Symbolic Computation of Scaling Invariant Lax Pairs in Operator Form for Integrable Systems

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Outline

- What are Lax pairs of nonlinear PDEs?
- Lax pairs in operator form
- Lax pairs in matrix form
- Reasons to compute Lax pairs
- Quick method to find Lax pairs
- More algorithmic approach
- Examples of Lax pairs of nonlinear PDEs
- Conclusions and future work

Peter D. Lax (1926-)

Seminal paper: Integrals of nonlinear equations of evolution and solitary waves

Commun. Pure Appl. Math. **21** (1968) 467-490

What are Lax Pairs of Nonlinear PDEs?

- Historical example: Korteweg-de Vries equation

$$\boxed{u_t + \alpha\, u u_x + u_{xxx} = 0}$$

- Key idea: Replace the nonlinear PDE with a compatible linear system (Lax pair):

$$\psi_{xx} + \left(\tfrac{1}{6}\alpha u - \lambda\right)\psi = 0$$

$$\psi_t + 4\psi_{xxx} + \alpha u \psi_x + \tfrac{1}{2}\alpha u_x \psi + a(t)\psi = 0$$

ψ is eigenfunction; λ is constant eigenvalue ($\lambda_t = 0$) (isospectral), and $a(t)$ is an arbitrary function. We will set $a(t) = 0$.

Class of Equations and Notation

- Consider a system of evolution equations:

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots, \mathbf{u}_{Mx})$$

with $\mathbf{u}(x,t) = (u^{(1)}, u^{(2)}, \ldots, u^{(N)})$ and where

$$u_{kx}^{(j)} = \frac{\partial^k u^{(j)}}{\partial x^k}$$

- In examples, the components of $\mathbf{u}$ are $u, v, \ldots$.
- Define the total derivative operator as

$$\mathrm{D}_t \bullet = \frac{\partial \bullet}{\partial t} + \sum_{j=1}^{N} \sum_{k=0}^{M} \frac{\partial \bullet}{\partial u_{kx}^{(j)}} D_x^k \left(u_t^{(j)} \right)$$

Lax Pairs in Operator Form

- Replace a completely integrable nonlinear PDE by a pair of linear equations (called a Lax pair):

$$\boxed{\mathcal{L}\psi = \lambda\psi} \qquad \text{and} \qquad \boxed{\mathrm{D}_t\psi = \mathcal{M}\psi}$$

- Require compatibility of both equations

$$\begin{aligned} \mathcal{L}_t\psi + \mathcal{L}\mathrm{D}_t\psi &= \lambda \mathrm{D}_t\psi \\ \mathcal{L}_t\psi + \mathcal{L}\mathcal{M}\psi &= \lambda\mathcal{M}\psi \\ &= \mathcal{M}\lambda\psi \\ &\doteq \mathcal{M}\mathcal{L}\psi \end{aligned}$$

Hence, $\quad \mathcal{L}_t\psi + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})\psi \doteq 0.$

- Lax equation: $\boxed{\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \doteq \mathcal{O}}$

with commutator $[\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}$.

Furthermore, $\mathcal{L}_t \psi = [\mathrm{D}_t, \mathcal{L}]\psi = \mathrm{D}_t(\mathcal{L}\psi) - \mathcal{L}\mathrm{D}_t\psi$

and $\doteq$ means "evaluated on solutions of the PDE."

Example: Lax operators for the KdV equation

- Recall:

$$\psi_{xx} + \left(\tfrac{1}{6}\alpha u - \lambda\right)\psi = 0$$

$$\psi_t + 4\psi_{xxx} + \alpha u \psi_x + \tfrac{1}{2}\alpha u_x \psi = 0$$

- Can be written in the form

$$\mathcal{L}\psi = \lambda\psi \quad \text{and} \quad \mathsf{D}_t\psi = \mathcal{M}\psi$$

with

$$\boxed{\mathcal{L} = \mathsf{D}_x^2 + \tfrac{1}{6}\alpha u\,\mathbf{I}}$$

$$\boxed{\mathcal{M} = -\left(4\,\mathsf{D}_x^3 + \alpha u \mathsf{D}_x + \tfrac{1}{2}\alpha u_x\,\mathbf{I}\right)}$$

