Continuous and Discrete Homotopy Operators with Applications in Integrability Testing

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Colloquium Talk

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OUTLINE

Part I: Continuous Case

Integration by Parts on the Jet Space (by hand) + Mathematica Experiment

Exactness or Integrability Criterion: Continuous Euler Operator

Continuous Homotopy Operator

Application of Continuous Homotopy Operator

Demo of Mathematica software

Part II: Discrete Case

Inverting the Total Difference Operator (by hand)

Exactness or 'Total Difference' Criterion: Discrete Euler Operator

Discrete Homotopy Operator

Application of Discrete Homotopy Operator

Demo of Mathematica Software

Future Research

Problem Statement

For continuous case:

Given, for example,

$$f = 3\,u'\,v^2\,\sin(u) - u'^3\,\sin(u) - 6\,v\,v'\,\cos(u) + 2\,u'\,u''\,\cos(u) + 8\,v'v''$$

Find F so that $f = D_x F$ or $F = \int f\,dx$.

Result:

$$F = 4\,v'^2 + u'^2\,\cos(u) - 3\,v^2\,\cos(u)$$

Can this be done without integration by parts?

Can the problem be reduced to a single integral in one variable?

For discrete case:

Given, for example,

$$f_n = -u_n\,u_{n+1}\,v_n - v_n^2 + u_{n+1}\,u_{n+2}\,v_{n+1} + v_{n+1}^2 + u_{n+3}\,v_{n+2} - u_{n+1}\,v_n$$

Find F_n so that $f_n = \Delta F_n = F_{n+1} - F_n$ or $F_n = \Delta^{-1} f_n$.

Result:

$$F_n = v_n^2 + u_n\,u_{n+1}\,v_n + u_{n+1}\,v_n + u_{n+2}\,v_{n+1}.$$

How can this be done algorithmically?

Can this be done in the same way as the continuous case?

Part I: Continuous Case

Integration by Parts on the Jet Space

- Given f involving $u(x)$ and $v(x)$ and their derivatives

  $f = 3\,u'\,v^2\,\sin(u) - u'^3\,\sin(u) - 6\,v\,v'\,\cos(u) + 2\,u'\,u''\,\cos(u) + 8\,v'v''$

- Find F so that $f = D_x F$ or $F = \int f\,dx$.

  Integrate by parts (compute F by hand)

$$
\begin{array}{lll}
8\,v'v'' & \longrightarrow & 4\,v'^2 \\
2\,u'\,u''\,\cos(u) & \longrightarrow & u'^2\,\cos(u) \\
-u'^3\,\sin(u) & & \\
-6\,v\,v'\,\cos(u) & \longrightarrow & -3\,v^2\,\cos(u) \\
3\,u'\,v^2\,\sin(u) & &
\end{array}
$$

- Integral:

$$F = 4\,v'^2 + u'^2\,\cos(u) - 3\,v^2\,\cos(u)$$

Remark: For simplicity we denote $f(\mathbf{u}, \mathbf{u}', \mathbf{u}'', \cdots, \mathbf{u}^{(m)})$ as $f(\mathbf{u})$.

- **Exactness Criterion:**

  Continuous Euler Operator (variational derivative)

  Definition (exactness):

  A function $f(\mathbf{u})$ is exact, i.e. can be integrated fully, if there exists a function $F(\mathbf{u})$, such that $f(\mathbf{u}) = \mathrm{D}_x F(\mathbf{u})$ or equivalently $F(\mathbf{u}) = \mathrm{D}_x^{-1} f(\mathbf{u}) = \int_x f(\mathbf{u})\, dx$.

  D_x is the (total) derivative with respect to x.

  Theorem (exactness or integrability test):

  A necessary and sufficient condition for a function f to be exact, i.e. the derivative of another function, is that $\mathcal{L}_{\mathbf{u}}^{(0)}(f) \equiv 0$ where $\mathcal{L}_{\mathbf{u}}^{(0)}$ is the continuous Euler operator (variational derivative) defined by

$$
\begin{aligned}
\mathcal{L}_{\mathbf{u}}^{(0)} &= \sum_{k=0}^{m_0} (-\mathrm{D}_x)^k \frac{\partial}{\partial \mathbf{u}^{(k)}} \\
&= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial \mathbf{u}'} + \mathrm{D}_x^2 \frac{\partial}{\partial \mathbf{u}''} + \cdots + (-1)^{m_0} \mathrm{D}_x^{m_0} \frac{\partial}{\partial \mathbf{u}^{(m_0)}}
\end{aligned}
$$

  where m_0 is the order (of f).

  Proof:

  See calculus of variations (derivation of Euler-Lagrange equations — the forgotten case!).

Example: Apply the continuous Euler operator to

$$f(\mathbf{u}) = 3u'\,v^2\,\sin(u) - u'^3\,\sin(u) - 6v\,v'\,\cos(u) + 2u'\,u''\,\cos(u) + 8v'v''$$

Here $\mathbf{u} = (u, v)$.

For component u (order 2):

$$\begin{aligned}
\mathcal{L}_u^{(0)}(f) &= \frac{\partial}{\partial u}(f) - \mathrm{D}_x \frac{\partial}{\partial u'}(f) + \mathrm{D}_x^2 \frac{\partial}{\partial u''}(f) \\
&= 3u'\,v^2\,\cos(u) - u'^3\,\cos(u) + 6v\,v'\,\sin(u) - 2u'\,u''\,\sin(u) \\
&\quad -\mathrm{D}_x[3v^2\,\sin(u) - 3u'^2\,\sin(u) + 2u''\,\cos(u)] + \mathrm{D}_x^2[2u'\,\cos(u)] \\
&= 3u'\,v^2\,\cos(u) - u'^3\,\cos(u) + 6v\,v'\,\sin(u) - 2u'\,u''\,\sin(u) \\
&\quad -[3u'v^2\,\cos(u) + 6v\,v'\,\sin(u) - 3u'^3\,\cos(u) - 6u\,u''\,\sin(u) \\
&\quad -2u'\,u''\,\sin(u) + 2u'''\,\cos(u)] \\
&\quad +[-2u'''\cos(u) - 6u'\,u''\sin(u) + 2u'''\cos(u)] \\
&\equiv 0
\end{aligned}$$

For component v (order 2):

$$\begin{aligned}
\mathcal{L}_v^{(0)}(f) &= \frac{\partial}{\partial v}(f) - \mathrm{D}_x \frac{\partial}{\partial v'}(f) + \mathrm{D}_x^2 \frac{\partial}{\partial v''}(f) \\
&= 6u'\,v\,\sin(u) - 6v'\,\cos(u) - \mathrm{D}_x[-6v\,\cos(u) + 8v''] + \mathrm{D}_x^2[8v'] \\
&= 6u'v\sin(u) - 6v'\cos(u) - [6u'v\sin(u) - 6v'\cos(u) + 8v'''] + 8v''' \\
&\equiv 0
\end{aligned}$$

- **Computation of the integral** F

  Definition (higher Euler operators):

  The continuous higher Euler operators are defined by

$$\mathcal{L}_{\mathbf{u}}^{(i)} = \sum_{k=i}^{m_i} \binom{k}{i} (-\mathrm{D}_x)^{k-i} \frac{\partial}{\partial \mathbf{u}^{(k)}}$$

  These Euler operators all terminate at some maximal order m_i.

