SOLITARY WAVE SOLUTIONS
OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS
USING A DIRECT METHOD AND MACSYMA

1. INTRODUCTION

A. Framework and Motivation

- Solitary Wave - Soliton

  Famous example: the Korteweg-de Vries equation (KdV)

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

$$u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)+\delta\right]$$

  or with $c = 4k^2$

$$u(x,t) = 2k^2\,\mathrm{sech}^2\left[k(x-4k^2t)+\delta\right]$$

- Appearance:

  - shallow water waves (channels and beaches)
  - ion-acoustic waves in plasma's
  - continuum limit of non-linear lattice (Toda lattice)
  - non-linear transmission lines (electrical circuit)

- Observations:

  - critical balance between nonlinearity and dispersion
  - no change in shape (solitary wave) or speed
  - speed $\propto$ amplitude
  - width $\propto \frac{1}{\sqrt{\text{amplitude}}}$
  - taller waves travel faster and are narrower
  - soliton behavior upon interaction

B. Examples

- Korteweg-de Vries equation and generalizations

$$u_t + au^n u_x + u_{xxx} = 0, \quad n \in \mathbb{N}$$

$$u(x,t) = \left\{ \frac{c(n+1)(n+2)}{2a} \operatorname{sech}^2 \left[\frac{n}{2}\sqrt{c}(x-ct) + \delta \right] \right\}^{\frac{1}{n}}$$

- Burgers equation

$$u_t + auu_x - u_{xx} = 0$$

$$u(x,t) = \frac{c}{a} \left\{ 1 - \tanh \left[\frac{c}{2}(x-ct) + \delta \right] \right\}$$

- Fisher equation and generalizations

$$u_t - u_{xx} - u(1-u^n) = 0, \quad n \in \mathbb{N}$$

$$u(x,t) = \left\{ \frac{1}{2} \left[1 - \tanh \left[\frac{n}{2\sqrt{2n+4}}(x - \frac{(n+4)}{\sqrt{2n+4}}t) + \delta \right] \right] \right\}^{\frac{2}{n}}$$

- Fitzhugh-Nagumo equation

$$u_t - u_{xx} + u(1-u)(a-u) = 0$$

$$u(x,t) = \frac{a}{2} \left\{ 1 + \tanh \left[\frac{a}{2\sqrt{2}}(x - \frac{(2-a)}{\sqrt{2}}t) + \delta \right] \right\}$$

- Kuramoto-Sivashinski equation

$$u_t + uu_x + au_{xx} + bu_{xxxx} = 0$$

$$\begin{aligned} u(x,t) &= c + \frac{165ak}{19}\left\{ \tanh^3\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \\ &- \frac{135ak}{19}\left\{ \tanh\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \end{aligned}$$

with $k = \sqrt{\frac{11a}{19b}}$

$$\begin{aligned} u(x,t) &= c - \frac{15ak}{19}\left\{ \tanh^3\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \\ &+ \frac{45ak}{19}\left\{ \tanh\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \end{aligned}$$

with $k = \sqrt{\frac{-a}{19b}}$

- Dym-Kruskal equation

$$u_t + (1-u)^3 u_{xxx} = 0$$

$$\begin{aligned} u(x,t) &= \operatorname{sech}^2\left[\frac{1}{2}\sqrt{c}\,[x - ct + \delta(x,t)]\right] \\ \delta(x,t) &= \frac{2}{\sqrt{c}} \tanh\left[\frac{\sqrt{c}}{2}\,[x - ct + \delta(x,t)]\right] \end{aligned}$$

- sine-Gordon equation

$$u_{tt} - u_{xx} - \sin u = 0$$

$$u(x,t) = 4\arctan\left\{\exp\left[\frac{1}{\sqrt{-c}}(x-ct) + \delta\right]\right\}$$

- Coupled Korteweg-de Vries equations

$$u_t - a(6uu_x + u_{xxx}) - 2b\, vv_x = 0,$$
$$v_t + 3uv_x + v_{xxx} = 0$$

$$\begin{aligned} u(x,t) &= 2\,c\,\mathrm{sech}^2\left[\sqrt{c}(x-ct)+\delta\right], \\ v(x,t) &= \pm c\sqrt{\frac{-2(4a+1)}{b}}\;\mathrm{sech}\left[\sqrt{c}(x-ct)+\delta\right], \end{aligned}$$

$$\begin{aligned} u(x,t) &= c\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \\ v(x,t) &= \frac{3}{\sqrt{6|b|}}\,u(x,t) = \frac{3\,c}{\sqrt{6|b|}}\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \end{aligned}$$

