E_n$ to remove negative shift $n - 1$.

Since $\tilde{E}_n = -\Delta \tilde{J}_n$, the expression $\tilde{E}_n$ must be a total difference.

Require

$$\begin{aligned} \mathcal{L}_{u_n}^{(0)}(\tilde{E}_n) &= \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2})(\tilde{E}_n) = \frac{\partial}{\partial u_n} (D + I + D^{-1})(E_n) \\ &= 2(3c_1 - c_2)u_n v_{n-1} + 2(c_3 - 3c_1)u_n v_n \\ &\quad + (c_2 - c_3)u_{n-1} v_{n-1} + (c_2 - c_3)u_{n+1} v_n \equiv 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{v_n}^{(0)}(\tilde{E}_n) &= \frac{\partial}{\partial v_n} (I + D^{-1})(\tilde{E}_n) = \frac{\partial}{\partial v_n} (D + I)(E_n) \\ &= (3c_1 - c_2)u_{n+1}^2 + (c_3 - c_2)v_{n+1} + (c_2 - c_3)u_n u_{n+1} \\ &\quad + 2(c_2 - c_3)v_n + (c_3 - 3c_1)u_n^2 + (c_3 - c_2)v_{n-1} \equiv 0. \end{aligned}$$

Solve the linear system

$$\mathcal{S} = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}.$$

Solution: $3c_1 = c_2 = c_3$ Choose $c_1 = \frac{1}{3}$, and $c_2 = c_3 = 1$.

Substitute in ρ_n

$$\rho_n = \frac{1}{3} u_n^3 + u_n(v_{n-1} + v_n)$$

Step 3: Compute the flux J_n .

Start from $-\tilde{E}_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2$

Apply the discrete homotopy operator to $f_n = -\tilde{E}_n$.

For component u_n (highest shift 2):

i	$\mathcal{L}_{u_n}^{(i+1)}(f_n)$	$(D - I)^i(u_n \mathcal{L}_{u_n}^{(i+1)}(-\tilde{E}_n))$
0	$u_{n-1}v_{n-1} + u_{n+1}v_n$	$u_n u_{n-1} v_{n-1} + u_n u_{n+1} v_n$
1	$u_{n-1}v_{n-1}$	$u_{n+1} u_n v_n - u_n u_{n-1} v_{n-1}$

Hence, $f_{1,n}(\mathbf{u}_n)(f_n) = 2u_n u_{n+1} v_n$

For component v_n (highest shift 1):

i	$\mathcal{L}_{v_n}^{(i+1)}(f_n)$	$(D - I)^i(v_n \mathcal{L}_{v_n}^{(i+1)}(-\tilde{E}_n))$
0	$u_n u_{n+1} + 2v_n$	$v_n u_n u_{n+1} + 2v_n^2$

Hence, $f_{2,n}(\mathbf{u}_n)(f_n) = u_n u_{n+1} v_n + 2v_n^2$

$$\begin{aligned}
\tilde{J}_n &= \mathcal{H}_{\mathbf{u}_n}(f_n) = \int_0^1 (f_{1,n}(\mathbf{u}_n)(f_n)[\lambda \mathbf{u}_n] + f_{2,n}(f_n)(\mathbf{u}_n)[\lambda \mathbf{u}_n]) \frac{d\lambda}{\lambda} \\
&= \int_0^1 (3\lambda^2 u_n u_{n+1} v_n + 2\lambda v_n^2) d\lambda \\
&= u_n u_{n+1} v_n + v_n^2.
\end{aligned}$$

Final Result:

$$J_n = D^{-1} \tilde{J}_n = u_{n-1} u_n v_{n-1} + v_{n-1}^2$$

Computer Demos

(1) Use discrete homotopy operator to compute $F_n = \Delta^{-1} f_n$ for

$$f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n$$

(2) Compute density of rank 4 and flux for Toda system:

$$\begin{aligned}\dot{u}_n &= v_{n-1} - v_n \\ \dot{v}_n &= v_n(u_n - u_{n+1})\end{aligned}$$

(3) Compute density of rank 2 for Ablowitz-Ladik system:

$$i \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \kappa u_n^* u_n (u_{n+1} + u_{n-1})$$

(u_n^* is the complex conjugate of u_n).