Examples (for component u) :

$$\mathcal{L}_u^{(0)} = \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u'} + \mathrm{D}_x^2 \frac{\partial}{\partial u''} - \mathrm{D}_x^3 \frac{\partial}{\partial u'''} + \cdots + (-1)^{m_0} \mathrm{D}_x^{m_0} \frac{\partial}{\partial u^{(m_0)}}$$

$$\mathcal{L}_u^{(1)} = \frac{\partial}{\partial u'} - 2\mathrm{D}_x \frac{\partial}{\partial u''} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u'''} - 4\mathrm{D}_x^3 \frac{\partial}{\partial u^{(4)}} + \cdots - (-1)^{m_1} m_1 \mathrm{D}_x^{m_1-1} \frac{\partial}{\partial u^{(m_1)}}$$

$$\mathcal{L}_u^{(2)} = \frac{\partial}{\partial u''} - 3\mathrm{D}_x \frac{\partial}{\partial u'''} + 6\mathrm{D}_x^2 \frac{\partial}{\partial u^{(4)}} - 10\mathrm{D}_x^3 \frac{\partial}{\partial u^{(5)}} + \cdots + (-1)^{m_2} \binom{m_2}{2} \mathrm{D}_x^{m_2-2} \frac{\partial}{\partial u^{(m_2)}}$$

$$\mathcal{L}_u^{(3)} = \frac{\partial}{\partial u'''} - 4\mathrm{D}_x \frac{\partial}{\partial u^{(4)}} + 10\mathrm{D}_x^2 \frac{\partial}{\partial u^{(5)}} - 20\mathrm{D}_x^3 \frac{\partial}{\partial u^{(6)}} + \cdots - (-1)^{m_3} \binom{m_3}{3} \mathrm{D}_x^{m_3-3} \frac{\partial}{\partial u^{(m_3)}}$$

Similar formulae for component $\mathcal{L}_v^{(i)}$

Definition (homotopy operator):

The continuous homotopy operator is defined by

$$\mathcal{H}(\mathbf{u}) = \int_0^1 \sum_{r=1}^{N} f_r(\mathbf{u})[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where

$$f_r(\mathbf{u}) = \sum_{i=0}^{p_r} \mathrm{D}_x^i [u_r \boldsymbol{\mathcal{L}}_{u_r}^{(i+1)}]$$

p_r is the maximum order of u_r in f

N is the number of dependent variables

$f_r(\mathbf{u})[\lambda \mathbf{u}]$ means that in $f_r(\mathbf{u})$ one replaces $\mathbf{u} \to \lambda \mathbf{u}$, $\mathbf{u}' \to \lambda \mathbf{u}'$, etc.

Example:

For a two-component system ($N = 2$) where $\mathbf{u} = (u, v)$:

$$\mathcal{H}(\mathbf{u}) = \int_0^1 \{ f_1(\mathbf{u})[\lambda \mathbf{u}] + f_2(\mathbf{u})[\lambda \mathbf{u}] \} \, \frac{d\lambda}{\lambda}$$

with

$$f_1(\mathbf{u}) = \sum_{i=0}^{p_1} \mathrm{D}_x^i [u \boldsymbol{\mathcal{L}}_u^{(i+1)}]$$

and

$$f_2(\mathbf{u}) = \sum_{i=0}^{p_2} \mathrm{D}_x^i [v \boldsymbol{\mathcal{L}}_v^{(i+1)}]$$

Theorem (integration via homotopy operator):

Given an integrable function f

$$F = \mathrm{D}_x^{-1} f = \int f \, dx = \mathcal{H}(\mathbf{u})(f)$$

Proof: Olver's book 'Applications of Lie Groups to Differential Equations', p. 372. Proof is given in terms of differential forms.

Work of Henri Poincaré (1854-1912), George de Rham (1950), and Ian Anderson & Tom Duchamp (1980).

Example: Apply the continuous homotopy operator to integrate

$$f(\mathbf{u}) = 3u'v^2\sin(u) - u'^3\sin(u) - 6vv'\cos(u) + 2u'u''\cos(u) + 8v'v''$$

For component u (order 2):

i	$\mathcal{L}_u^{(i+1)}(f)$	$\mathrm{D}_x^i\left(u\mathcal{L}_u^{(i+1)}(f)\right)$
0	$\frac{\partial}{\partial u'}f - 2\mathrm{D}_x(\frac{\partial}{\partial u''}f)$ $=3v^2\sin u - 3u'^2\sin u + 2u''\cos u$ $-2\mathrm{D}_x(2u'\cos u)$ $=3v^2\sin u + u'^2\sin u - 2u''\cos u$	$3uv^2\sin u + uu'^2\sin u - 2uu''\cos u$
1	$\frac{\partial}{\partial u''}f = 2u'\cos u$	$\mathrm{D}_x[2uu'\cos u]$ $=2u'^2\cos u + 2uu''\cos u - 2uu'^2\sin u$

Hence, $f_1(\mathbf{u})(f) = 3uv^2\sin(u) - uu'^2\sin(u) + 2u'^2\cos(u)$

For component v (order 2):

i	$\mathcal{L}_v^{(i+1)}(f)$	$\mathrm{D}_x^i[v\mathcal{L}_v^{(i+1)}(f)]$
0	$-6v\cos(u) + 8v'' - 2\mathrm{D}_x[8v']$ $= -6v\cos(u) - 8v''$	$-6v^2\cos(u) - 8vv''$
1	$8v'$	$\mathrm{D}_x[8vv'] = 8v'^2 + 8vv''$