C. The Direct Method

- Goal: Exact solutions

  Single solitary wave or soliton solutions
  N-solitons
  Implicit solutions

- Applicable to:

  Single nonlinear evolution and wave equations
  Systems of nonlinear PDEs
  Nonlinear ODEs

- Method:

  Hirota's direct method
  Rosales' perturbation method
  Trace method
  Hereman *et al* real exponential approach
  Frobenius method
  Phase space analysis: Poincaré and Liapunov

- Requirements :

  Based on physical principles
  Simple and straightforward
  Programmable in MACSYMA, REDUCE,
  MATHEMATICA, SCRATCHPAD II, etc.

- Publications:

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W. Hereman & S. Angenent, MACSYMA Newsletter **6**, 11-18 (1989)

P. Banerjee *et al*, J. Phys. A: Math. & Gen. **23**, 521-536 (1990)

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M. Coffey, SIAM J. Appl. Math. **50**, 2 papers in press (1990)

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2. THE ALGORITHM

- Step 1: One equation or a system of coupled nonlinear PDEs

$$\begin{aligned} \mathcal{F}(u, v, u_t, u_x, v_t, v_x, u_{tx}, ..., u_{mx}, v_{nx}) &= 0, \\ \mathcal{G}(u, v, u_t, u_x, v_t, v_x, u_{tx}, ..., u_{px}, v_{qx}) &= 0, \quad (m, n, p, q \in \mathbb{N}) \end{aligned}$$

  $\mathcal{F}$ and $\mathcal{G}$ are polynomials in their arguments and

$$u_t = \frac{\partial u}{\partial t}, \quad u_{nx} = \frac{\partial^n u}{\partial x^n}$$

  EXAMPLE: The Korteweg-de Vries equation

$$\mathcal{F}(u, u_t, u_x, u_{3x}) = u_t + 6uu_x + u_{3x} = 0$$

- Step 2:
  - Introduce the variable $\xi = x - ct$, (c is the constant velocity)
  - Integrate the system of ODEs for $\phi(\xi) \equiv u(x,t)$ and $\psi(\xi) \equiv v(x,t)$ with respect to ξ to reduce the order
  - Ignore integration constants by assuming that ϕ and ψ and their derivatives vanish at $\xi = \pm\infty$
  - Carry out a nonlinear transformation (Painlevé analysis)

$$\phi = \tilde{\phi}^{\alpha}, \quad \psi = \tilde{\psi}^{\beta}$$

EXAMPLE: KdV type ODE in ϕ

$$-c\phi_{\xi} + 6\phi\phi_{\xi} + \phi_{3\xi} = 0$$

Integrate w.r.t. ξ

$$-c\phi + 3\phi^2 + \phi_{2\xi} = 0$$

No transformation needed, $\phi = \tilde{\phi}$

- Step 3:
  - Expand $\tilde{\phi}$ and $\tilde{\psi}$ in a power series

$$\tilde{\phi} = \sum_{n=1}^{\infty} a_n \, g^n, \quad \tilde{\psi} = \sum_{n=1}^{\infty} b_n \, g^n$$

  - $g(\xi) = \exp[-K(c)\xi]$ solves the linear part of at least one of the equations
  - Consider the dispersion laws $K(c)$ of (one of) the linearized equations
  - Substitute the expansions into the full nonlinear system
  - Use Cauchy's rule for multiple series to rearrange the multiple sums
  - Equate the coefficient of g^n to get the coupled recursion relations for a_n and b_n

EXAMPLE:

$g(\xi) = \exp[-K(c)\xi]$ solves $-c\phi + \phi_{2\xi} = 0$

if $K(c) = \sqrt{c}$ (dispersion law)

With $\phi = \frac{c}{3} \sum_{n=1}^{\infty} a_n \, g^n$

and Cauchy's rule

$$\sum_{n=1}^{\infty}(n^2 - 1)a_n \, g^n + \sum_{n=2}^{\infty} \sum_{l=1}^{n-1} a_l a_{n-l} \, g^n = 0$$