- Note: $\mathcal{L}_t\psi + [\mathcal{L}, \mathcal{M}]\psi = \frac{1}{6}\alpha\left(u_t + \alpha u u_x + u_{xxx}\right)\psi.$

Alternate Operator Formulations

- Define $\tilde{\mathcal{L}} = \mathcal{L} - \lambda \mathrm{I}$ and $\tilde{\mathcal{M}} = \mathcal{M} - \mathsf{D}_t$.
- Then, the Lax pair becomes

$$\tilde{\mathcal{L}}\psi = 0 \qquad \text{and} \qquad \tilde{\mathcal{M}}\psi = 0$$

and the Lax equation becomes $[\tilde{\mathcal{L}}, \tilde{\mathcal{M}}] \doteq \mathcal{O}$.

Challenge: Find commuting operators modulo the (nonlinear) PDE

- If S is an arbitrary invertible operator, then

$$\boxed{\hat{\mathcal{L}} = S\,\mathcal{L}\,S^{-1}} \qquad \boxed{\hat{\mathcal{M}} = S\,\mathcal{M}\,S^{-1}} \qquad \hat{\mathsf{D}}_t = S\,\mathsf{D}_t\,S^{-1}$$

satisfy $\hat{\mathcal{L}}_t + [\hat{\mathcal{L}}, \hat{\mathcal{M}}] \doteq \mathcal{O}$.

Lax Pairs in Matrix Form

- Express compatibility of

$$\begin{aligned} \mathsf{D}_x \boldsymbol{\Psi} &= \mathbf{X}\, \boldsymbol{\Psi} \\ \mathsf{D}_t \boldsymbol{\Psi} &= \mathbf{T}\, \boldsymbol{\Psi} \end{aligned}$$

where $\boldsymbol{\Psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix}, \quad \mathbf{X} \text{ and } \mathbf{T} \text{ are } N \times N \text{ matrices}$

- Lax equation (zero-curvature equation):

$$\boxed{\mathsf{D}_t \mathbf{X} - \mathsf{D}_x \mathbf{T} + [\mathbf{X}, \mathbf{T}] \doteq \mathbf{0}}$$

with commutator $[\mathbf{X}, \mathbf{T}] = \mathbf{X}\mathbf{T} - \mathbf{T}\mathbf{X}$.

- Example: Lax pair for the KdV equation

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6}\alpha u & 0 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \frac{1}{6}\alpha u_x & -4\lambda - \frac{1}{3}\alpha u \\ -4\lambda^2 + \frac{1}{3}\alpha\lambda u + \frac{1}{18}\alpha^2 u^2 + \frac{1}{6}\alpha u_{2x} & -\frac{1}{6}\alpha u_x \end{bmatrix}$$

Substitution into the Lax equation yields

$$\mathrm{D}_t\mathbf{X} - \mathrm{D}_x\mathbf{T} + [\mathbf{X}, \mathbf{T}] = -\frac{1}{6}\alpha \begin{bmatrix} 0 & 0 \\ u_t + \alpha u u_x + u_{3x} & 0 \end{bmatrix}$$

Equivalence under Gauge Transformations

- Lax pairs are equivalent under a gauge transformation:

If $(\mathbf{X}, \mathbf{T})$ is a Lax pair then so is $(\tilde{\mathbf{X}}, \tilde{\mathbf{T}})$ with

$$\begin{aligned} \tilde{\mathbf{X}} &= \mathbf{G}\,\mathbf{X}\,\mathbf{G}^{-1} + \mathsf{D}_x(\mathbf{G})\,\mathbf{G}^{-1} \\ \tilde{\mathbf{T}} &= \mathbf{G}\,\mathbf{T}\,\mathbf{G}^{-1} + \mathsf{D}_t(\mathbf{G})\,\mathbf{G}^{-1} \end{aligned}$$

$\mathbf{G}$ is arbitrary invertible matrix and $\tilde{\mathbf{\Psi}} = \mathbf{G}\mathbf{\Psi}$.