Hence, $f_2(\mathbf{u})(f) = -6v^2\cos(u) + 8v'^2$

The homotopy operator leads to an integral for (one) variable λ. (Use standard integration by parts to work the integral).

$$\begin{aligned} F(\mathbf{u}) &= \int_0^1 \{f_1(\mathbf{u})(f)[\lambda\mathbf{u}] + f_2(\mathbf{u})(f)[\lambda\mathbf{u}]\}\,\frac{d\lambda}{\lambda} \\ &= \int_0^1 [3\lambda^2uv^2\sin(\lambda u) - \lambda^2uu'^2\sin(\lambda u) + 2\lambda u'^2\cos(\lambda u) \\ &\qquad -6\lambda v^2\cos(\lambda u) + 8\lambda v'^2]\,d\lambda \\ &= 4v'^2 + u'^2\cos(u) - 3v^2\cos(u) \end{aligned}$$

- **Application: Conserved densities and fluxes for PDEs with transcendental nonlinearities**

Definition (conservation law):

$$\mathrm{D}_t\rho + \mathrm{D}_x J = 0 \qquad \text{(on PDE)}$$

conserved density ρ and flux J.

Example: Sine-Gordon system (type $\mathbf{u}_t = \mathbf{F}$)

$$\begin{aligned} u_t &= v \\ v_t &= u_{xx} + \alpha\sin(u) \end{aligned}$$

has scaling symmetry

$$(t, x, u, v, \alpha) \to (\lambda^{-1}t, \lambda^{-1}x, \lambda^0 u, \lambda v, \lambda^2\alpha)$$

In terms of weights:

$$w(\mathrm{D}_x) = 1,\ w(\mathrm{D}_t) = 1,\ w(u) = 0,\ w(v) = 1,\ w(\alpha) = 2$$

Conserved densities and fluxes

$$\begin{aligned} \rho_{(1)} &= 2\alpha\cos(u) + v^2 + {u_x}^2 \qquad\qquad J_{(1)} = -2u_x v \\ \rho_{(2)} &= u_x v \qquad\qquad J_{(2)} = -[\frac{1}{2}v^2 + \frac{1}{2}{u_x}^2 - \alpha\cos(u)] \\ \rho_{(3)} &= 12\cos(u)vu_x + 2v^3u_x + 2v{u_x}^3 - 16v_xu_{2x} \\ \rho_{(4)} &= 2\cos^2(u) - 2\sin^2(u) + v^4 + 6v^2{u_x}^2 + {u_x}^4 + 4\cos(u)v^2 \\ &\quad +20\cos(u){u_x}^2 - 16{v_x}^2 - 16{u_{2x}}^2. \end{aligned}$$

are all scaling invariant!

Remark: $J_{(3)}$ and $J_{(4)}$ are not shown (too long).

- **Algorithm for Conserved Densities and Fluxes**

Example: Density and flux of rank 2 for sine-Gordon system

Step 1: **Construct the form of the density**

$$\rho = \alpha h_1(u) + h_2(u)v^2 + h_3(u){u_x}^2 + h_4(u)u_x v$$

where $h_i(u)$ are unknown functions.

Step 2: **Determine the functions** h_i

Compute

$$\begin{aligned} E = \mathrm{D}_t\rho &= \frac{\partial \rho}{\partial t} + \rho'(\mathbf{u})[\mathbf{F}] \quad \text{(on PDE)} \\ &= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{m_1} \frac{\partial \rho}{\partial u_{kx}} \mathrm{D}_x^k u_t + \sum_{k=0}^{m_2} \frac{\partial \rho}{\partial v_{kx}} \mathrm{D}_x^k v_t \end{aligned}$$

Since $E = \mathrm{D}_t\rho = -\mathrm{D}_x J$, the expression E must be integrable.

Require that $\mathcal{L}_u^{(0)}(E) \equiv 0$ and $\mathcal{L}_v^{(0)}(E) \equiv 0$.

Solve the system of linear mixed system (algebraic eqs. and ODEs):

$$\begin{aligned}
&h_2(u) - h_3(u) = 0 \\
&{h_2}'(u) = 0 \\
&{h_3}'(u) = 0 \\
&{h_4}'(u) = 0 \\
&{h_2}''(u) = 0 \\
&{h_4}''(u) = 0 \\
&2{h_2}'(u) - {h_3}'(u) = 0 \\
&2{h_2}''(u) - {h_3}''(u) = 0 \\
&{h_1}'(u) + 2\sin(u)h_2(u) = 0 \\
&{h_1}''(u) + 2\sin(u){h_2}'(u) + 2\cos(u)h_2(u) = 0
\end{aligned}$$

Solution:

$$\begin{aligned} h_1(u) &= 2c_1\cos(u) + c_3 \\ h_2(u) &= h_3(u) = c_1 \\ h_4(u) &= c_2 \end{aligned}$$

(with arbitrary constants c_i).

Substitute in ρ

$$\rho = 2c_1\alpha\cos(u) + c_1 v^2 + c_1 {u_x}^2 + c_2 u_x v + \alpha c_3$$

Step 3: Compute the flux J

First, compute

$$\begin{aligned} E = \mathrm{D}_t\rho &= c_1(-2\alpha u_t \sin u + 2vv_t + 2u_x u_{xt}) + c_2(u_{xt}v + u_x v_t) \\ &= c_1(-2\alpha v \sin u + 2v\,(u_{2x} + \alpha\sin u) + 2u_x v_x) \\ &\quad + c_2(v_x v + u_x\,(u_{2x} + \alpha\sin u)) \\ &= c_1(2u_{2x}v + 2u_x v_x) + c_2(vv_x + u_x u_{2x} + \alpha u_x \sin u) \end{aligned}$$

Since $E = \mathrm{D}_t\rho = -\mathrm{D}_x J$, one must integrate.

Apply the homotopy operator for each component of $\mathbf{u} = (u, v)$.

