Symbolic Computation of Lax Pairs of Two-Dimensional Nonlinear Partial Difference Equations

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Nonlinear Waves Seminar

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This presentation is made in `TeXpower`

Outline

- What are (nonlinear) maps in 1D and 2D?
- What are nonlinear PΔEs?
- Classification of integrable nonlinear PΔEs in 2D
- Lax pair of nonlinear PDEs
- Lax pair of nonlinear PΔEs
- Algorithm (Nijhoff 2001, Bobenko & Suris 2001)
- Software demonstration
- Additional examples
- Conclusions and future work

.

What are maps in 1D and 2D?

- Examples of maps in one dimension (1D):

  Example 1:

$$x_n = x_{n-1} + x_{n-2}, \quad x_n \in \mathbb{R}, \quad n \geq 2, \quad x_0 = x_1 = 1$$

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Fibonacci numbers: 1,1,2,3,5,8,13,21,...,

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  Fibonacci numbers: 1,1,2,3,5,8,13,21,...,

  Example 2:

  $$z_{n+1} = z_n^2 + c, \quad z_n \in \mathbb{C}, \quad c \in \mathbb{C}, \quad z_0 = 0$$

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Fibonacci numbers: 1,1,2,3,5,8,13,21,...,

Example 2:

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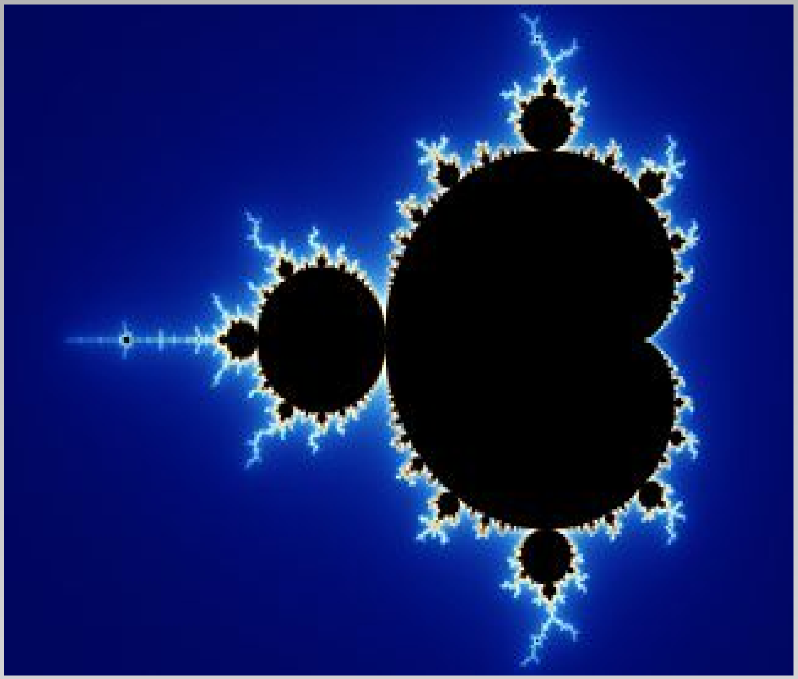
Mandelbrot set!



Benoit Mandelbrot

(1924-)

Mandelbrot set

Fractal image

Example 3:

$$x_{n+1} = r\, x_n(1 - x_n), \quad x_n \in \mathbb{R}^+, \quad x_0 = 0.2, \quad r \in [0, 4]$$

.