Thus,

$$\boxed{\tilde{\mathbf{X}}_t - \tilde{\mathbf{T}}_x + [\tilde{\mathbf{X}}, \tilde{\mathbf{T}}] \doteq \mathbf{0}}$$

- Example: For the KdV equation

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6}\alpha u & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{X}} = \begin{bmatrix} -ik & \frac{1}{6}\alpha u \\ -1 & ik \end{bmatrix}$$

Here,

$$\tilde{\mathbf{X}} = \mathbf{G}\;\mathbf{X}\;\mathbf{G}^{-1} \quad \text{and} \quad \tilde{\mathbf{T}} = \mathbf{G}\;\mathbf{T}\;\mathbf{G}^{-1}$$

with

$$\mathbf{G} = \begin{bmatrix} -i\,k & 1 \\ -1 & 0 \end{bmatrix}$$

where $\lambda = -k^2$.

Reasons to Compute a Lax Pair

- Compatible linear system is the starting point for application of the IST and the Riemann-Hilbert method for boundary value problems.
- Confirm the complete integrability of the PDE.
- Zero-curvature representation of the PDE.
- Compute conservation laws of the PDE.
- Discover families of completely integrable PDEs.

Question: How to find a Lax pair of a completely integrable PDE?

Answer: There is no completely systematic method.

Dilation Invariance and Weights

- KdV equation is invariant under dilation symmetry

$$(x,t,u) \rightarrow (\kappa^{-1}x, \kappa^{-3}t, \kappa^{2}u) = (\tilde{x}, \tilde{t}, \tilde{u})$$

where κ is an arbitrary parameter. Indeed,

$$u_t + \alpha u u_x + u_{xxx} = 0 \;\rightarrow\; \frac{1}{\kappa^5}(\tilde{u}_{\tilde{t}} + \alpha \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}) = 0$$

- The weight W of a variable is the exponent of κ in the symmetry. Thus, $W(x) = -1, W(t) = -3$, or

$$W(\mathsf{D}_x) = 1, \qquad W(\mathsf{D}_t) = 3, \qquad W(u) = 2.$$

- The total weight of the KdV equation is 5 because each monomial scales with κ^5.

Key Observation

- The Lax operators for the KdV equation are scaling invariant.

  Indeed,
  $$\mathcal{L} = \mathsf{D}_x^2 + \tfrac{1}{6}\alpha u \, \mathrm{I}$$
  is uniform of weight 2.
  $$\mathcal{M} = -\left(4\mathsf{D}_x^3 + \alpha u \mathsf{D}_x + \tfrac{1}{2}\alpha u_x \, \mathrm{I}\right)$$
  is uniform of weight 3.

- Furthermore, $\mathcal{L}\psi = \lambda\psi$ and $\mathsf{D}_t\psi = \mathcal{M}\psi$ are uniform in weight if $W(\lambda) = W(\mathcal{L}) = 2$ and $W(\mathcal{M}) = W(\mathsf{D}_t) = 3$.

Elementary Method to Compute Lax Pairs

Using the KdV equation as an example

- Select $W(\mathcal{L}) = 2$. Here $W(\mathcal{M}) = 3$. In general, $W(\mathcal{L}) \geq W(u)$ and $W(\mathcal{M}) = W(\mathsf{D}_t)$.
- Build $\mathcal{L}$ and $\mathcal{M}$ as linear combinations of scaling invariant terms with undetermined coefficients:

$$\begin{aligned} \mathcal{L} &= \mathsf{D}_x^2 + c_1 u\,\mathrm{I} \\ \mathcal{M} &= c_2 \mathsf{D}_x^3 + c_3 u \mathsf{D}_x + c_4 u_x\,\mathrm{I} \end{aligned}$$

- Substitute into $\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \doteq \mathcal{O}$, thus replacing u_t by $-(\alpha u u_x + u_{3x})$.