For component u (order 2):

i	$\mathcal{L}_u^{(i+1)}(-E)$	$\mathrm{D}_x^i\left(u\mathcal{L}_u^{(i+1)}(-E)\right)$
0	$2c_1 v_x + c_2(u_{2x} - \alpha\sin u)$	$2c_1 u v_x + c_2(u u_{2x} - \alpha u \sin u)$
1	$-2c_1 v - c_2 u_x$	$-2c_1(u_x v + u v_x) - c_2(u_x^2 + u u_{2x})$

Hence, $f_1(\mathbf{u})(f) = 2c_1 u_x v - c_2(u_x^2 + \alpha u \sin u)$

For component v (order 1):

i	$\mathcal{L}_v^{(i+1)}(-E)$	$\mathrm{D}_x^i\left(v\mathcal{L}_v^{(i+1)}(-E)\right)$
0	$-2c_1u_x - c_2v$	$-2c_1u_xv - c_2v^2$

Hence, $f_2(\mathbf{u})(f) = -2c_1u_xv - c_2v^2$

The homotopy operator leads to an integral for (one) variable λ :

$$\begin{aligned} J(\mathbf{u}) &= \int_0^1 (f_1(\mathbf{u})(f)[\lambda\mathbf{u}] + f_2(\mathbf{u})(f)[\lambda\mathbf{u}]) \ \frac{d\lambda}{\lambda} \\ &= -\int_0^1 \left(4c_1\lambda u_xv + c_2(\lambda u_x^2 + \alpha u\sin(\lambda u) + \lambda v^2)\right) \ d\lambda \\ &= -2c_1u_xv - c_2\left(\frac{1}{2}v^2 + \frac{1}{2}u_x^2 - \alpha\cos u\right) \end{aligned}$$

Split the density and flux in independent pieces (for c_1 and c_2):

$$\begin{aligned} \rho_{(1)} &= 2\alpha\cos u + v^2 + {u_x}^2 \qquad J_{(1)} = -u_xv \\ \rho_{(2)} &= u_xv \qquad J_{(2)} = -\frac{1}{2}v^2 - \frac{1}{2}u_x^2 + \alpha\cos u \end{aligned}$$

Remark: Computation of $J_{(3)}$ and $J_{(4)}$ requires integration with the homotopy operator!

Computer Demos

(1) Use continuous homotopy operator to integrate

$$f = 3\,u'\,v^2\,\sin(u) - u'^3\,\sin(u) - 6\,v\,v'\,\cos(u) + 2\,u'\,u''\,\cos(u) + 8\,v'v''$$

(2) Compute densities of rank 8 and fluxes for 5th-order Korteweg-de Vries equation with three parameters:

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

(α, β, γ are nonzero constant parameters).

(3) Compute density of rank 4 and flux for sine-Gordon system:

$$\begin{aligned} u_t &= v \\ v_t &= u_{xx} + \alpha \sin(u) \end{aligned}$$

Analogy PDEs and DDEs

	Continuous Case (PDEs)	Semi-discrete Case (DDEs)
System	$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$	$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$
Conservation Law	$\mathrm{D}_t \rho + \mathrm{D}_x J = 0$	$\dot{\rho}_n + J_{n+1} - J_n = 0$
Symmetry	$\mathrm{D}_t \mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}]$ $= \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u} + \epsilon \mathbf{G})\|_{\epsilon=0}$	$\mathrm{D}_t \mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}]$ $= \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u}_n + \epsilon \mathbf{G})\|_{\epsilon=0}$
Recursion Operator	$\mathrm{D}_t \mathcal{R} + [\mathcal{R}, \mathbf{F}'(u)] = 0$	$\mathrm{D}_t \mathcal{R} + [\mathcal{R}, \mathbf{F}'(\mathbf{u}_n)] = 0$

Table 1: Conservation Laws and Symmetries

	KdV Equation	Volterra Lattice
Equation	$u_t = 6uu_x + u_{3x}$	$\dot{u}_n = u_n\,(u_{n+1} - u_{n-1})$
Densities	$\rho = u, \quad \rho = u^2$ $\rho = u^3 - \frac{1}{2}u_x^2$	$\rho_n = u_n, \quad \rho_n = u_n(\frac{1}{2}u_n + u_{n+1})$ $\rho_n = \frac{1}{3}u_n^3 + u_n u_{n+1}(u_n + u_{n+1} + u_{n+2})$
Symmetries	$G = u_x, \quad G = 6uu_x + u_{3x}$ $G = 30u^2 u_x + 20u_x u_{2x}$ $+10uu_{3x} + u_{5x}$	$G = u_n u_{n+1}\,(u_n + u_{n+1} + u_{n+2})$ $-u_{n-1}u_n(u_{n-2} + u_{n-1} + u_n)$
Recursion Operator	$\mathcal{R} = \mathrm{D}_x^2 + 4u + 2u_x \mathrm{D}_x^{-1}$	$\mathcal{R} = u_n(\mathrm{I} + \mathrm{D})(u_n \mathrm{D} - \mathrm{D}^{-1} u_n)$ $(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{u_n}$

Table 2: Prototypical Examples

Part II: Discrete Case

Definitions (shift and total difference operators):

D is the **up-shift** (forward or right-shift) operator if for F_n

$$\mathrm{D}F_n = F_{n+1} = F_{n|_{n\to n+1}}$$

D^{-1} the **down-shift** (backward or left-shift) operator if

$$\mathrm{D}^{-1}F_n = F_{n-1} = F_{n|_{n\to n-1}}$$

$\Delta = \mathrm{D} - \mathrm{I}$ is the total **difference operator**

$$\Delta F_n = (\mathrm{D} - \mathrm{I})F_n = F_{n+1} - F_n$$

D (up-shift operator) corresponds the differential operator D_x

$$\mathrm{D}_x F(x) \to \frac{F_{n+1} - F_n}{\Delta x} = \frac{\mathrm{D}F_n}{\Delta x} \quad (\text{set } \Delta x = 1)$$

For $k > 0$, $\mathrm{D}^k = \mathrm{D} \circ \mathrm{D} \circ \cdots \circ \mathrm{D} \quad (k \text{ times})$.

Similarly, $\mathrm{D}^{-k} = \mathrm{D}^{-1} \circ \mathrm{D}^{-1} \circ \cdots \circ \mathrm{D}^{-1}$.

Problem to be solved:

Continuous case:

Given f. Find F so that $f = \mathrm{D}_x F$ or $F = \mathrm{D}_x^{-1} f = \int f\, dx$.

Discrete case:

Given f_n. Find F_n so that $f_n = \Delta F_n = F_{n+1} - F_n$ or $F_n = \Delta^{-1} f_n$.

Inverting the Δ Operator

- Given f_n involving u_n and v_n and their shifts:

  $$f_n = -u_n\,u_{n+1}\,v_n - v_n^2 + u_{n+1}\,u_{n+2}\,v_{n+1} + v_{n+1}^2 + u_{n+3}\,v_{n+2} - u_{n+1}\,v_n$$

- Find F_n so that $f_n = \Delta F_n = F_{n+1} - F_n$ or $F_n = \Delta^{-1} f_n$.

