Symbolic Computation of Travelling Wave Solutions of Nonlinear Partial Differential and Differential-Difference Equations

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Centre de Recherches Mathématiques
Tuesday, May 29, 2007, 15:30.

# Acknowledgements

Collaborators:

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Several undergraduate students.

Research supported in part by NSF
under Grants DMS-9912293 & CCR-9901929

This presentation was made in `TeXpower`

“Make Things as Simple as Possible but No Simpler”

Albert Einstein

Outline

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Part II: Tanh Method for DDEs (Lattices)

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- Conclusions and Future Research
- Research Papers and Software

- Develop and implement various **methods** to find exact solutions of nonlinear PDEs and DDEs: direct methods, Lie symmetry methods, similarity methods, etc.
- Fully automate the hyperbolic and elliptic function methods to compute travelling solutions of nonlinear **PDEs**.
- Fully automate the tanh method to compute travelling wave solutions of nonlinear **DDEs** (lattices).
- Class of nonlinear PDEs and DDEs solvable with such methods includes famous evolution and wave equations, and lattices.

Examples PDEs: Korteweg-de Vries, Boussinesq, and Kuramoto-Sivashinsky equations.
Fisher and FitzHugh-Nagumo equations.

Examples ODEs: Duffing and nonlinear oscillator equations.

Examples DDEs: Volterra, Toda, and Ablowitz-Ladik lattices.

- **PDEs:** Solutions of tanh (kink) or sech (pulse) type **model** solitary waves in fluids, plasmas, circuits, optical fibers, bio-genetics, etc.

  DDEs: discretizations of PDEs, lattice theory, queing and network problems, solid state and quantum physics.

- Benchmark solutions for numerical PDE and DDE solvers.

- **Research aspect**: Design high-quality application packages to compute solitary wave solutions of large classes of nonlinear evolution and wave equations and lattices.

- **Educational aspect**: Software as course ware for courses in nonlinear PDEs and DDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software.

  REU projects of NSF. Extreme Programming!

- **Users** scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.

Typical Examples of ODEs and PDEs

- The Duffing equation

$$u'' + u + \alpha u^3 = 0$$

Solutions in terms of elliptic functions

$$u(x) = \pm \frac{\sqrt{c_1^2 - 1}}{\sqrt{\alpha}} \, \mathrm{cn}(c_1 x + \Delta; \frac{c_1^2 - 1}{2c_1^2}),$$

and

$$u(x) = \pm \frac{\sqrt{2(c_1^2 - 1)}}{\sqrt{\alpha}} \, \mathrm{sn}(c_1 x + \Delta; \frac{1 - c_1^2}{c_1^2}).$$

- The Korteweg-de Vries (KdV) equation

$$u_t + 6\alpha u u_x + u_{3x} = 0.$$

Solitary wave solution

$$u(x,t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2 \left[c_1 x + c_2 t + \Delta\right],$$

or, equivalently,

$$u(x,t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \operatorname{sech}^2 \left[c_1 x + c_2 t + \Delta\right].$$

Cnoidal wave solution

$$u(x,t) = \frac{4c_1^3(1-2m) - c_2}{\alpha c_1} + \frac{12m\, c_1^2}{\alpha} \operatorname{cn}^2(c_1 x + c_2 t + \Delta; m),$$

modulus m.

- The modified Korteweg-de Vries (mKdV) equation

$$u_t + \alpha u^2 u_x + u_{3x} = 0.$$

Solitary wave solution

$$u(x,t) = \pm\sqrt{\frac{6}{\alpha}}\, c_1 \operatorname{sech}\left[c_1 x - c_1^3 t + \Delta\right].$$

- Three-dimensional modified Korteweg-de Vries equation

$$u_t + 6u^2 u_x + u_{xyz} = 0.$$

Solitary wave solution

$$u(x,y,z,t) = \pm\sqrt{c_2 c_3} \operatorname{sech}\left[c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t + \Delta\right]$$

- The Fisher equation

$$u_t - u_{xx} - u\,(1-u) = 0.$$

Solitary wave solution

$$u(x,t) = \frac{1}{4} \pm \frac{1}{2}\tanh\xi + \frac{1}{4}\tanh^2\xi,$$

with

$$\xi = \pm\frac{1}{2\sqrt{6}}\,x \pm \frac{5}{12}\,t + \Delta.$$

- The generalized Kuramoto-Sivashinski equation

$$u_t + u u_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0.$$

Solitary wave solutions

(ignoring symmetry $u \to -u, x \to -x, \sigma \to -\sigma)$:

For $\boxed{\sigma = 4}$

$$u(x,t) = 9 - 2c_2 - 15 \tanh \xi \, (1 + \tanh \xi - \tanh^2 \xi),$$

with $\xi = \frac{x}{2} + c_2 t + \Delta.$

For $\boxed{\sigma = \frac{12}{\sqrt{47}}}$

$$
\begin{aligned}
u(x,t) \quad = \quad & \frac{45 \mp 4418 c_2}{47\sqrt{47}} \pm \frac{45}{47\sqrt{47}} \tanh \xi \\
& - \frac{45}{47\sqrt{47}} \tanh^2 \xi \pm \frac{15}{47\sqrt{47}} \tanh^3 \xi,
\end{aligned}
$$

with $\xi = \pm \frac{1}{2\sqrt{47}}\, x + c_2 t + \Delta.$

For $\boxed{\sigma = 16/\sqrt{73}}$

$$u(x,t) = \frac{2\,(30 \mp 5329c_2)}{73\sqrt{73}} \pm \frac{75}{73\sqrt{73}}\tanh\xi - \frac{60}{73\sqrt{73}}\tanh^2\xi \pm \frac{15}{73\sqrt{73}}\tanh^3\xi,$$

with $\xi = \pm\frac{1}{2\sqrt{73}}\,x + c_2 t + \Delta.$

For $\boxed{\sigma = 0}$

$$u(x,t) = -2\sqrt{\frac{19}{11}}c_2 - \frac{135}{19}\sqrt{\frac{11}{19}}\tanh\xi + \frac{165}{19}\sqrt{\frac{11}{19}}\tanh^3\xi,$$

with $\xi = \frac{1}{2}\sqrt{\frac{11}{19}}\,x + c_2 t + \Delta.$

- The Boussinesq (wave) equation

$$u_{tt} - u_{2x} + 3uu_{2x} + 3{u_x}^2 + \alpha u_{4x} = 0,$$

or written as a first-order system (v auxiliary variable):

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0. \end{aligned}$$

Solitary wave solution

$$\begin{aligned} u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[c_1 x + c_2 t + \Delta\right], \\ v(x,t) &= b_0 + 4\alpha c_1 c_2 \tanh^2\left[c_1 x + c_2 t + \Delta\right]. \end{aligned}$$

- sine-Gordon equation (light cone coordinates)

$$\Phi_{xt} = \sin\Phi.$$

Set $u = \Phi_x, \quad v = \cos(\Phi) - 1,$

$$\begin{aligned} u_{xt} - u - u\,v &= 0, \\ u_t^2 + 2v + v^2 &= 0. \end{aligned}$$

Solitary wave solution (kink)

$$\begin{aligned} u &= \pm\frac{1}{\sqrt{-c}}\mathrm{sech}[\frac{1}{\sqrt{-c}}(x - ct) + \Delta], \\ v &= 1 - 2\mathrm{sech}^2[\frac{1}{\sqrt{-c}}(x - ct) + \Delta]. \end{aligned}$$

Solution:

$$\begin{aligned}\Phi(x,t) &= \int u(x,t)\,dx \\ &= \pm 4 \arctan\left[\exp\left(\frac{1}{\sqrt{-c}}(x-ct)+\Delta\right)\right].\end{aligned}$$

Typical Examples of DDEs (lattices)

- The Volterra lattice

$$\dot{u}_n = u_n(v_n - v_{n-1}),$$
$$\dot{v}_n = v_n(u_{n+1} - u_n).$$

Travelling wave solution:

$$u_n(t) = -c_1 \coth(d_1) + c_1 \tanh\left[d_1 n + c_1 t + \delta\right],$$
$$v_n(t) = -c_1 \coth(d_1) - c_1 \tanh\left[d_1 n + c_1 t + \delta\right].$$

- The Toda lattice

$$\ddot{u}_n = (1 + \dot{u}_n)\,(u_{n-1} - 2u_n + u_{n+1})\,.$$

Travelling wave solution:

$$u_n(t) = a_{10} \pm \sinh(d_1)\tanh\left[d_1 n \pm \sinh(d_1)\,t + \delta\right].$$

- The Ablowitz-Ladik lattice

$$\begin{aligned} \dot{u}_n(t) &= (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n, \\ \dot{v}_n(t) &= -(\alpha + u_n v_n(v_{n+1} + v_{n-1}) + 2\alpha v_n. \end{aligned}$$

Travelling wave solution:

$$u_n(t) = \frac{\alpha \sinh^2(d_1)}{a_{21}} \left(\pm 1 - \tanh\left[d_1 n + 2\alpha t \sinh^2(d_1) + \delta \right] \right),$$

$$v_n(t) = a_{21}(\pm 1 + \tanh\left[d_1 n + 2\alpha \sinh^2(d_1) t + \delta \right]).$$