Note that a_1 is arbitrary

Recursion relation:

$$(n^2 - 1)a_n \; + \sum_{l=1}^{n-1} a_l a_{n-l} = 0 \quad (n \geq 2)$$

- Step 4:
  - Assume that a_n and b_n are polynomials in n
  - Determine their degrees δ_1 and δ_2
  - Substitute

    $$a_n = \sum_{j=0}^{\delta_1} A_j \, n^j, \quad b_n = \sum_{j=0}^{\delta_2} B_j \, n^j$$

    into the recursion relations

- Compute the sums by using the formulae for

$$S_k = \sum_{i=1}^{n} i^k, \quad (k = 0, 1, 2, ...)$$

- Examples:

$$S_0 = n, \quad S_1 = \frac{n(n+1)}{2}$$

$$S_2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{etc.}$$

- Equate to zero the coefficients of the polynomial in n
- Solve the algebraic (nonlinear) equations for the constant coefficients A_j and B_j

EXAMPLE:

a_n is of degree $\delta_1 = 1$

Substitute $a_n = A_1 n + A_0$

Use S_0, S_1 and S_2

Solve the algebraic equations for A_1 and A_0:

$$A_0(A_1 + 1) = 0$$

$$A_1{}^2 + 6A_0A_1 + 6A_1 - 6A_0{}^2 = 0$$

$$A_0(A_1 + 1) = 0$$

$$A_1(A_1 + 6) = 0$$

Nontrivial solution: $A_1 = -6, \quad A_0 = 0$

Hence, $a_n = -6\ n\ (a_0)^n = (-1)^{n+1}\ 6\ n\ (\frac{a_1}{6})^n$

- Step 5:
  - Find the closed forms for

$$\tilde{\phi} = \sum_{n=1}^{\infty}\sum_{j=0}^{\delta_1} A_j\, n^j\, g^n \equiv \sum_{j=0}^{\delta_1} A_j\, F_j(g)$$

$$\tilde{\psi} = \sum_{n=1}^{\infty}\sum_{j=0}^{\delta_2} B_j\, n^j\, g^n \equiv \sum_{j=0}^{\delta_2} B_j\, F_j(g)$$

    with

$$F_j(g) \equiv \sum_{n=1}^{\infty} n^j g^n$$

$$F_{j+1}(g) = gF_j'(g) \quad (j = 0, 1, 2, ...)$$

  - Examples:

$$F_0(g) = \frac{g}{1-g}, \quad F_1(g) = \frac{g}{(1-g)^2}$$

$$F_2(g) = \frac{g(1+g)}{(1-g)^3}, \quad \text{etc.}$$

  - Return to ϕ and ψ and then to the original variables x and t to obtain the travelling wave solution(s) $u(x,t)$ and $v(x,t)$

EXAMPLE:

$$\phi = \frac{c}{3}\sum_{n=1}^{\infty}(-1)^{n+1}\, 6\, n\, (\frac{a_1}{6})^n g^n$$

Use $F_1 = \frac{g}{(1-g)^2}$

$$\phi = 2c\frac{ag}{(1+ag)^2}, \quad a = \frac{a_1}{6}$$

Return to the original variables, $g = \exp[-\sqrt{c}(x-ct)]$

$$u(x,t) = \phi = \frac{c}{2}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)+\delta\right]$$

where $\delta = -\ln|a| = -\ln|\frac{a_1}{6}|$

3. EXAMPLE 1: The Kuramoto-Sivashinski Equation

- Step 1:
  - Consider the KS equation in $1+1$ dimension

$$u_t + uu_x + au_{2x} + bu_{4x} = 0$$

    with $a, b \in \mathbb{R}$

- Step 2:
  - Introduce the variable $\xi = x - ct$, c is constant
  - Replace $\phi(\xi) \equiv u(x,t)$ by $C + \tilde{\phi}$
  - Integrate the equation w.r.t. ξ

$$(C - c)\tilde{\phi} + \frac{1}{2}\tilde{\phi}^2 + a\tilde{\phi}_\xi + b\tilde{\phi}_{3\xi} = 0$$