  Invert the Δ operator (compute F_n by hand)

$$\begin{array}{lcl}
-v_n^2 & \longrightarrow & v_n^2 \\
 & & \\
v_{n+1}^2 & & \\
 & & \\
-u_n\,u_{n+1}\,v_n & \longrightarrow & u_n\,u_{n+1}\,v_n \\
 & & \\
u_{n+1}\,u_{n+2}\,v_{n+1} & & \\
 & & \\
-u_{n+1}\,v_n & \longrightarrow & u_{n+1}\,v_n \\
 & & \\
+u_{n+2}\,v_{n+1} & & \\
-u_{n+2}\,v_{n+1} & \longrightarrow & u_{n+2}\,v_{n+1} \\
 & & \\
u_{n+3}\,v_{n+2} & &
\end{array}$$

- Result:

$$F_n = v_n^2 + u_n\,u_{n+1}\,v_n + u_{n+1}\,v_n + u_{n+2}\,v_{n+1}.$$

Remarks: We denote $f(\mathbf{u}_n, \mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+p})$ as $f(\mathbf{u}_n)$.

Assume that all negative shifts have been removed via up-shifting

Replace $f_n = u_{n-2}\,v_n\,v_{n+3}$ by $\tilde{f}_n = \mathrm{D}^2 f_n = u_n\,v_{n+2}\,v_{n+5}$.

- **'Total Difference' Criterion:**

  Discrete Euler Operator (variational derivative)

  Definition (exactness):

  A function $f_n(\mathbf{u}_n)$ is exact, i.e. a total difference, if there exists a function $F_n(\mathbf{u}_n)$, such that $f_n = \Delta\, F_n$ or equivalently $F_n = \Delta^{-1} f_n$.

  D is the up-shift operator.

  Theorem (exactness or total difference test):

  A necessary and sufficient condition for a function f_n to be exact, i.e. a total difference, is that $\mathcal{L}^{(0)}_{\mathbf{u}_n}(f_n) \equiv 0$, where $\mathcal{L}^{(0)}_{\mathbf{u}_n}$ is the discrete Euler operator (variational derivative) defined by

$$
\begin{aligned}
\mathcal{L}^{(0)}_{\mathbf{u}_n} &= \sum_{k=0}^{m_0} \mathrm{D}^{-k} \frac{\partial}{\partial \mathbf{u}_{n+k}} \\
&= \frac{\partial}{\partial \mathbf{u}_n} + \mathrm{D}^{-1}\left(\frac{\partial}{\partial \mathbf{u}_{n+1}}\right) + \mathrm{D}^{-2}\left(\frac{\partial}{\partial \mathbf{u}_{n+2}}\right) + \cdots + \mathrm{D}^{-m_0}\left(\frac{\partial}{\partial \mathbf{u}_{n+m_0}}\right) \\
&= \frac{\partial}{\partial \mathbf{u}_n}\left(\sum_{k=0}^{m_0} \mathrm{D}^{-k}\right) \\
\mathcal{L}^{(0)}_{\mathbf{u}_n} &= \frac{\partial}{\partial \mathbf{u}_n}(\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \cdots + \mathrm{D}^{-m_0})
\end{aligned}
$$

  where m_0 is the highest forward shift (in f_n).

Example: Apply the discrete Euler operator to

$$f_n(\mathbf{u}_n) = -u_n\, u_{n+1}\, v_n - v_n^2 + u_{n+1}\, u_{n+2}\, v_{n+1} + v_{n+1}^2 + u_{n+3}\, v_{n+2} - u_{n+1}\, v_n$$

Here $\mathbf{u}_n = (u_n, v_n)$.

For component u_n (highest shift 3):

$$\begin{aligned}\mathcal{L}_{u_n}^{(0)}(f_n) &= \frac{\partial}{\partial u_n}[\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \mathrm{D}^{-3}](f_n) \\ &= [-u_{n+1}v_n] + [-u_{n-1}v_{n-1} + u_{n+1}v_n - v_{n-1}] + [u_{n-1}v_{n-1}] + [v_{n-1}] \\ &\equiv 0\end{aligned}$$

For component v_n (highest shift 2):

$$\begin{aligned}\mathcal{L}_{v_n}^{(0)}(f_n) &= \frac{\partial}{\partial v_n}[\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2}](f_n) \\ &= [-u_n u_{n+1} - 2v_n - u_{n+1}] + [u_n u_{n+1} + 2v_n] + [u_{n+1}] \\ &\equiv 0\end{aligned}$$

- **Computation of F_n**

  Definition (higher Euler operators):

  The discrete higher Euler operators are defined by

$$\mathcal{L}^{(i)}_{\mathbf{u}_n} = \frac{\partial}{\partial \mathbf{u}_n} \Big(\sum_{k=i}^{m_i} \binom{k}{i} \mathrm{D}^{-k} \Big)$$

  These Euler operators all terminate at some maximal shifts m_i.

  Examples (for component u_n):

$$\mathcal{L}^{(0)}_{u_n} = \frac{\partial}{\partial u_n} (\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \mathrm{D}^{-3} + \cdots + \mathrm{D}^{-m_0})$$

$$\mathcal{L}^{(1)}_{u_n} = \frac{\partial}{\partial u_n} (\mathrm{D}^{-1} + 2\mathrm{D}^{-2} + 3\mathrm{D}^{-3} + 4\mathrm{D}^{-4} + \cdots + m_1 \, \mathrm{D}^{-m_1})$$

$$\mathcal{L}^{(2)}_{u_n} = \frac{\partial}{\partial u_n} (\mathrm{D}^{-2} + 3\mathrm{D}^{-3} + 6\mathrm{D}^{-4} + 10\mathrm{D}^{-5} + \cdots + \frac{1}{2} m_2 (m_2 - 1) \, \mathrm{D}^{-m_2})$$

$$\mathcal{L}^{(3)}_{u_n} = \frac{\partial}{\partial u_n} (\mathrm{D}^{-3} + 4\mathrm{D}^{-4} + 10\mathrm{D}^{-5} + 20\mathrm{D}^{-6} + \cdots + \binom{m_3}{3} \mathrm{D}^{-m_3})$$

  Similar formulae for $\mathcal{L}^{(i)}_{v_n}$.