- 2D Toda lattice

$$\frac{\partial^2 u_n}{\partial x \partial t}(x,t) = \left(\frac{\partial u_n}{\partial t} + 1 \right) (u_{n-1} - 2u_n + u_{n+1})\,.$$

Travelling wave solution:

$$u_n(x,t) =$$
$$a_{10} + \frac{1}{c_2} \sinh^2(d_1) \tanh \left[d_1 n + \frac{\sinh^2(d_1)}{c_2} x + c_2 t + \delta \right].$$

Algorithm for Tanh Solutions of PDEs

System of nonlinear PDEs of order m

$$\Delta(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \cdots \mathbf{u}^{(m)}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable $\mathbf{u}$ has M components u_i (or $u, v, w, ...$).

Independent variable $\mathbf{x}$ has N components x_j (or $x, y, z, ..., t$).

Step T1

- Seek solution $\mathbf{u}(\mathbf{x}) = \mathbf{U}(T)$, with

$$T = \tanh \xi = \tanh \left[\sum_j^N c_j x_j + \delta \right].$$

- Observe $\tanh' \xi = 1 - \tanh^2 \xi$ or $T' = 1 - T^2$.
  Hence, all derivatives of T are polynomial in T.
  For example, $T'' = -2T(1 - T^2)$, etc.

- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{\partial \xi}{\partial x_j} \frac{dT}{d\xi} \frac{d\bullet}{dT} = c_j (1 - T^2) \frac{d\bullet}{dT}$$

Produces a nonlinear system of ODEs

$$\mathbf{\Delta}(T, \mathbf{U}(T), \mathbf{U}'(T), \mathbf{U}''(T), \ldots, \mathbf{U}^{(m)}(T)) = \mathbf{0}.$$

Compare with ultra-spherical (linear) ODE

$$(1 - x^2)y''(x) - (2\alpha + 1)xy'(x) + n(n + 2\alpha)y(x) = 0,$$

with integer $n \geq 0$ and α real.

Includes:

* Legendre equation ($\alpha = \frac{1}{2}$),
* ODE for Chebeyshev polynomials of type I ($\alpha = 0$),
* ODE for Chebeyshev polynomials of type II ($\alpha = 1$).

- Example: For the Boussinesq system

$$
\begin{aligned}
u_t + v_x &= 0, \\
v_t + u_x - 3uu_x - \alpha u_{3x} &= 0,
\end{aligned}
$$

after cancelling common factors $1 - T^2$,

$$
\begin{aligned}
& c_2 U' + c_1 V' = 0, \\
& c_2 V' + c_1 U' - 3c_1 UU' + \alpha c_1^3 \Big[2(1 - 3T^2) U' \\
& \quad + 6T(1 - T^2) U'' - (1 - T^2)^2 U''' \Big] = 0.
\end{aligned}
$$

Step T2

- Seek polynomial solutions

$$U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j.$$

Determine the highest exponents $M_i \geq 1$.

Substitute $U_i(T) = T^{M_i}$ into the LHS of ODE.

Gives polynomial $\mathbf{P}(T)$.

For every P_i consider all possible balances of the highest exponents in T.

Solve the resulting linear system(s) for the unknowns M_i.

- Example: Balance highest exponents for the Boussinesq system

$$M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1.$$

So, $M_1 = M_2 = 2$.

Hence,

$$\begin{aligned} U(T) &= a_{10} + a_{11}T + a_{12}T^2, \\ V(T) &= a_{20} + a_{21}T + a_{22}T^2. \end{aligned}$$

Step T3

- Derive algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the power terms in T.
- Example: Algebraic system for Boussinesq case

$$a_{11}c_1(3a_{12} + 2\alpha c_1^2) = 0,$$

$$a_{12}c_1(a_{12} + 4\alpha c_1^2) = 0,$$

$$a_{21}c_1 + a_{11}c_2 = 0,$$

$$a_{22}c_1 + a_{12}c_2 = 0,$$

$$a_{11}c_1 - 3a_{10}a_{11}c_1 + 2\alpha a_{11}c_1^3 + a_{21}c_2 = 0,$$

$$6a_{10}a_{12}c_1 + 16\alpha a_{12}c_1^3 + 2a_{22}c_2 = 0.$$

Step T4

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for Boussinesq system

$$
\begin{aligned}
a_{10} &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2}, \quad a_{11} = 0, \\
a_{12} &= -4\alpha c_1^2, \quad a_{20} = \text{free}, \\
a_{21} &= 0, \quad a_{22} = 4\alpha c_1 c_2.
\end{aligned}
$$

Step T5

- Return to the original variables.
  Test the final solution(s) of PDE.
  Reject trivial solutions.