- Step 3:
  - Solve the linearized equation

$$(C - c)\tilde{\psi} + a\tilde{\psi}_\xi + b\tilde{\phi}_{3\xi} = 0$$

    for $\tilde{\psi} = \exp[-K(c)\xi]$
  - Dispersion relation:

$$f(K, c) = (C - c) - aK - bK^3 = 0$$

  - Since $c, K(c)$ and C are unknowns one cannot solve for $K(c)$
  - Rely on the physical idea and take $g = \exp[-\frac{K(c)}{s}\xi]$, where $s \in \mathbb{N}$
  - Assume that the roots K_1, K_2 and K_3 are integer multiples of a common $\tilde{K}$, i.e. $K_1 = s_1\tilde{K}, K_2 = s_2\tilde{K}$ and $K_3 = s_3\tilde{K}$
  - Seek a solution of the form

$$\tilde{\phi} = \sum_{n=1}^{\infty} a_n g^n$$

  - Substitute this expansion and use Cauchy's rule for multiple series

– Recursion relation:

$$[(C-c) - \frac{n}{s}Ka - (\frac{n}{s})^3 bK^3]a_n + \frac{1}{2}\sum_{l=1}^{n-1} a_l a_{n-l} = 0 \quad (n \geq 2)$$

with a_1 arbitrary if $s = 1$ and $a_1 = 0$ if $s \neq 1$

– Replace $b\,K^3$ from the dispersion law

$$\begin{aligned} 2 \quad & (\frac{n}{s} - 1)\Big\{[(\frac{n}{s})^2 + (c-C)(\frac{n}{s}) + 1] + \frac{aKn}{s}(\frac{n}{s}+1)\Big\}a_n \\ + \quad & \sum_{l=1}^{n-1} a_l a_{n-l} = 0 \end{aligned}$$

- Step 4:
  - Assume that a_n is a polynomial in n
  - Calculate the degree $\delta_1 = 2$ of a_n in n
  - Substitute $a_n = A_2 n^2 + A_1 n + A_0$
  - Apply the formulae for S_0 through S_4
  - Set the six coefficients of the polynomial of degree 5 in n equal to zero
  - Solve the resulting nonlinear system:

$$\begin{aligned} A_0[2(C-c) - A_0] &= 0 \\ 60A_1 s(C-c) - {A_2}^2 s + 10A_0A_2 s - 5{A_1}^2 s - 30A_0A_1\ s & \\ +30{A_0}^2 s - 60aA_0K &= 0 \\ 12A_2 s(C-c) - A_1A_2 s - 6A_0A_2 s + 6A_0A_1 s - 12aA_1K &= 0 \\ 12A_0(C-c) - 4A_0A_2 s^3 - {A_1}^2 s^3 + 12aA_2Ks^2 - 12aA_0K &= 0 \\ A_1[12(C-c) - A_2 s^3 - 12aK] &= 0 \\ A_2[60(C-c) - A_2 s^3 - 60aK] &= 0 \end{aligned}$$

- Step 5:
  - Find the closed form for $\tilde{\phi}$

– Use $F_0(g) = \dfrac{g}{1-g}$ and $F_2(g) = \dfrac{g(1+g)}{(1-g)^3}$

$$\begin{aligned}
\tilde{\phi} &= \sum_{n=1}^{\infty}(A_2 n^2 + A_0)(a_0 g)^n \\
&= A_2 F_2(a_0 g) + A_0 F_0(a_0 g) \\
&= A_2 \frac{g(1+a_0 g)}{(1-a_0 g)^3} + A_0 \frac{a_0 g}{(1-a_0 g)} \\
&= \frac{1}{4} A_2 \frac{(1+a_0 g)^3}{(1-a_0 g)^3} + \frac{1}{4}(2A_0 - A_2)\frac{(1+a_0 g)}{(1-a_0 g)} - \frac{1}{2}A_0
\end{aligned}$$

– Select the constant $a_0 = -\exp(-\Delta) < 0$

– Absorb the arbitrary phase Δ into the exponential g

$$\tilde{\phi} = \frac{1}{4}A_2 \tanh^3\Big(\frac{\frac{K}{s}\xi + \Delta}{2}\Big) + \frac{1}{4}(2A_0 - A_2)\tanh\Big(\frac{\frac{K}{s}\xi + \Delta}{2}\Big) - \frac{1}{2}A_0$$