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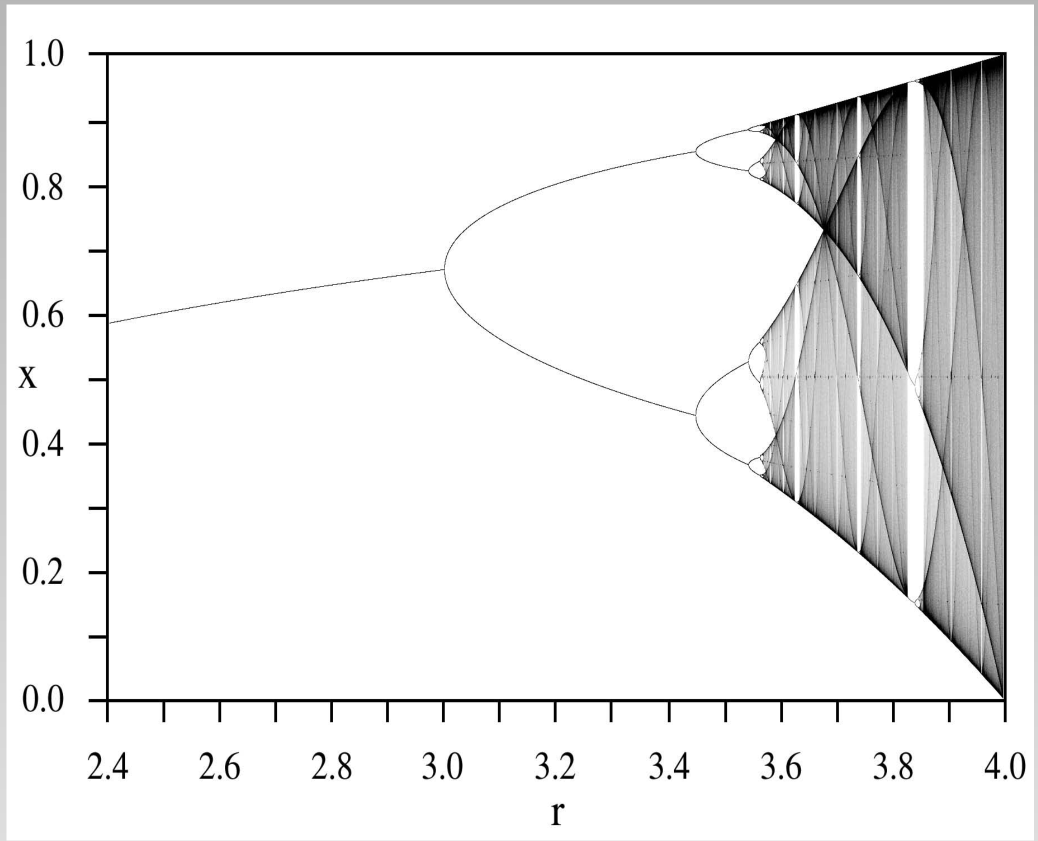
Logistic or Verhulst map!

As r increases: repeated period doubling and....
eventually chaos (when $r = 3.569946...$)



Bifurcation diagram for logistic map

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- Examples of maps in two-dimensions (2D):

Example 4:

$$\begin{aligned} x_{n+1} &= y_n + 1 - a\,x_n^2 \\ y_{n+1} &= b\,x_n, \quad a, b, x_n, y_n \in \mathbb{R}, \;\; x_0, y_0 \geq 0 \end{aligned}$$

.

- Examples of maps in two-dimensions (2D):

Example 4:

$$
\begin{aligned}
x_{n+1} &= y_n + 1 - a\,x_n^2 \\
y_{n+1} &= b\,x_n, \quad a, b, x_n, y_n \in \mathbb{R}, \;\; x_0, y_0 \geq 0
\end{aligned}
$$

Hénon map! Strange attractors!

.

Strange attractor for Hénon map

$(a = 1.4, b = 0.3)$

Example 5:

$$
\begin{aligned}
y_{n+1} &= \frac{1 - \sqrt[4]{1 - y_n^4}}{1 + \sqrt[4]{1 - y_n^4}} \\
a_{n+1} &= (1 + y_{n+1})^4 a_n - 2^{2n+3} y_{n+1}(1 + y_{n+1} + y_{n+1}^2)
\end{aligned}
$$

$$
\begin{aligned}
y_0 &= \sqrt{2} - 1 \\
a_0 &= 6 - 4\sqrt{2}
\end{aligned}
$$

.

Example 5:

$$
\begin{aligned}
y_{n+1} &= \frac{1-\sqrt[4]{1-y_n^4}}{1+\sqrt[4]{1-y_n^4}} \\
a_{n+1} &= (1+y_{n+1})^4 a_n - 2^{2n+3} y_{n+1}(1+y_{n+1}+y_{n+1}^2)
\end{aligned}
$$

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The π map!