- Set the coefficients of $\mathsf{D}_x^2, \mathsf{D}_x$, and I equal to zero.
- Set the coefficients of like monomial terms in u, u_x, u_{xx}, etc. equal to zero.
- Reduce the nonlinear algebraic system

$$2c_3 - 3c_1c_2 = 0, \quad 2c_4 + c_3 - 3c_1c_2 = 0,$$
$$c_1(c_3 + \alpha) = 0, \quad c_1 - c_4 + c_1c_2 = 0$$

with the Gröbner basis method into

$$c_1(6c_1 - \alpha) = 0, \quad c_1(c_2 + 4) = 0, \quad c_1(c_3 + \alpha) = 0,$$
$$c_1(2c_4 + \alpha) = 0, \quad 6c_1 + c_3 = 0, \quad 3c_1 + c_4 = 0$$

- Solve: $c_1 = \frac{1}{6}\alpha, \quad c_2 = -4, \quad c_3 = -\alpha, \quad c_4 = -\frac{1}{2}\alpha$

- Substitute the coefficients into $\mathcal{L}$ and $\mathcal{M}$:

$$\mathcal{L} = \mathsf{D}_x^2 + \tfrac{1}{6}\alpha u\,\mathbf{I}$$

$$\mathcal{M} = -\left(4\mathsf{D}_x^3 + \alpha u \mathsf{D}_x + \tfrac{1}{2}\alpha u_x\,\mathbf{I}\right)$$

- In complicated cases the nonlinear algebraic systems are long and hard to solve (too many solution branches).
- A divide and conquer strategy is needed!

Algorithm to Compute Lax Pairs

Using the KdV equation as an example

- Step 1: Compute the weights

$$W(\mathsf{D}_x) = 1, \qquad W(\mathsf{D}_t) = 3, \qquad W(u) = 2.$$

- Step 2: Build a candidate Lax pair

  Select $W(\mathcal{L}) = 2$. Here $W(\mathcal{M}) = 3$.

  The candidate Lax pair is

$$\begin{aligned} \mathcal{L} &= \mathsf{D}_x^2 + f_1\,\mathsf{D}_x + f_0\,\mathrm{I} \\ \mathcal{M} &= c_3\,\mathsf{D}_x^3 + g_2\,\mathsf{D}_x^2 + g_1\,\mathsf{D}_x + g_0\,\mathrm{I} \end{aligned}$$

with undetermined functions f_0, f_1, g_0, g_1, g_2 and undetermined constant coefficient c_3.

- Step 3: Substitute into the Lax equation

$$\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] =$$

$$\Big(2\mathsf{D}_x g_2 - 3c_3 \mathsf{D}_x f_1\Big)\mathsf{D}_x^3$$

$$+\Big(\mathsf{D}_x^2 g_2 - 3c_3 \mathsf{D}_x^2 f_1 + f_1 \mathsf{D}_x g_2 + 2\mathsf{D}_x g_1 - 2g_2 \mathsf{D}_x f_1$$

$$-3c_3 \mathsf{D}_x f_0\Big)\mathsf{D}_x^2$$

$$+\Big(\mathsf{D}_t f_1 - c_3 \mathsf{D}_x^3 f_1 + \mathsf{D}_x^2 g_1 - g_2 \mathsf{D}_x^2 f_1 - 3c_3 \mathsf{D}_x^2 f_0$$

$$+f_1 \mathsf{D}_x g_1 + 2\mathsf{D}_x g_0 - g_1 \mathsf{D}_x f_1 - 2g_2 \mathsf{D}_x f_0\Big)\mathsf{D}_x$$

$$+\Big(\mathsf{D}_t f_0 - c_3 \mathsf{D}_x^3 f_0 + \mathsf{D}_x^2 g_0 - g_2 \mathsf{D}_x^2 f_0 + f_1 \mathsf{D}_x g_0 - g_1 \mathsf{D}_x f_0\Big)\,\mathrm{I}$$

- Step 4: Solve the kinematic constraints (i.e., equations not involving D_t).