- **Definition** (homotopy operator):

  The discrete homotopy operator is defined by

$$\mathcal{H}(\mathbf{u}_n) = \int_0^1 \sum_{r=1}^{N} f_{r,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] \frac{d\lambda}{\lambda}$$

where

$$f_{r,n}(\mathbf{u}_n) = \sum_{i=0}^{p_r} (\mathrm{D} - \mathrm{I})^i [u_{r,n} \mathcal{L}_{u_{r,n}}^{(i+1)}]$$

p_r is the maximum shift of $u_{r,n}$ in f_n
N is the number of dependent variables
$f_{r,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n]$ means that in $f_{r,n}(\mathbf{u}_n)$ one replaces $\mathbf{u}_n \to \lambda \mathbf{u}_n$, $\mathbf{u}_{n+1} \to \lambda \mathbf{u}_{n+1}$, etc.

Example:
For a two-component system ($N = 2$) where $\mathbf{u}_n = (u_n, v_n)$:

$$\mathcal{H}(\mathbf{u}_n) = \int_0^1 \{ f_{1,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] + f_{2,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] \} \frac{d\lambda}{\lambda}$$

with

$$f_{1,n}(\mathbf{u}_n) = \sum_{i=0}^{p_1} (\mathrm{D} - \mathrm{I})^i [u_n \mathcal{L}_{u_n}^{(i+1)}]$$

and

$$f_{2,n}(\mathbf{u}_n) = \sum_{i=0}^{p_2} (\mathrm{D} - \mathrm{I})^i [v_n \mathcal{L}_{v_n}^{(i+1)}]$$

Theorem (total difference via homotopy operator):

Given a function f_n which is a total difference, then

$$F_n = \Delta^{-1} f_n = \mathcal{H}(\mathbf{u}_n)(f_n)$$

Proof: Recent work by Mansfield and Hydon on discrete variational bi-complexes. Proof is given in terms of differential forms.

Higher Euler Operators Side by Side

Continuous Case (for component u)

$$\mathcal{L}_u^{(0)} = \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{2x}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{3x}} + \cdots$$
$$\mathcal{L}_u^{(1)} = \frac{\partial}{\partial u_x} - 2\mathrm{D}_x \frac{\partial}{\partial u_{2x}} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x^3 \frac{\partial}{\partial u_{4x}} + \cdots$$
$$\mathcal{L}_u^{(2)} = \frac{\partial}{\partial u_{2x}} - 3\mathrm{D}_x \frac{\partial}{\partial u_{3x}} + 6\mathrm{D}_x^2 \frac{\partial}{\partial u_{4x}} - 10\mathrm{D}_x^3 \frac{\partial}{\partial u_{5x}} + \cdots$$
$$\mathcal{L}_u^{(3)} = \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x \frac{\partial}{\partial u_{4x}} + 10\mathrm{D}_x^2 \frac{\partial}{\partial u_{5x}} - 20\mathrm{D}_x^3 \frac{\partial}{\partial u_{6x}} + \cdots$$

Discrete Case (for component u_n)

$$\mathcal{L}_{u_n}^{(0)} = \frac{\partial}{\partial u_n}(\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \mathrm{D}^{-3} + \cdots)$$
$$\mathcal{L}_{u_n}^{(1)} = \frac{\partial}{\partial u_n}(\mathrm{D}^{-1} + 2\mathrm{D}^{-2} + 3\mathrm{D}^{-3} + 4\mathrm{D}^{-4} + \cdots)$$
$$\mathcal{L}_{u_n}^{(2)} = \frac{\partial}{\partial u_n}(\mathrm{D}^{-2} + 3\mathrm{D}^{-3} + 6\mathrm{D}^{-4} + 10\mathrm{D}^{-5} + \cdots)$$
$$\mathcal{L}_{u_n}^{(3)} = \frac{\partial}{\partial u_n}(\mathrm{D}^{-3} + 4\mathrm{D}^{-4} + 10\mathrm{D}^{-5} + 20\mathrm{D}^{-6} + \cdots)$$

Homotopy Operators Side by Side

Continuous Case (for components u and v)

$$\mathcal{H}(\mathbf{u}) = \int_0^1 \{ f_1(\mathbf{u})[\lambda \mathbf{u}] + f_2(\mathbf{u})[\lambda \mathbf{u}] \} \, \frac{d\lambda}{\lambda}$$

with

$$f_1(\mathbf{u}) = \sum_{i=0}^{p_1} \mathrm{D}_x^i [u \mathcal{L}_u^{(i+1)}]$$

and

$$f_2(\mathbf{u}) = \sum_{i=0}^{p_2} \mathrm{D}_x^i [v \mathcal{L}_v^{(i+1)}]$$

Discrete Case (for components u_n and v_n)

$$\mathcal{H}(\mathbf{u}_n) = \int_0^1 \{ f_{1,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] + f_{2,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] \} \frac{d\lambda}{\lambda}$$

with

$$f_{1,n}(\mathbf{u}_n) = \sum_{i=0}^{p_1} (\mathrm{D} - \mathrm{I})^i [u_n \mathcal{L}_{u_n}^{(i+1)}]$$

and

$$f_{2,n}(\mathbf{u}_n) = \sum_{i=0}^{p_2} (\mathrm{D} - \mathrm{I})^i [v_n \mathcal{L}_{v_n}^{(i+1)}]$$

Example: Apply the discrete homotopy operator to

$$f_n(\mathbf{u}_n) = -u_n\, u_{n+1}\, v_n - v_n^2 + u_{n+1}\, u_{n+2}\, v_{n+1} + v_{n+1}^2 + u_{n+3}\, v_{n+2} - u_{n+1}\, v_n$$

For component u_n (highest shift 3):

i	$\mathcal{L}_{u_n}^{(i+1)}(f_n)$	$(\mathrm{D}-\mathrm{I})^i[u_n \mathcal{L}_{u_n}^{(i+1)}(f_n)]$
0	$u_{n-1}v_{n-1}+u_{n+1}v_n+2v_{n-1}$	$u_{n-1}u_nv_{n-1}+u_nu_{n+1}v_n+2u_nv_{n-1}$
1	$u_{n-1}v_{n-1}+3v_{n-1}$	$u_nu_{n+1}v_n+3u_{n+1}v_n-u_{n-1}u_nv_{n-1}-3u_nv_{n-1}$
2	v_{n-1}	$u_{n+2}v_{n+1}-u_{n+1}v_n-u_{n+1}v_n+u_nv_{n-1}$

Hence, $f_{1,n}(\mathbf{u}_n)(f_n) = 2u_nu_{n+1}v_n + u_{n+1}v_n + u_{n+2}v_{n+1}$