– Return to the original variables x and t

CASE 1: $s = 1$:

$$\phi = c + \frac{15aK}{19}\Big[11\,\tanh^3\Big(\frac{K\xi + \Delta}{2}\Big) - 9\,\tanh\Big(\frac{K\xi + \Delta}{2}\Big)\Big]$$

where $K = \sqrt{\frac{11a}{19b}}$

CASE 2: $s = 2, 3$ or -5:

$$\phi = c - \frac{15a\tilde{K}}{19}\Big[\tanh^3\Big(\frac{\tilde{K}\xi + \Delta}{2}\Big) - 3\,\tanh\Big(\frac{\tilde{K}\xi + \Delta}{2}\Big)\Big]$$

with $\tilde{K} = \sqrt{\frac{-a}{19b}}$

4. EXAMPLE 2: The sine-Gordon Equation

- Step 1:
  - Consider the SG in light cone coordinates

$$u_{xt} = \sin(u)$$

  - Remove the transcendental nonlinearity and transform it into a coupled system with strictly polynomial terms

$$\begin{aligned} \Phi_{xt} - \Phi - \Phi\Psi &= 0 \\ 2\,\Psi + \Psi^2 + \Phi_t^2 &= 0 \end{aligned}$$

    where $\Phi = u_x$ and $\Psi = \cos(u) - 1$

- Step 2 and 3:
  - Substitute the scaled expansions

$$\begin{aligned} \Phi(x-ct) &= \phi(\xi) = \frac{1}{\sqrt{-c}} \sum_{n=1}^{\infty} a_n\, g^n(\xi) \\ \Psi(x-ct) &= \psi(\xi) = \sum_{n=1}^{\infty} b_n\, g^n(\xi) \end{aligned}$$

    with $g(\xi) = \exp[-K(c)\xi]$
  - Use the dispersion law $K^2 = -c$ for $c < 0$
  - Coupled recursion relations:

$$\begin{aligned} (n^2 - 1)\, a_n - \sum_{l=1}^{n-1} a_l\, b_{n-l} &= 0 \\ 2\, b_n + \sum_{l=1}^{n-1} [b_l\, b_{n-l} + l\,(n-l)\, a_l\, a_{n-l}] &= 0 \end{aligned}$$

    for $n \geq 2$, a_1 is arbitrary, $b_1 = 0$

- Step 4:
  - Solve the coupled system of the recursion relations:

$$\begin{aligned} a_{2n} &= 0, \quad b_{2n} = 8\,(-1)^n\, n\, {a_0}^{2n}, \quad n = 1, 2, \ldots\,, \\ a_{2n+1} &= 4\,(-1)^n\, {a_0}^{2n+1}, \quad b_{2n+1} = 0, \quad n = 0, 1, \ldots \end{aligned}$$

  with $a_0 = a_1/4 \ > 0$

- Step 5:
  - Find the closed form for ϕ and ψ
  - Use the formulae for F_0 and F_1

$$\begin{aligned} \phi &= \frac{4}{\sqrt{-c}} \sum_{n=0}^{\infty} (-1)^n\, (a_0\, g)^{2n+1} = \frac{4}{\sqrt{-c}}\, \frac{a_0\, g}{[1 + (a_0\, g)^2]} \\ \psi &= -8 \sum_{n=1}^{\infty} (-1)^{n+1}\, n\, (a_0\, g)^{2n} = \frac{-8\,(a_0\, g)^2}{[1 + (a_0\, g)^2]^2} \end{aligned}$$

  - Return to the original variables x and t

$$\begin{aligned} \cos[u(x,t)] - 1 &= 1 - 2\, \mathrm{sech}^2[\frac{1}{\sqrt{-c}}(x - ct) + \delta] \\ u(x,t) &= \pm \frac{2}{\sqrt{-c}} \int \mathrm{sech}[\frac{1}{\sqrt{-c}}(x - ct) + \delta]\, dx \\ &= \pm 4\ \arctan\left\{ \exp[\frac{1}{\sqrt{-c}}(x - ct) + \delta] \right\} \end{aligned}$$