  Equate the coefficients of D_x^3 and D_x^2 to zero and solve, yielding

$$
\begin{aligned}
g_2 &= \tfrac{3}{2} c_3 f_1, \\
g_1 &= \tfrac{3}{4} c_3 \mathsf{D}_x f_1 + \tfrac{3}{8} c_3 f_1^2 + \tfrac{3}{2} c_3 f_0
\end{aligned}
$$

- The candidate $\mathcal{M}$ operator reduces to

$$
\mathcal{M} = c_3 \mathsf{D}_x^3 + \tfrac{3}{2} c_3 f_1 \mathsf{D}_x^2 + \tfrac{3}{8} c_3 \left(2\mathsf{D}_x f_1 + f_1^2 + 4 f_0 \right) \mathsf{D}_x + g_0 \,\mathrm{I}
$$

- The candidate $\mathcal{L}$ remains unchanged.

- Step 5: Solve the dynamical equations (i.e., equations that do involve D_t).

  The coefficients of I and D_x yield

$$\mathsf{D}_t f_1 + 2\mathsf{D}_x g_0 - \tfrac{1}{8} c_3 \mathsf{D}_x \Big(2\mathsf{D}_x^2 f_1 + 12 \mathsf{D}_x f_0 - f_1^3 + 12 f_1 f_0 \Big) = 0$$

$$\mathsf{D}_t f_0 + \mathsf{D}_x^2 g_0 + f_1 \mathsf{D}_x g_0 - c_3 \Big(\mathsf{D}_x^3 f_0 + \tfrac{3}{2} f_1 \mathsf{D}_x^2 f_0 + \tfrac{3}{4} \mathsf{D}_x f_1 \mathsf{D}_x f_0 + \tfrac{3}{8} f_1^2 \mathsf{D}_x f_0 + \tfrac{3}{2} f_0 \mathsf{D}_x f_0 \Big) = 0$$

- Because $W(\mathcal{L}) = 2$ one has $f_1 = 0$. Thus,

$$2\mathsf{D}_x g_0 - \tfrac{3}{2} c_3 \mathsf{D}_x^2 f_0 = 0$$

$$\mathsf{D}_t f_0 + \mathsf{D}_x^2 g_0 - c_3 \Big(\mathsf{D}_x^3 f_0 + \tfrac{3}{2} f_0 \mathsf{D}_x f_0 \Big) = 0$$

- Step 5: continued

  Solving these equations gives

$$g_0 = \tfrac{3}{4} c_3 \mathsf{D}_x f_0 \qquad \text{and} \qquad f_0 = b_0 u$$

- Replace u_t by $-(\alpha u u_x + u_{3x})$,

$$\left(\alpha + \tfrac{3}{2} c_3 b_0\right) u u_x + \left(1 + \tfrac{1}{4} c_3\right) u_{3x} = 0$$

- Hence,

$$c_3 = -4, \;\; b_0 = \tfrac{1}{6}\alpha, \;\; f_0 = \tfrac{1}{6}\alpha u, \;\; f_1 = 0, \; g_0 = -\tfrac{1}{2}\alpha u_x$$

- Step 6: Substitute the coefficients into the undetermined functions and these into the candidate pair.

Thus,

$$\mathcal{L} = \mathsf{D}_x^2 + \tfrac{1}{6}\alpha u \,\mathrm{I}$$

and

$$\mathcal{M} = -\left(4\,\mathsf{D}_x^3 + \alpha u\,\mathsf{D}_x + \tfrac{1}{2}\alpha u_x\,\mathrm{I}\right)$$

form a Lax pair for the KdV equation.

Algorithm for Computing Lax Pairs

- Compute the scaling symmetry of the PDE
- Select $W(\mathcal{L}) = l \geq 1.$

  From the Lax equation: $W(\mathcal{M}) = W(\partial_t) = m.$
- Build a candidate Lax pair of the form

$$
\begin{aligned}
\mathcal{L} &= \mathsf{D}_x^l + f_{l-1}\mathsf{D}_x^{l-1} + \ldots + f_0\,\mathrm{I} \\
\mathcal{M} &= c_m\mathsf{D}_x^m + g_{m-1}\mathsf{D}_x^{m-1} + \ldots + g_0\,\mathrm{I}
\end{aligned}
$$

  for a constant c_m.
- Substitute into the Lax equation.