- Example: Solitary wave solution for Boussinesq system

$$\begin{aligned} u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[c_1 x + c_2 t + \delta\right], \\ v(x,t) &= a_{20} + 4\alpha c_1 c_2 \tanh^2\left[c_1 x + c_2 t + \delta\right]. \end{aligned}$$

Other Types of Solutions for PDEs

Case	ODE $(y' = \frac{dy}{d\xi})$	Chain Rule
$\tanh(\xi)$	$y' = 1 - y^2$	$\frac{\partial \bullet}{\partial x_j} = c_j (1 - \mathsf{T}^2) \frac{d\bullet}{d\mathsf{T}}$
$\operatorname{sech}(\xi)$	$y' = -y\sqrt{1 - y^2}$	$\frac{\partial \bullet}{\partial x_j} = -c_j \mathsf{S}\sqrt{1 - \mathsf{S}^2} \frac{d\bullet}{d\mathsf{S}}$
$\tan(\xi)$	$y' = 1 + y^2$	$\frac{\partial \bullet}{\partial x_j} = c_j (1 + \mathsf{TAN}^2) \frac{d\bullet}{d\mathsf{TAN}}$
$\exp(\xi)$	$y' = y$	$\frac{\partial \bullet}{\partial x_j} = c_j \mathsf{E} \frac{d\bullet}{d\mathsf{E}}$
$\operatorname{cn}(\xi; m)$	$y' = -\sqrt{(1-y^2)(1-m+m\,y^2)}$	$\frac{\partial \bullet}{\partial x_j} = -c_j \sqrt{(1-\mathsf{CN}^2)(1-m+m\mathsf{CN}^2)} \frac{d\bullet}{d\mathsf{CN}}$
$\operatorname{sn}(\xi; m)$	$y' = \sqrt{(1-y^2)(1-m\,y^2)}$	$\frac{\partial \bullet}{\partial x_j} = c_j \sqrt{(1-\mathsf{SN}^2)(1-m\mathsf{SN}^2)} \frac{d\bullet}{d\mathsf{SN}}$

Algorithm for Jacobi Cn and Sn Solutions of PDEs

Given: System of nonlinear PDEs of order m

$$\Delta(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \cdots \mathbf{u}^{(m)}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable $\mathbf{u}$ has M components u_i (or $u, v, w, ...$).

Independent variable $\mathbf{x}$ has N components x_j (or $x, y, z, ..., t$).

Step CN1

- Seek solution $\mathbf{u}(\mathbf{x}) = \mathbf{U}(CN)$, with

$$CN = \mathrm{cn}(\xi; m) = \mathrm{cn}(\left[\sum_{j}^{N} c_j x_j + \Delta\right]; m).$$

  with modulus m.

- Observe $\mathrm{cn}'(\xi; m) = -\mathrm{sn}(\xi; m)\mathrm{dn}(\xi; m)$.

  Using

$$\mathrm{sn}^2(\xi; m) = 1 - \mathrm{cn}^2(\xi; m),\ \mathrm{dn}^2(\xi; m) = 1 - m + m\,\mathrm{cn}^2(\xi; m),$$

  one has

$$\mathsf{CN}' = -\sqrt{(1 - \mathsf{CN}^2)(1 - m + m\,\mathsf{CN}^2)}.$$

- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d\bullet}{d\mathrm{CN}} \frac{d\mathrm{CN}}{d\xi} \frac{\partial \xi}{\partial x_j} = -c_j \sqrt{(1-\mathrm{CN}^2)(1-m+m\,\mathrm{CN}^2)} \frac{d\bullet}{d\mathrm{CN}},$$

produces a nonlinear ODE:

$$\mathbf{\Delta}(CN, \mathbf{U}(CN), \mathbf{U}'(CN), \mathbf{U}''(CN), \ldots, \mathbf{U}^{(m)}(CN)) = \mathbf{0}.$$

- Example: The KdV equation

$$u_t + \alpha u u_x + u_{xxx} = 0,$$

transforms into

$$\Big(c_1^3(1-2m+6m\,\mathrm{CN}^2) - c_2 - \alpha c_1 U_1\Big) U_1'$$
$$+3c_1^3 \mathrm{CN}(1-2m+2m\,\mathrm{CN}^2) U_1''$$
$$-c_1^3(1-\mathrm{CN}^2)(1-m+m\,\mathrm{CN}^2) U_1''' = 0.$$

Step CN2

- Seek polynomial solutions

$$U_i(CN) = \sum_{j=0}^{M_i} a_{ij} CN^j.$$

  Determine the highest exponents $M_i \geq 1$.