This is an integrable discretization of the NLS equation:

$$iu_t + u_{xx} + \kappa u^2 u^* = 0$$

Take equation and its complex conjugate.

Treat u_n and $v_n = u_n^*$ as dependent variables. Absorb i in t :

$$\begin{aligned}\dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}) \\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}).\end{aligned}$$

Future Research

- Generalize continuous homotopy operator in multi-dimensions $(x, y, z, \dots)$.
- Problem (in three dimensions):
Given $E = \nabla \cdot \mathbf{J} = J_x^{(1)} + J_y^{(2)} + J_z^{(3)}$.
Find $\mathbf{J} = (J^{(1)}, J^{(2)}, J^{(3)})$.
- Application:
Compute densities and fluxes of multi-dimensional systems of PDEs
(in t, x, y, z).
- Generalize discrete homotopy operator in multi-dimensions $(n, m, \dots)$.

Higher Euler Operators Side by Side

Continuous Case (for component u)

$$\begin{aligned}
 \mathcal{L}_u^{(0)} &= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{2x}} - D_x^3 \frac{\partial}{\partial u_{3x}} + \dots \\
 \mathcal{L}_u^{(1)} &= \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} - 4D_x^3 \frac{\partial}{\partial u_{4x}} + \dots \\
 \mathcal{L}_u^{(2)} &= \frac{\partial}{\partial u_{2x}} - 3D_x \frac{\partial}{\partial u_{3x}} + 6D_x^2 \frac{\partial}{\partial u_{4x}} - 10D_x^3 \frac{\partial}{\partial u_{5x}} + \dots \\
 \mathcal{L}_u^{(3)} &= \frac{\partial}{\partial u_{3x}} - 4D_x \frac{\partial}{\partial u_{4x}} + 10D_x^2 \frac{\partial}{\partial u_{5x}} - 20D_x^3 \frac{\partial}{\partial u_{6x}} + \dots
 \end{aligned}$$

Discrete Case (for component u_n)

$$\begin{aligned}
 \mathcal{L}_{u_n}^{(0)} &= \frac{\partial}{\partial u_n} (I + D^{-1} + D^{-2} + D^{-3} + \dots) \\
 \mathcal{L}_{u_n}^{(1)} &= \frac{\partial}{\partial u_n} (D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \dots) \\
 \mathcal{L}_{u_n}^{(2)} &= \frac{\partial}{\partial u_n} (D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \dots) \\
 \mathcal{L}_{u_n}^{(3)} &= \frac{\partial}{\partial u_n} (D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \dots)
 \end{aligned}$$

Homotopy Operators Side by Side

Continuous Case (for components u and v)

$$\mathcal{H}_{\mathbf{u}} = \int_0^1 \{f_1(\mathbf{u})[\lambda \mathbf{u}] + f_2(\mathbf{u})[\lambda \mathbf{u}]\} \frac{d\lambda}{\lambda}$$

with

$$f_1(\mathbf{u}) = \sum_{i=0}^{p_1} D_x^i [u \mathcal{L}_u^{(i+1)}]$$

and

$$f_2(\mathbf{u}) = \sum_{i=0}^{p_2} D_x^i [v \mathcal{L}_v^{(i+1)}]$$

Discrete Case (for components u_n and v_n)

$$\mathcal{H}_{\mathbf{u}_n} = \int_0^1 \{f_{1,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] + f_{2,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n]\} \frac{d\lambda}{\lambda}$$

with

$$f_{1,n}(\mathbf{u}_n) = \sum_{i=0}^{p_1} (D - I)^i [u_n \mathcal{L}_{u_n}^{(i+1)}]$$

and

$$f_{2,n}(\mathbf{u}_n) = \sum_{i=0}^{p_2} (D - I)^i [v_n \mathcal{L}_{v_n}^{(i+1)}]$$