.

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\end{aligned}
$$

$$
\begin{aligned}
y_0 &= \sqrt{2}-1 \\
a_0 &= 6-4\sqrt{2}
\end{aligned}
$$

The π map!

Compute:

$y_0 \longrightarrow y_1, (a_0, y_1) \longrightarrow a_1, \;\; y_1 \longrightarrow y_2, (a_1, y_2) \longrightarrow a_2,$ etc.

.

Then,

- $\frac{1}{a_1} = 3.1415926$ (8 digits)
- $\frac{1}{a_2} = 3.1415926535897932384626433832795028841971$ (41 digits)
- $\frac{1}{a_3} = \pi$ (171 digits)
- $\frac{1}{a_4} = \pi$ (694 digits)
- ....
- $\frac{1}{a_{15}} = \pi$ (2 billion digits)

Number of correct digits quadruples in each iteration

.

What are nonlinear PΔEs?

- Nonlinear maps with two lattice points!

  Origin: full discretizations of PDEs in 2D

.

- Example: discrete potential Korteweg-de Vries (pKdV) equation

$$\boxed{(v_{n,m} - v_{n+1,m+1})(v_{n+1,m} - v_{n,m+1}) - p^2 + q^2 = 0}$$

or

$$v_{n+1,m+1} = v_{n,m} - \frac{p^2 - q^2}{v_{n+1,m} - v_{n,m+1}}$$

- Notation:

v is dependent variable or field

n and m are lattice points

p and q are parameters

- For brevity,

$(v_{n,m}, v_{n+1,m}, v_{n,m+1}, v_{n+1,m+1}) = (x, x_1, x_2, x_{12})$

- discrete pKdV equation:

$$\boxed{(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0}$$

.

- Background: Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad \alpha \in \mathbb{R}$$

for $u(x,t)$, or in subscript notation

$$u_t + \alpha u u_x + u_{3x} = 0$$

- Potential form of the KdV equation:

Set $u = v_x$ and integrate with respect to x

$$v_t + \frac{\alpha}{2} v_x^2 + v_{xxx} = 0$$

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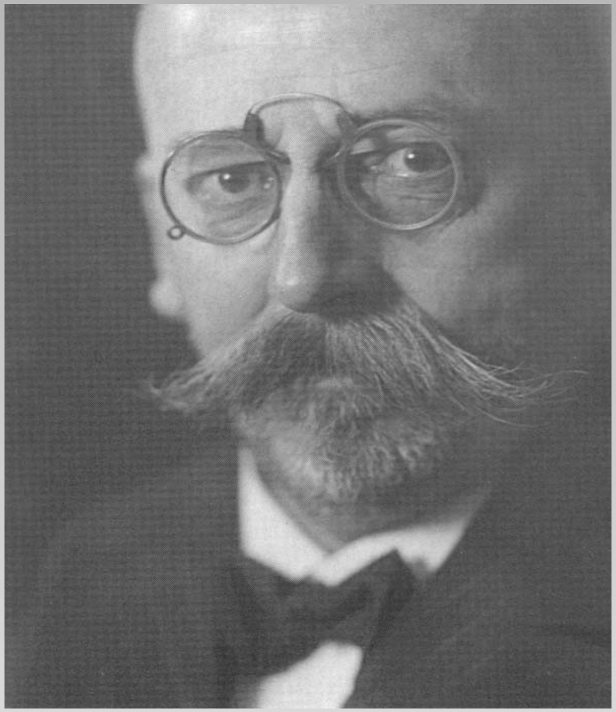
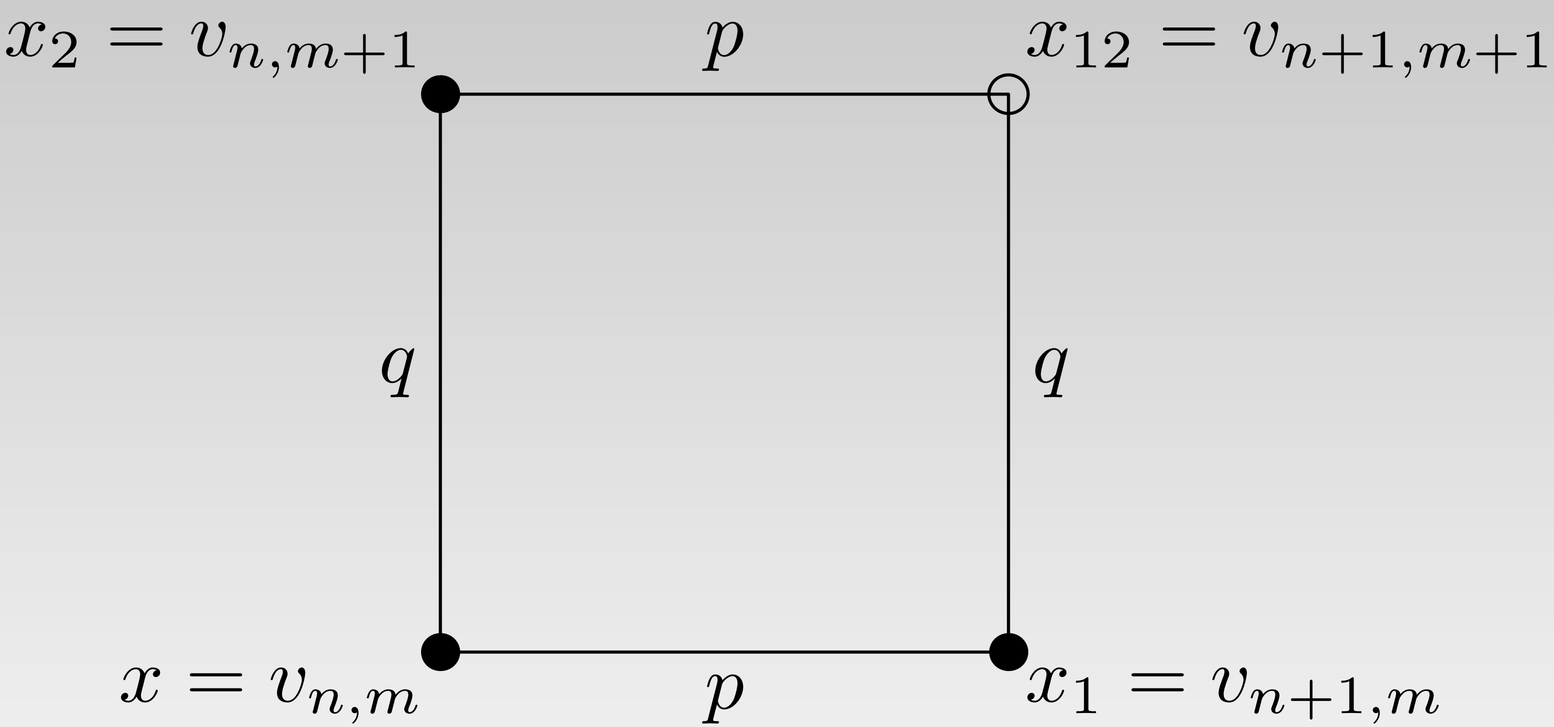


Diederik Korteweg Gustav de Vries

(1848-1941) (1866-1934)

$$\boxed{(v_{n,m} - v_{n+1,m+1})(v_{n+1,m} - v_{n,m+1}) - p^2 + q^2 = 0}$$

$$\boxed{(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0}$$



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Classification of 2D nonlinear integrable PΔEs

Adler, Bobenko, Suris (2003)