- Example: For KdV case: $M_1 = 2$. Thus,

$$U_1(\mathrm{CN}) = a_{10} + a_{11}\mathrm{CN} + a_{12}\mathrm{CN}^2.$$

Step CN3

- Derive the algebraic system for the coefficients a_{ij}.
- Example: Algebraic system for KdV case

$$
\begin{aligned}
&-3\,a_{11}\,c_1\,(\alpha\,a_{12} - 2\,m\,c_1^2) = 0,\\
&-2\,a_{12}\,c_1\,(\alpha\,a_{12} - 12\,m\,c_1^2) = 0,\\
&-a_{11}\,(\alpha\,a_{10}\,c_1 - c_1^3 + 2\,m\,c_1^3 + c_2) = 0,\\
&-\alpha\,a_{11}^2\,c_1 - a_{12}\,(2\,\alpha\,a_{10}\,c_1 - 16m\,c_1^3 - 8\,c_1^3 + 2\,c_2) = 0.
\end{aligned}
$$

Note: modulus m is extra parameter.

Step CN4

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for KdV system

$$
\begin{aligned}
a_{10} &= \frac{4c_1^3\,(1-2\,m)-c_2}{\alpha\,c_1},\\
a_{11} &= 0,\\
a_{12} &= \frac{12\,m\,c_1^2}{\alpha}.
\end{aligned}
$$

Step CN5

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.

- Example: Cnoidal solution for the KdV equation

$$u(x,t) = \frac{4c_1^3(1-2m) - c_2}{\alpha c_1} + \frac{12m\, c_1^2}{\alpha}\, \mathrm{cn}^2(c_1 x + c_2 t + \Delta; m).$$

NOTE: For Jacobi sn solutions, use

$$\begin{aligned} \mathrm{cn}^2(\xi;m) &= 1-\mathrm{sn}^2(\xi;m), \\ \mathrm{dn}^2(\xi;m) &= 1-m\,\mathrm{sn}^2(\xi;m), \\ \mathrm{sn}'(\xi;m) &= \mathrm{cn}(\xi;m)\,\mathrm{dn}(\xi;m). \end{aligned}$$

Hence,

$$\mathsf{SN}' = \sqrt{(1-\mathsf{SN}^2)(1-m\,\mathsf{SN}^2)},$$

with $\mathsf{SN} = \mathrm{sn}(\xi;m)$.

Chain rule:

$$\frac{\partial \bullet}{\partial x_j} = \frac{d\bullet}{d\mathsf{SN}}\frac{d\mathsf{SN}}{d\xi}\frac{\partial \xi}{\partial x_j} = c_j\sqrt{(1-\mathsf{SN}^2)(1-m\,\mathsf{SN}^2)}\frac{d\bullet}{d\mathsf{SN}}.$$

Algorithm for Tanh Solutions of DDEs

Nonlinear DDEs of order m

$$\mathbf{\Delta}(\mathbf{u}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}), \mathbf{u}_{\mathbf{n}+\mathbf{p}_2}(\mathbf{x}), \cdots, \mathbf{u}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}), \mathbf{u}'_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}), \mathbf{u}'_{\mathbf{n}+\mathbf{p}_2}(\mathbf{x}),$$
$$\cdots, \mathbf{u}'_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x}), \cdots, \mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(\mathbf{x}), \mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_2}(\mathbf{x}), \cdots, \mathbf{u}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable $\mathbf{u_n}$ has M components $u_{i,\mathbf{n}}$ (or $u_\mathbf{n}, v_\mathbf{n}, w_\mathbf{n}, \cdots$)

Independent variable $\mathbf{x}$ has N components x_i (or $t, x, y, \cdots$).

Shift vectors $\mathbf{p}_i \in \mathbb{Z}^Q$.

$\mathbf{u}^{(r)}(\mathbf{x})$ is collection of mixed derivatives of order r.

Simplest case for independent variable (t), and one lattice point (n):

$$\Delta(\ldots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \ldots, \dot{\mathbf{u}}_{n-1}, \dot{\mathbf{u}}_n, \dot{\mathbf{u}}_{n+1}, \ldots, \mathbf{u}_{n-1}^{(r)}, \mathbf{u}_n^{(r)}, \mathbf{u}_{n+1}^{(r)}, \ldots) = \mathbf{0}.$$

Step D1

- Seek solution $\mathbf{u_n}(\mathbf{x}) = \mathbf{U_n}(T_\mathbf{n})$, with $T_\mathbf{n} = \tanh(\xi_\mathbf{n})$,

$$\xi_\mathbf{n} = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \delta = \mathbf{d} \cdot \mathbf{n} + \mathbf{c} \cdot \mathbf{x} + \delta.$$