For component v_n (highest shift 2):

i	$\mathcal{L}_{v_n}^{(i+1)}(f_n)$	$(\mathrm{D}-\mathrm{I})^i[v_n \mathcal{L}_{v_n}^{(i+1)}(f_n)]$
0	$u_nu_{n+1}+2v_n+2u_{n+1}$	$u_nu_{n+1}v_n+2v_n^2+2u_{n+1}v_n$
1	u_{n+1}	$u_{n+2}v_{n+1}-u_{n+1}v_n$

Hence, $f_{2,n}(\mathbf{u}_n)(f_n) = u_nu_{n+1}v_n + 2v_n^2 + u_{n+1}v_n + u_{n+2}v_{n+1}$

The homotopy operator leads to an integral for (one) variable λ. (Use standard integration by parts to work the integral).

$$\begin{aligned} F_n(\mathbf{u}_n) &= \int_0^1 \{f_{1,n}(\mathbf{u}_n)(f_n)[\lambda \mathbf{u}_n] + f_{2,n}(\mathbf{u}_n)(f_n)[\lambda \mathbf{u}_n]\} \, \frac{d\lambda}{\lambda} \\ &= \int_0^1 [2\lambda v_n^2 + 3\lambda^2 u_nu_{n+1}v_n + 2\lambda u_{n+1}v_n + 2\lambda u_{n+2}v_{n+1}] \, d\lambda \\ &= v_n^2 + u_nu_{n+1}v_n + u_{n+1}v_n + u_{n+2}v_{n+1} \end{aligned}$$

- **Application: Conserved densities and fluxes for DDEs**

Definition (conservation law):

$$\mathrm{D}_t \rho_n + \Delta J_n = \mathrm{D}_t \rho_n + J_{n+1} - J_n = 0 \qquad \text{(on DDE)}$$

conserved density ρ_n and flux J_n.

Example The Toda lattice (type $\dot{\mathbf{u}}_n = \mathbf{F}$) :

$$\begin{aligned} \dot{u}_n &= v_{n-1} - v_n \\ \dot{v}_n &= v_n(u_n - u_{n+1}) \end{aligned}$$

has scaling symmetry

$$(t, u_n, v_n) \rightarrow (\lambda^{-1} t, \lambda u_n, \lambda^2 v_n).$$

In terms of weights:

$w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1,\ w(u_n) = w(u_{n+1}) = 1,\ w(v_n) = w(v_{n-1}) = 2.$

Conserved densities and fluxes

$$\rho_n^{(0)} = \ln(v_n) \qquad J_n^{(0)} = u_n$$

$$\rho_n^{(1)} = u_n \qquad J_n^{(1)} = v_{n-1}$$

$$\rho_n^{(2)} = \tfrac{1}{2} u_n^2 + v_n \qquad J_n^{(2)} = u_n v_{n-1}$$

$$\rho_n^{(3)} = \tfrac{1}{3} u_n^3 + u_n(v_{n-1} + v_n) \qquad J_n^{(3)} = u_{n-1} u_n v_{n-1} + v_{n-1}^2$$

are all scaling invariant!

- **Algorithm for Conserved Densities and Fluxes**

Example: Density of rank 3 for Toda system

Step 1: **Construct the form of the density.**

$$\rho_n = c_1\, u_n^3 + c_2\, u_n v_{n-1} + c_3\, u_n v_n$$

where c_i are unknown constants.

Step 2: **Determine the constants c_i.**

Compute

$$\begin{aligned} E_n = \mathrm{D}_t \rho_n &= \frac{\partial \rho_n}{\partial t} + \rho_n'(\mathbf{u}_n)[\mathbf{F}] \quad \text{(on DDE)} \\ &= (3c_1 - c_2) u_n^2 v_{n-1} + (c_3 - 3c_1) u_n^2 v_n + (c_3 - c_2) v_{n-1} v_n \\ &\quad + c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2 \end{aligned}$$

Compute $\tilde{E}_n = \mathrm{D} E_n$ to remove negative shift $n-1$.

Since $\tilde{E}_n = -\Delta \tilde{J}_n$, the expression $\tilde{E}_n$ must be a total difference.

Require

$$\begin{aligned} \mathcal{L}^{(0)}_{u_n}(\tilde{E}_n) &= \frac{\partial}{\partial u_n}(\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2})(\tilde{E}_n) = \frac{\partial}{\partial u_n}(\mathrm{D} + \mathrm{I} + \mathrm{D}^{-1})(E_n) \\ &= 2(3c_1 - c_2) u_n v_{n-1} + 2(c_3 - 3c_1) u_n v_n \\ &\quad + (c_2 - c_3) u_{n-1} v_{n-1} + (c_2 - c_3) u_{n+1} v_n \equiv 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{(0)}_{v_n}(\tilde{E}_n) &= \frac{\partial}{\partial v_n}(\mathrm{I} + \mathrm{D}^{-1})(\tilde{E}_n) = \frac{\partial}{\partial v_n}(\mathrm{D} + \mathrm{I})(E_n) \\ &= (3c_1 - c_2) u_{n+1}^2 + (c_3 - c_2) v_{n+1} + (c_2 - c_3) u_n u_{n+1} \\ &\quad + 2(c_2 - c_3) v_n + (c_3 - 3c_1) u_n^2 + (c_3 - c_2) v_{n-1} \equiv 0. \end{aligned}$$

Solve the linear system

$$\mathcal{S} = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}.$$

Solution: $3c_1 = c_2 = c_3$ Choose $c_1 = \frac{1}{3}$, and $c_2 = c_3 = 1$.

Substitute in ρ_n

$$\rho_n = \frac{1}{3}\, u_n^3 + u_n(v_{n-1} + v_n)$$

Step 3: **Compute the flux J_n.**

Start from $-\tilde{E}_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2$

Apply the discrete homotopy operator!