- Separate into kinematic constraints and dynamical equations.
- Solve the kinematic equations.
- Solve the dynamical equations.
- Substitute the coefficients into undetermined functions and these into the candidate Lax pair.
- Test the Lax pair.

- Example 1: The modified KdV (mKdV) equation

$$u_t + \alpha u^2 u_x + u_{3x} = 0$$

has weights of $W(u) = W(\mathsf{D}_x) = 1$ and $W(\mathsf{D}_t) = 3.$

- Selecting $W(\mathcal{L}) = 1$ gives a trivial Lax pair.
- Select $W(\mathcal{L}) = 2$, as in the KdV case, yields

$$\begin{aligned} \mathcal{L} &= \mathsf{D}_x^2 + f_1 \mathsf{D}_x + f_0 \, \mathrm{I} \\ \mathcal{M} &= c_3 \mathsf{D}_x^3 + g_2 \mathsf{D}_x^2 + g_1 \mathsf{D}_x + g_0 \, \mathrm{I} \end{aligned}$$

- Requiring uniform weights gives

$$f_1 = b_0 u, \quad f_0 = b_1 u^2 + b_2 u_x, \quad g_0 = a_1 u^3 + a_2 u u_x + a_3 u_{xx}$$

- Example 1: The mKdV equation – continued
- Solving the kinematic constraints and dynamical equations gives the Lax pair

$$\mathcal{L} = \mathrm{D}_x^2 + 2\epsilon u \mathrm{D}_x + \tfrac{1}{6}\left(\left(6\epsilon^2 + \alpha\right) u^2 + \left(6\epsilon \pm \sqrt{-6\alpha}\right) u_x\right) \mathrm{I}$$

$$\begin{aligned}\mathcal{M} = {} & -4\mathrm{D}_x^3 - 12\epsilon u \mathrm{D}_x^2 \\ & - \left(\left(12\epsilon^2 + \alpha\right) u^2 + \left(12\epsilon \pm \sqrt{-6\alpha}\right) u_x\right) \mathrm{D}_x \\ & -\Big((4\epsilon^3 + \tfrac{2}{3}\epsilon\alpha)u^3 + (12\epsilon^2 \pm \epsilon\sqrt{-6\alpha} + \alpha)uu_x \\ & +(3\epsilon \pm \tfrac{1}{2}\sqrt{-6\alpha})u_{xx}\Big) \mathrm{I}\end{aligned}$$

[M. Wadati, J. Phys. Soc. Jpn., 1972-1973].

- Example 2: The Boussinesq system

$$u_t - v_x = 0$$

$$v_t - \beta u_x + 3uu_x + \alpha u_{3x} = 0$$

has $W(\mathsf{D}_x) = 1, W(\mathsf{D}_t) = W(u) = W(\beta) = 2, W(v) = 3$

- Select $W(\mathcal{L}) = 3$. Then,

$$\begin{aligned} \mathcal{L} &= \mathsf{D}_x^3 + f_1 \mathsf{D}_x + f_0 \, \mathbf{I} \\ \mathcal{M} &= c_2 \mathsf{D}_x^2 + g_0 \, \mathbf{I} \end{aligned}$$

- The kinematic constraint yields $g_0 = \frac{2}{3} c_2 f_1 + c_0 \beta$

  The dynamical equations then become

$$\begin{aligned} \mathsf{D}_t f_1 &= c_2 \left(2 \mathsf{D}_x f_0 - \mathsf{D}_x^2 f_1 \right) \\ \mathsf{D}_t f_0 &= c_2 \left(\mathsf{D}_x^2 f_0 - \frac{2}{3} \mathsf{D}_x^3 f_1 - \frac{2}{3} f_1 \mathsf{D}_x f_1 \right) \end{aligned}$$