  with $\delta = \ln|4/a_1|$

5. CONSTRUCTION OF N-SOLITON SOLUTIONS

Example: The sine-Gordon Equation

- Consider the SG in light cone coordinates

$$u_{xt} = \sin(u)$$

- Transform the SG into a coupled system with polynomial terms

$$\begin{aligned} \Phi_{xt} - \Phi - \Phi\Psi &= 0 \\ 2\,\Psi + \Psi^2 + \Phi_t^2 &= 0 \end{aligned}$$

  where $\Phi = u_x$ and $\Psi = \cos(u) - 1$

- Substitute

$$\begin{aligned} \Phi^{(1)} &= \sum_{i=1}^{N} c_i\, g_i(x,t) \\ &= \sum_{i=1}^{N} c_i\, a_i\, \exp(K_i x - \Omega_i t) \end{aligned}$$

  into the linear part of the first equation

- Dispersion law:

$$\Omega_i = -\frac{1}{K_i}, \quad i = 1, 2, ..., N$$

- Starting term in the expansion of Ψ, say $\Psi^{(2)}$, must be of the form

$$\begin{aligned} \Psi^{(2)} &= \sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij}\, g_i\, g_j \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij}\, a_i\, a_j\, \exp[(K_i+K_j)x-(\Omega_i+\Omega_j)t] \end{aligned}$$

  so that $-2\,\Psi^{(2)}$ balances

$$\begin{aligned} {\Phi^2}_t^{(1)} &= \sum_{i=1}^{N}\sum_{j=1}^{N} c_i\, c_j\, \Omega_i\, \Omega_j\, g_i\, g_j \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} c_i\, c_j\, a_i\, a_j\, \Omega_i\, \Omega_j\, \exp[(K_i+K_j)x-(\Omega_i+\Omega_j)t] \end{aligned}$$

- Use the dispersion law and find

$$\begin{aligned} d_{ij} &= -\frac{1}{2}\, c_i\, c_j\, \Omega_i\, \Omega_j \\ &= -\frac{1}{2}\, \frac{c_i\, c_j}{K_i\, K_j} \\ &= \frac{1}{2}\, c_i\, c_j\, \frac{\Omega_i + \Omega_j}{K_i + K_j} \end{aligned}$$

- Note that Φ (Ψ, respectively) will only have an odd (even, respectively) number of $g's$
- Consider the action of the linear operator

$$L\bullet = \frac{\partial^2 \bullet}{\partial x \partial t} - 1\bullet$$

on the $(2n+1)^{\text{th}}$ term in the expansion of Φ

$$\Phi^{(2n+1)} = \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} \dots \sum_{s=1}^{N}}_{2n+1 \text{ summations}} c_{ij\dots s}\; g_i\; g_j\; \dots\; g_s$$

with $n = 0, 1, \dots$

- Balance with the second term, written in its most symmetric form:

$$\begin{aligned} \tfrac{1}{2} \quad & \sum_{l=0}^{n-1} \Phi^{(2l+1)} \underbrace{(K_i, K_j, \dots, K_o)}_{2l+1 \text{ arguments}}\; \Psi^{(2n-2l)} \underbrace{(K_p, K_q, \dots, K_s)}_{2(n-l) \text{ arguments}} \\ + \quad & \Psi^{(2n-2l)} \underbrace{(K_i, K_j, \dots, K_l)}_{2(n-l) \text{ arguments}}\; \Phi^{(2l+1)} \underbrace{(K_m, K_n, \dots, K_s)}_{2l+1 \text{ arguments}} \end{aligned}$$

- Similarly, from the second equation, one has

$$\begin{aligned}
\Psi^{(2n)} &= \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} \dots \sum_{r=1}^{N}}_{2n \text{ summations}} d_{ij\dots r}\, g_i\, g_j\, \dots\, g_r \\
&= -\frac{1}{2}\sum_{l=1}^{n-1} \Phi^{(2l)} \underbrace{(K_i, K_j, \dots, K_n)}_{2l \text{ arguments}}\ \Psi^{(2n-2l)} \underbrace{(K_o, K_p, \dots, K_r)}_{2(n-l) \text{ arguments}} \\
&+ \sum_{l=0}^{n-1} {\Phi_t}^{(2l+1)} \underbrace{(K_i, K_j, \dots, K_o)}_{2l+1 \text{ arguments}} {\Phi_t}^{(2n-2l-1)} \underbrace{(K_p, K_q, \dots, K_s)}_{2(n-l)-1 \text{ arguments}}
\end{aligned}$$