For component u_n (highest shift 2):

i	$\mathcal{L}_{u_n}^{(i+1)}(-\tilde{E}_n)$	$(\mathrm{D}-\mathrm{I})^i(u_n \mathcal{L}_{u_n}^{(i+1)}(-\tilde{E}_n))$
0	$u_{n-1}v_{n-1}+u_{n+1}v_n$	$u_n u_{n-1} v_{n-1} + u_n u_{n+1} v_n$
1	$u_{n-1}v_{n-1}$	$u_{n+1}u_n v_n - u_n u_{n-1} v_{n-1}$

Hence, $\tilde{j}_{1,n}(\mathbf{u}_n) = 2u_n u_{n+1} v_n$

For component v_n (highest shift 1):

i	$\mathcal{L}_{v_n}^{(i+1)}(-\tilde{E}_n)$	$(\mathrm{D}-\mathrm{I})^i(v_n \mathcal{L}_{v_n}^{(i+1)}(-\tilde{E}_n))$
0	$u_n u_{n+1} + 2v_n$	$v_n u_n u_{n+1} + 2v_n^2$

Hence, $\tilde{j}_{2,n}(\mathbf{u}_n) = u_n u_{n+1} v_n + 2v_n^2$

$$\begin{aligned}
\tilde{J}_n &= \int_0^1 (\tilde{j}_{1,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] + \tilde{j}_{2,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n])\, \frac{d\lambda}{\lambda} \\
&= \int_0^1 (3\lambda^2 u_n u_{n+1} v_n + 2\lambda v_n^2)\, d\lambda \\
&= u_n u_{n+1} v_n + v_n^2.
\end{aligned}$$

Final Result:

$$J_n = \mathrm{D}^{-1} \tilde{J}_n = u_{n-1} u_n v_{n-1} + v_{n-1}^2$$

Computer Demos

(1) Use discrete homotopy operator to compute $F_n = \Delta^{-1} f_n$ for

$$f_n = -u_n\, u_{n+1}\, v_n - v_n^2 + u_{n+1}\, u_{n+2}\, v_{n+1} + v_{n+1}^2 + u_{n+3}\, v_{n+2} - u_{n+1}\, v_n$$

(2) Compute density of rank 4 and flux for Toda system:

$$\begin{aligned} \dot{u}_n &= v_{n-1} - v_n \\ \dot{v}_n &= v_n(u_n - u_{n+1}) \end{aligned}$$

(3) Compute density of rank 2 for Ablowitz-Ladik system:

$$i\, \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \kappa u_n^* u_n (u_{n+1} + u_{n-1})$$

(u_n^* is the complex conjugate of u_n).

This is an integrable discretization of the NLS equation:

$$i u_t + u_{xx} + \kappa u^2 u^* = 0$$

Take equation and its complex conjugate.
Treat u_n and $v_n = u_n^*$ as dependent variables. Absorb i in t:

$$\begin{aligned} \dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}) \\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

Future Research

- Generalize continuous homotopy operator in multi-dimensions $(x, y, z, ...)$.

- Problem (in three dimensions):

  Given $E = \nabla\cdot\mathbf{J} = J_x^{(1)} + J_y^{(2)} + J_z^{(3)}$.

  Find $\mathbf{J} = (J^{(1)},\ J^{(2)},\ J^{(3)})$.

- Application:

  Compute densities and fluxes of multi-dimensional systems of PDEs (in t, x, y, z).

- Generalize discrete homotopy operator in multi-dimensions $(n, m, ...)$.

Higher Euler Operators Side by Side

Continuous Case (for component u)

$$\mathcal{L}_u^{(0)} = \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{2x}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{3x}} + \cdots$$
$$\mathcal{L}_u^{(1)} = \frac{\partial}{\partial u_x} - 2\mathrm{D}_x \frac{\partial}{\partial u_{2x}} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x^3 \frac{\partial}{\partial u_{4x}} + \cdots$$
$$\mathcal{L}_u^{(2)} = \frac{\partial}{\partial u_{2x}} - 3\mathrm{D}_x \frac{\partial}{\partial u_{3x}} + 6\mathrm{D}_x^2 \frac{\partial}{\partial u_{4x}} - 10\mathrm{D}_x^3 \frac{\partial}{\partial u_{5x}} + \cdots$$
$$\mathcal{L}_u^{(3)} = \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x \frac{\partial}{\partial u_{4x}} + 10\mathrm{D}_x^2 \frac{\partial}{\partial u_{5x}} - 20\mathrm{D}_x^3 \frac{\partial}{\partial u_{6x}} + \cdots$$

Discrete Case (for component u_n)

$$\mathcal{L}_{u_n}^{(0)} = \frac{\partial}{\partial u_n}(\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \mathrm{D}^{-3} + \cdots)$$
$$\mathcal{L}_{u_n}^{(1)} = \frac{\partial}{\partial u_n}(\mathrm{D}^{-1} + 2\mathrm{D}^{-2} + 3\mathrm{D}^{-3} + 4\mathrm{D}^{-4} + \cdots)$$
$$\mathcal{L}_{u_n}^{(2)} = \frac{\partial}{\partial u_n}(\mathrm{D}^{-2} + 3\mathrm{D}^{-3} + 6\mathrm{D}^{-4} + 10\mathrm{D}^{-5} + \cdots)$$
$$\mathcal{L}_{u_n}^{(3)} = \frac{\partial}{\partial u_n}(\mathrm{D}^{-3} + 4\mathrm{D}^{-4} + 10\mathrm{D}^{-5} + 20\mathrm{D}^{-6} + \cdots)$$

Homotopy Operators Side by Side

Continuous Case (for components u and v)

$$\mathcal{H}(\mathbf{u}) = \int_0^1 \{f_1(\mathbf{u})[\lambda\mathbf{u}] + f_2(\mathbf{u})[\lambda\mathbf{u}]\} \, \frac{d\lambda}{\lambda}$$

with

$$f_1(\mathbf{u}) = \sum_{i=0}^{p_1} \mathrm{D}_x^i [u \boldsymbol{\mathcal{L}}_u^{(i+1)}]$$

and

$$f_2(\mathbf{u}) = \sum_{i=0}^{p_2} \mathrm{D}_x^i [v \boldsymbol{\mathcal{L}}_v^{(i+1)}]$$

Discrete Case (for components u_n and v_n)

$$\mathcal{H}(\mathbf{u}_n) = \int_0^1 \{f_{1,n}(\mathbf{u}_n)[\lambda\mathbf{u}_n] + f_{2,n}(\mathbf{u}_n)[\lambda\mathbf{u}_n]\} \frac{d\lambda}{\lambda}$$

with

$$f_{1,n}(\mathbf{u}_n) = \sum_{i=0}^{p_1} (\mathrm{D} - \mathrm{I})^i [u_n \boldsymbol{\mathcal{L}}_{u_n}^{(i+1)}]$$

and

$$f_{2,n}(\mathbf{u}_n) = \sum_{i=0}^{p_2} (\mathrm{D} - \mathrm{I})^i [v_n \boldsymbol{\mathcal{L}}_{v_n}^{(i+1)}]$$