- Example 2: The Boussinesq system – continued
- The uniform weight ansatz gives

$$
\begin{aligned}
f_1 &= a_1 u + a_2 \beta \\
f_0 &= a_3 u_x + \mathsf{D}_x^{-1} \left(a_4 u^2 + a_5 \beta u + a_6 v_x + a_7 \beta^2 \right)
\end{aligned}
$$

- Solving the dynamical equations gives

$$
\begin{aligned}
\mathcal{L} &= \mathsf{D}_x^3 + \tfrac{1}{4\alpha} \, (3u - \beta) \, \mathsf{D}_x + \tfrac{3}{8\alpha^2} \left(\alpha u_x \pm \tfrac{1}{3} \sqrt{3\alpha} v \right) \mathrm{I} \\
\mathcal{M} &= \pm \sqrt{3\alpha} \, \mathsf{D}_x^2 \pm \tfrac{\sqrt{3\alpha}}{2\alpha} \, u \, \mathrm{I}
\end{aligned}
$$

[V. E. Zakharov, Sov. Phys. JETP, 1974].

- Example 3: The coupled KdV system (Hirota & Satsuma)

$$u_t - 6\beta u u_x + 6 v v_x - \beta u_{3x} = 0$$
$$v_t + 3 u v_x + v_{3x} = 0$$

has $W(\mathrm{D}_x) = 1, W(\mathrm{D}_t) = 3, W(u) = W(v) = 2.$

- Select $W(\mathcal{L}) = 4$. If $\beta = \frac{1}{2}$, then

$$\begin{aligned} \mathcal{L} &= \mathrm{D}_x^4 + 2u\mathrm{D}_x^2 + 2(u_x - v_x)\mathrm{D}_x \\ &\quad + (u^2 - v^2 + u_{2x} - v_{2x})\,\mathrm{I} \\ \mathcal{M} &= 2\mathrm{D}_x^3 + 3u\mathrm{D}_x + 3\left(\tfrac{1}{2}u_x - v_x\right)\mathrm{I} \end{aligned}$$

[R. K. Dodd & A. Fordy, Phys. Lett. A, 1982].

- Example 4: The Drinfel'd-Sokolov-Wilson system

$$u_t + 3vv_x = 0, \qquad v_t + 2uv_x + \alpha u_x v + 2v_{3x} = 0$$

has $W(\mathrm{D}_x) = 1, W(\mathrm{D}_t) = 3, W(u) = W(v) = 2.$

- Select $W(\mathcal{L}) = 6$. If $\alpha = 1$, then

$$\begin{aligned}\mathcal{L} = \mathrm{D}_x^6 + 2u\mathrm{D}_x^4 + (4u_x - 3v_x)\,\mathrm{D}_x^3 \\ &+ \left(\tfrac{9}{2}\,(u_{2x} - v_{2x}) - u^2 - v^2\right)\mathrm{D}_x^2 \\ &+ \left(\tfrac{5}{2}\,(u_{3x} - v_{3x}) + 2\,(uu_x - vv_x) + u_x v - uv_x\right)\mathrm{D}_x \\ &+ \left(\tfrac{1}{2}(u_{4x} - v_{4x}) + \tfrac{1}{2}(u+v)(u_{2x} - v_{2x}) + \tfrac{1}{4}(u_x^2 - v_x^2)\right)\mathrm{I}\end{aligned}$$

$$\mathcal{M} = \mathrm{D}_x^3 + u\mathrm{D}_x - \tfrac{1}{2}(3v_x - u_x)\,\mathrm{I}$$

[G. Wilson, Phys. Lett. A, 1974].