- Repeatedly apply the operator rule

$$\frac{d\bullet}{dx_j} = \frac{\partial \xi_{\mathbf{n}}}{\partial x_j} \frac{dT_{\mathbf{n}}}{d\xi_{\mathbf{n}}} \frac{d\bullet}{dT_{\mathbf{n}}} = c_j(1 - T_{\mathbf{n}}^2) \frac{d\bullet}{dT_{\mathbf{n}}},$$

transforms DDE into

$$\mathbf{\Delta}(\mathbf{U}_{\mathbf{n}+\mathbf{p}_1}(T_{\mathbf{n}}), \cdots, \mathbf{U}_{\mathbf{n}+\mathbf{p}_k}(T_{\mathbf{n}}), \mathbf{U}'_{\mathbf{n}+\mathbf{p}_1}(T_{\mathbf{n}}), \cdots,$$
$$\mathbf{U}'_{\mathbf{n}+\mathbf{p}_k}(T_{\mathbf{n}}), \cdots \mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_1}(T_{\mathbf{n}}), \cdots, \mathbf{U}^{(r)}_{\mathbf{n}+\mathbf{p}_k}(T_{\mathbf{n}})) = \mathbf{0}.$$

Note: $\mathbf{U}_{\mathbf{n}+\mathbf{p}_s}$ is function of $T_{\mathbf{n}}$ not of $T_{\mathbf{n}+\mathbf{p}_s}$.

- Example: Toda lattice

$$\ddot{u}_n = (1 + \dot{u}_n)\,(u_{n-1} - 2u_n + u_{n+1})$$

transforms into

$$c_2^2(1 - T^2)\left[2TU_n' - (1 - T^2)U_n''\right]$$
$$+\left[1 + c_2(1 - T^2)U_n'\right][U_{n-1} - 2U_n + U_{n+1}] = 0.$$

Step D2

- Seek polynomial solutions

$$U_{i,\mathbf{n}}(T_{\mathbf{n}}) = \sum_{j=0}^{M_i} a_{ij} T_{\mathbf{n}}^{j}.$$

Use

$$\tanh(x+y) = \frac{\tanh\, x + \tanh\, y}{1 + \tanh\, x \tanh\, y}$$

to deal with the shift:

$$T_{\mathbf{n}+\mathbf{p}_s} = \frac{T_{\mathbf{n}} + \tanh\phi_s}{1 + T_{\mathbf{n}} \tanh\phi_s},$$

where

$$\phi_s = \mathbf{p}_s \cdot \mathbf{d} = p_{s1} d_1 + p_{s2} d_2 + \cdots + p_{sQ} d_Q,$$

Substitute $U_{i,\mathbf{n}}(T_{\mathbf{n}}) = T_{\mathbf{n}}^{M_i}$, and

$$U_{i,\mathbf{n}+\mathbf{p}_s}(T_{\mathbf{n}}) = T_{\mathbf{n}+\mathbf{p}_s}^{M_i} = \left[\frac{T_{\mathbf{n}} + \tanh\phi_s}{1 + T_{\mathbf{n}}\tanh\phi_s}\right]^{M_i},$$

and balance the highest exponents in $T_{\mathbf{n}}$ to determine M_i.

Note: $U_{i,\mathbf{n}+\mathbf{0}}(T_{\mathbf{n}}) = T_{\mathbf{n}}^{M_i}$ is of degree M_i in $T_{\mathbf{n}}$.

$U_{i,\mathbf{n}+\mathbf{p}_s}(T_{\mathbf{n}}) = \left[\frac{T_{\mathbf{n}}+\tanh\phi_s}{1+T_{\mathbf{n}}\tanh\phi_s}\right]^{M_i}$ is of degree zero in $T_{\mathbf{n}}$.

- Example: Balance of exponents for Toda lattice

$$2M_1 + 1 = M_1 + 2.$$

So, $M_1 = 1$. Hence,

$$\begin{aligned} U_n(T_n) &= a_{10} + a_{11} T_n, \\ U_{n\pm1}(T(n\pm1)) &= a_{10} + a_{11} T(n\pm1) \\ &= a_{10} + a_{11} \frac{T_n \pm \tanh(d_1)}{1 \pm T_n \tanh(d_1)}. \end{aligned}$$

Step D3

- Determine the algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the powers in T_n.

- Example: Algebraic system for Toda lattice

$$c_1^2 - \tanh^2(d_1) - a_{11}c_1 \tanh^2(d_1) = 0,$$
$$c_1 - a_{11} = 0.$$

Step D4

- Solve the nonlinear algebraic system with parameters.
- Example: Solution of algebraic system for Toda lattice

$$a_{10} = \text{free}, \quad a_{11} = \pm \sinh(d_1), \quad c_1 = \pm \sinh(d_1).$$

Step D5

- Return to the original variables. Test solution(s) of DDE.
  Reject trivial ones.