  with $n = 1, 2, \dots$

- Determine the coefficients $d_{ij\dots r}$

- For example, c_{ijk} is computed by equating

$$\begin{aligned}
L\Phi^{(3)} &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}[-(\Omega_i+\Omega_j+\Omega_k)(K_i+K_j+K_k)-1]c_{ijk}g_ig_jg_k \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}(K_i+K_j)(K_j+K_k)(K_k+K_i)\frac{c_{ijk}}{K_i\, K_j\, K_k}g_ig_jg_k
\end{aligned}$$

  to

$$\begin{aligned}
&\tfrac{1}{2}\ \left\{\Phi^{(1)}(K_i)\Psi^{(2)}(K_j, K_k) + \Psi^{(2)}(K_i, K_j)\, \Phi^{(1)}(K_k)\right\} \\
&= -\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}(K_i+K_j)(K_j+K_k)(K_k+K_i)\frac{c_ic_jc_k}{K_i\, K_j\, K_k}g_ig_jg_k
\end{aligned}$$

- Thus, with $c_i = 1$,

$$c_{ijk} = \frac{-1}{4(K_i + K_j)(K_j + K_k)}$$

- After some lengthy calculations, with MACSYMA, one obtains

$$\begin{aligned}
c_{ij\,\dots\,s} &= \frac{(-1)^n}{4^n(K_i + K_j)(K_j + K_k)\,\dots\,(K_r + K_s)}, \quad n = 0, 1, \dots \\
d_{ij\,\dots\,r} &= \frac{-2(-1)^n(\Omega_i + \Omega_j +\ \dots\ + \Omega_r)}{4^n(K_i + K_j)(K_j + K_k)\,\dots\,(K_q + K_r)}, \quad n = 1, 2, \dots
\end{aligned}$$

- Find the closed form of

$$\begin{aligned}\Phi &= \sum_{n=0}^{\infty} \Phi^{(2n+1)} \\ &= \sum_{i=1}^{N} g_i + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}(\frac{-1}{4})\frac{g_i\, g_j\, g_k}{(K_i{+}K_j)(K_j{+}K_k)} + \ldots \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N}\ldots\sum_{s=1}^{N}(\frac{-1}{4})^n \frac{g_i\, g_j\, \ldots\, g_s}{(K_i{+}K_j)(K_j{+}K_k)\ldots(K_r{+}K_s)} + \ldots\end{aligned}$$

- Similar expression for Ψ
- Introduce the $N \times N$ identity matrix I and the $N \times N$ matrix B with elements

$$B_{ij} = \frac{1}{2}\frac{\sqrt{a_i\, a_j}}{(K_i + K_j)} \exp\left\{\frac{1}{2}[(K_i + K_j)x - (\Omega_i + \Omega_j)t]\right\}$$

- N-soliton solution of the SG equation is

$$\begin{aligned}\Phi(x,t) &= 4\,[\mathrm{Tr}(\arctan B)]_x, \\ \Psi(x,t) &= -2\left\{\ln[\det(I + B^2)]\right\}_{xt}\end{aligned}$$

  where Tr stands for trace
- Finally, with $\Phi = u_x$ one has

$$u(x,t) = \pm 4\ \mathrm{Tr}(\arctan B) = \pm(\frac{2}{i})\ \mathrm{Tr}\left\{\ln\left[\frac{I + iB}{I - iB}\right]\right\}$$

- Special case: The two-soliton solution:

$$u(x,t) = 4\ \arctan\left\{\left(\frac{K_1 + K_2}{K_1 - K_2}\right) F(x,t)\right\}$$

  with

$$F(x,t) = \left[\frac{\exp[K_1 x - \Omega_1 t + \delta_1] - \exp[K_2 x - \Omega_2 t + \delta_2]}{1 + \exp[(K_1 + K_2)x - (\Omega_1 + \Omega_2)t + \delta_1 + \delta_2]}\right]$$

- Figure: select $K_1 = 1, K_2 = \sqrt{2}$, thus $\Omega_1 = 1, \Omega_2 = -\frac{\sqrt{2}}{2}$, and with $\delta_1 = \delta_2 = 0$