- Example 5: Class of fifth-order KdV equations

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0$$

includes several completely integrable equations:

Parameter ratios $\left(\frac{\alpha}{\gamma^2}, \frac{\beta}{\gamma}\right)$	Commonly used values (α, β, γ)	Equation name
$(\frac{3}{10}, 2)$	$(30, 20, 10), (120, 40, 20),$ $(270, 60, 30)$	Lax
$(\frac{1}{5}, 1)$	$(5, 5, 5), (180, 30, 30),$ $(45, 15, 15)$	Sawada-Kotera
$(\frac{1}{5}, \frac{5}{2})$	$(20, 25, 10)$	Kaup-Kupershmidt

- Example 5: Fifth-order equations – continued
- For $W(\mathcal{L}) = 2$, only Lax's equation has a Lax pair

$$\mathcal{L} = \mathrm{D}_x^2 + \tfrac{1}{10}\gamma u \, \mathrm{I}$$

$$\mathcal{M} = -16\,\mathrm{D}_x^5 - 4\gamma u\,\mathrm{D}_x^3 - 6\gamma u_x\,\mathrm{D}_x^2 - \gamma\left(5u_{xx} + \tfrac{3}{10}\gamma u^2\right)\mathrm{D}_x$$
$$\qquad - \gamma\left(\tfrac{3}{2}u_{3x} + \tfrac{3}{10}\gamma u u_x\right)\mathrm{I}$$

[P. Lax, Commun. Pure Appl. Math., 1968].

- Example 5: Fifth-order equations – continued
- For $W(\mathcal{L}) = 3$, the Sawada-Kotera and Kaup-Kupershmidt equations have Lax pairs.
- For the Kaup-Kupershmidt equation:

$$\mathcal{L} = \mathsf{D}_x^3 + \tfrac{1}{5}\gamma u \, \mathsf{D}_x + \tfrac{1}{10}\gamma u_x \, \mathrm{I}$$

$$\mathcal{M} = 9\,\mathsf{D}_x^5 + 3\gamma u \, \mathsf{D}_x^3 + \tfrac{9}{2}\gamma u_x \, \mathsf{D}_x^2 + \left(\tfrac{1}{5}\gamma^2 u^2 + \tfrac{7}{2}\gamma u_{xx}\right) + \left(\tfrac{1}{5}\gamma^2 u u_x + \gamma u_{3x}\right) \mathrm{I}$$

[A. Fordy & J. Gibbons, J. Math. Phys., 1980].

- Example 5: Fifth-order equations – continued
- For the Sawada-Kotera equation with $W(\mathcal{L}) = 3$:

$$\mathcal{L} = \mathsf{D}_x^3 + \tfrac{1}{5}\gamma u\,\mathsf{D}_x$$

$$\mathcal{M} = 9\,\mathsf{D}_x^5 + 3\gamma u\,\mathsf{D}_x^3 + 3\gamma u_x\,\mathsf{D}_x^2 + \left(\tfrac{1}{5}\gamma^2 u^2 + 2\gamma u_{2x}\right)\mathsf{D}_x$$

[R. K. Dodd & J. D. Gibbon, Proc. R. Soc. Lond. A, 1978].

Computations also resulted in:

$$\tilde{\mathcal{L}} = \mathsf{D}_x^3 + \tfrac{1}{5}\gamma u\,\mathsf{D}_x + \tfrac{1}{5}\gamma u_x\,\mathbf{I} \;= \mathrm{D}_x\,\mathcal{L}\,\mathrm{D}_x^{-1}$$

$$\tilde{\mathcal{M}} \;=\; 9\,\mathsf{D}_x^5 + 3\gamma u\,\mathsf{D}_x^3 + 6\gamma u_x\,\mathsf{D}_x^2 + \left(\tfrac{1}{5}\gamma^2 u^2 + 5\gamma u_{2x}\right)\mathsf{D}_x + \left(\tfrac{2}{5}\gamma^2 u u_x + 2\gamma u_{3x}\right)\mathbf{I} \;= \mathrm{D}_x\,\mathcal{M}\,\mathrm{D}_x^{-1}$$

Conclusions and Future Work

- Scaling invariant Lax pairs in operator form are fairly easy to construct.
- Scaling invariant Lax pairs in matrix form are hard to construct.
- Gauge equivalence: Which Lax pairs are useful, which ones are not?
- Compare with Wahlquist & Estabrook method, pseudo-differential operator method, etc.
- Implementation in Mathematica.

Thank You