- Example: Solitary wave solution for Toda lattice

$$u_n(t) = a_{10} \pm \sinh(d_1) \tanh\left[d_1 n \pm \sinh(d_1)\, t + \delta\right].$$

Example: Relativistic Toda Lattice

$$\begin{aligned} \dot{u}_n &= (1+\alpha u_n)(v_n - v_{n-1}), \\ \dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}). \end{aligned}$$

Change of variables

$$u_n(t) = U_n(T_n), \qquad v_n(t) = V_n(T_n),$$

with

$$T_n(t) = \tanh\left[d_1 n + c_1 t + \delta\right].$$

gives

$$c_1(1-T^2)U_n' - (1+\alpha U_n)(V_n - V_{n-1}) = 0,$$
$$c_1(1-T^2)V_n' - V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) = 0.$$

Seek polynomial solutions

$$U_n(T_n) = \sum_{j=0}^{M_1} a_{1j} T_n^j, \qquad V_n(T_n) = \sum_{j=0}^{M_2} a_{2j} T_n^j.$$

Balance the highest exponents in T_n to determine M_1, and M_2 :

$$M_1 + 1 = M_1 + M_2, \quad M_2 + 1 = M_1 + M_2.$$

So, $M_1 = M_2 = 1$. Hence,

$$U_n = a_{10} + a_{11} T_n, \qquad V_n = a_{20} + a_{21} T_n.$$

Algebraic system for a_{ij} :

$$-a_{11}\,c_1 + a_{21}\,\tanh(d_1) + \alpha\,a_{10}\,a_{21}\,\tanh(d_1) = 0,$$

$$a_{11}\,\tanh(d_1)\,(\alpha\,a_{21} + c_1) = 0,$$

$$-a_{21}\,c_1 + a_{11}\,a_{20}\,\tanh(d_1) + 2\alpha\,a_{20}\,a_{21}\,\tanh(d_1) = 0,$$

$$\tanh(d_1)\,(a_{11}\,a_{21} + 2\alpha\,a_{21}^2 - a_{11}\,a_{20}\,\tanh(d_1)) = 0,$$

$$a_{21}\,\tanh^2(d_1)\,(c_1 - a_{11}) = 0.$$

Solution of the algebraic system

$$
\begin{aligned}
a_{10} &= -\frac{1}{\alpha} - c_1 \coth(d_1), \\
a_{11} &= c_1, \\
a_{20} &= \frac{c_1 \ \coth(d_1)}{\alpha}, \\
a_{21} &= -\frac{c_1}{\alpha}.
\end{aligned}
$$

Solitary wave solution in original variables:

$$
\begin{aligned}
u_n(t) &= -\frac{1}{\alpha} - c_1 \ \coth(d_1) + c_1 \tanh\left[d_1 n + c_1 t + \Delta\right], \\
v_n(t) &= \frac{c_1 \ \coth(d_1)}{\alpha} - \frac{c_1}{\alpha} \tanh\left[d_1 n + c_1 t + \Delta\right].
\end{aligned}
$$

Example: 2D Toda Lattice

2D Toda lattice:

$$\frac{\partial^2 y_n}{\partial x \partial t} = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1}),$$

$y_n(x,t)$ is displacement from equilibrium of the n-th unit mass under an exponential decaying interaction force
between nearest neighbors.

Set

$$\frac{\partial u_n}{\partial t} = \exp(y_{n-1} - y_n) - 1. \qquad (*)$$

Then,

$$\exp(y_{n-1} - y_n) = \frac{\partial u_n}{\partial t} + 1,$$

and the 2D-Toda lattice becomes

$$\frac{\partial^2 y_n}{\partial x \partial t} = \frac{\partial u_n}{\partial t} + 1 - \left(\frac{\partial u_{n+1}}{\partial t} + 1 \right) = \frac{\partial u_n}{\partial t} - \frac{\partial u_{n+1}}{\partial t}.$$

Integrate with respect to t to get

$$\frac{\partial y_n}{\partial x} = u_n - u_{n+1}.$$

Differentiate (*) with respect to x and

$$
\begin{aligned}
\frac{\partial^2 u_n}{\partial x \partial t} &= \frac{\partial}{\partial x}\left(\exp(y_{n-1} - y_n) - 1\right) \\
&= \exp(y_{n-1} - y_n)\left(\frac{\partial y_{n-1}}{\partial x} - \frac{\partial y_n}{\partial x}\right), \\
&= \left(\frac{\partial u_n}{\partial t} + 1\right)\left[(u_{n-1} - u_n) - (u_n - u_{n+1})\right], \\
&= \left(\frac{\partial u_n}{\partial t} + 1\right)\left(u_{n-1} - 2u_n + u_{n+1}\right).
\end{aligned}
$$

So, the 2D Toda lattice is written in polynomial form:

$$\frac{\partial^2 u_n}{\partial x \partial t}(x,t) = \left(\frac{\partial u_n}{\partial t} + 1\right)(u_{n-1} - 2u_n + u_{n+1})\,.$$

Travelling wave solution:

$$u_n(x,t) = a_{10} + \frac{1}{c_2}\sinh^2(d_1)\tanh\left[d_1 n + \frac{\sinh^2(d_1)}{c_2}x + c_2 t + \delta\right].$$

Example: Ablowitz-Ladik Lattice

The Ablowitz-Ladik lattice:

$$\begin{aligned} \dot{u}_n(t) &= (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n, \\ \dot{v}_n(t) &= -(\alpha + u_n v_n(v_{n+1} + v_{n-1}) + 2\alpha v_n. \end{aligned}$$

Travelling wave solution:

$$\begin{aligned} u_n(t) &= \frac{\alpha \sinh^2(d_1)}{a_{21}} \left(\pm 1 - \tanh\left[d_1 n + 2\alpha t \sinh^2(d_1) + \delta \right] \right) \\ v_n(t) &= a_{21}(\pm 1 + \tanh\left[d_1 n + 2\alpha \sinh^2(d_1) t + \delta \right]). \end{aligned}$$

Solving Nonlinear Parameterized Systems

- Assumptions
  - All $c_i \neq 0$ and $d_i \neq 0$ (and modulus $m \neq 0$).
  - Parameters $(\alpha, \beta, \gamma, ...)$. Otherwise the maximal exponents M_i may change.
  - All $M_i \geq 1$.
  - All $a_{i\,M_i} \neq 0$. Highest power terms in U_i must be present, except in mixed sech-tanh-method.
  - Solve for a_{ij}, then $c_i, \tanh(d_i)$, and m. Then find conditions on parameters.

- Strategy followed by hand
  - Solve all linear equations in a_{ij} first (branching).
    Start with the ones without parameters.
    Capture constraints in the process.
  - Solve linear equations in $c_i, \tanh(d_i), m$ if they are free of a_{ij}.
  - Solve linear equations in parameters if they free of $a_{ij}, c_i, \tanh(d_i), m$.
  - Solve quasi-linear equations for $a_{ij}, c_i,$ $\tanh(d_i), m$.

- Solve quadratic equations for $a_{ij}, c_i, \tanh(d_i), m$.
- Eliminate cubic terms for $a_{ij}, c_i, \tanh(d_i), m$, without solving.
- Show remaining equations, if any.

- Alternatives
  - Use (adapted) Gröbner bases techniques.
  - Use Ritt-Wu characteristic sets method.
  - Use combinatorics on coefficients $a_{ij} = 0$ or $a_{ij} \neq 0$.

- Other applications (of the nonlinear algebraic solver)

  Computation of conservation laws, symmetries, first integrals, etc. leading to **linear** parameterized systems for unknowns coefficients (see InvariantsSymmetries by Göktaş and Hereman).

Demonstration and Future Work

- Demonstration of Mathematica package for hyperbolic and elliptic function methods for PDEs and tanh function for DDEs.
- Long term goal: Develop PDESolve and DDESolve for analytical solutions of nonlinear PDEs and DDEs.
- Implement various methods: Lie symmetry methods, etc.

- Consider other types of explicit solutions involving
  - other hyperbolic and elliptic functions $\sinh$, $\cosh$, $\mathrm{dn}, \ldots$.
  - complex exponentials combined with sech, $\tanh$.

Papers and Software

Papers

- D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J.C. Miller, Symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDEs, Journal of Symbolic Computation **37** (2004) 669–705.
- D. Baldwin, Ü. Göktaş, and W. Hereman, Symbolic computation of exact tanh solutions of nonlinear differential-difference equations, Computer Physics Communications **162** (2004) 203–217.

Software

- D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J.C. Miller, **PDESpecialSolutions.m**: A Mathematica program for the symbolic computation of exact solutions expressible in hyperbolic and elliptic functions for systems of nonlinear partial differential equations (2001-2006).
  Available on the Internet
  URL: http://www.mines.edu/fs_home/whereman/

- D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J.C. Miller, **DDESpecialSolutions.m**: A Mathematica program for the symbolic computation of tanh solutions for systems of nonlinear differential-difference equations (2001-2006). Available on the Internet URL: http://www.mines.edu/fs_home/whereman/