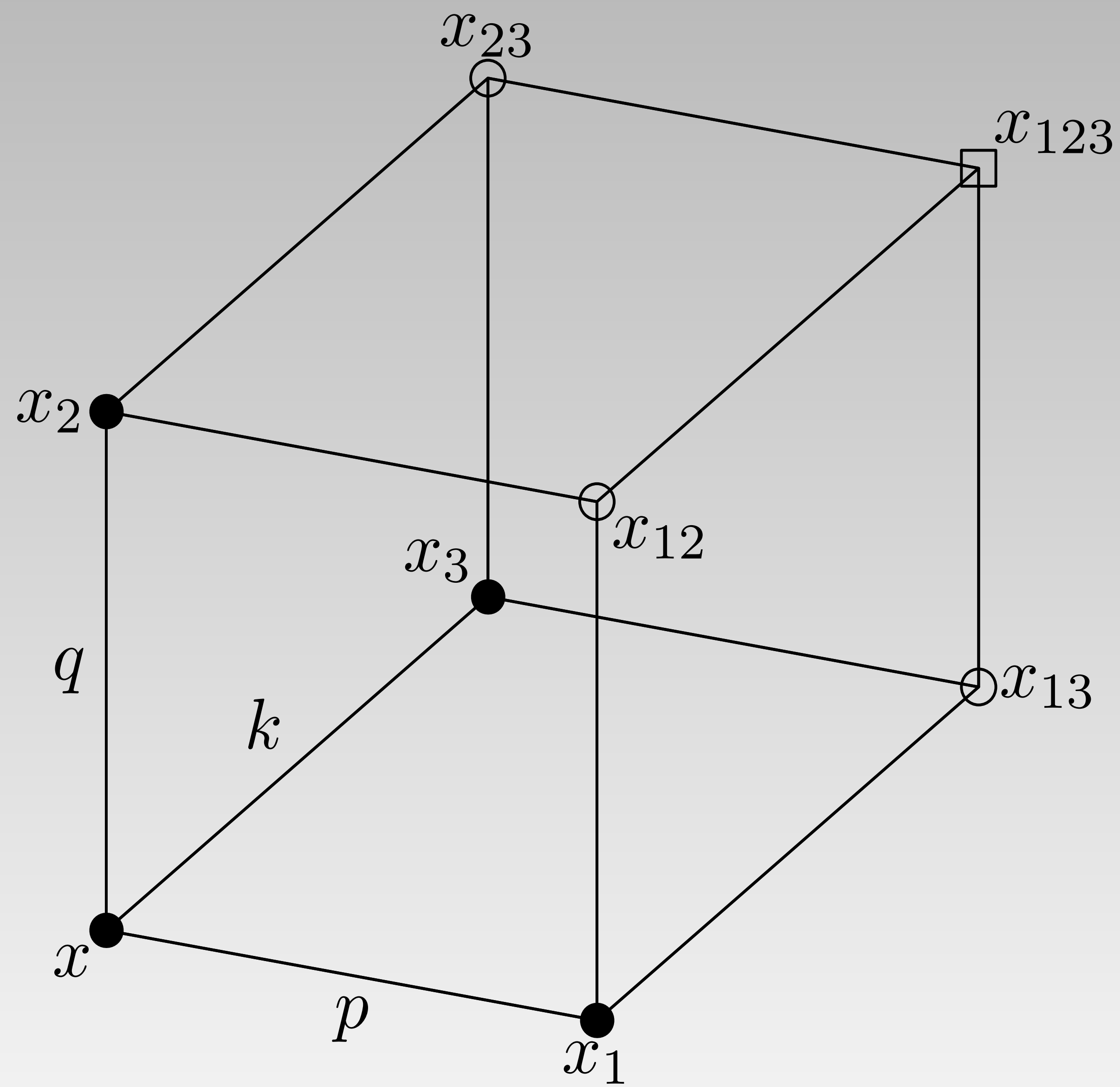
- Family of nonlinear PΔEs in two dimensions:

$$Q(x, x_1, x_2, x_{12}; p, q) = 0$$

- Assumpsions:
  1. Linear in each argument (affine linear):
$$\begin{aligned} Q(x, x_1, x_2, x_{12}; p, q) &= a_1 x x_1 x_2 x_{12} + a_2 x x_1 x_2 + \\ &\quad \ldots + a_{14} x_2 + a_{15} x_{12} + a_{16} \end{aligned}$$
  2. Invariant under D_4 (symmetries of square)
$$\begin{aligned} Q(x, x_1, x_2, x_{12}; p, q) &= \epsilon Q(x, x_2, x_1, x_{12}; q, p) \\ &= \sigma Q(x_1, x, x_{12}, x_2; p, q) \end{aligned}$$

$$\epsilon, \sigma = \pm 1$$

3. Consistency around the cube:



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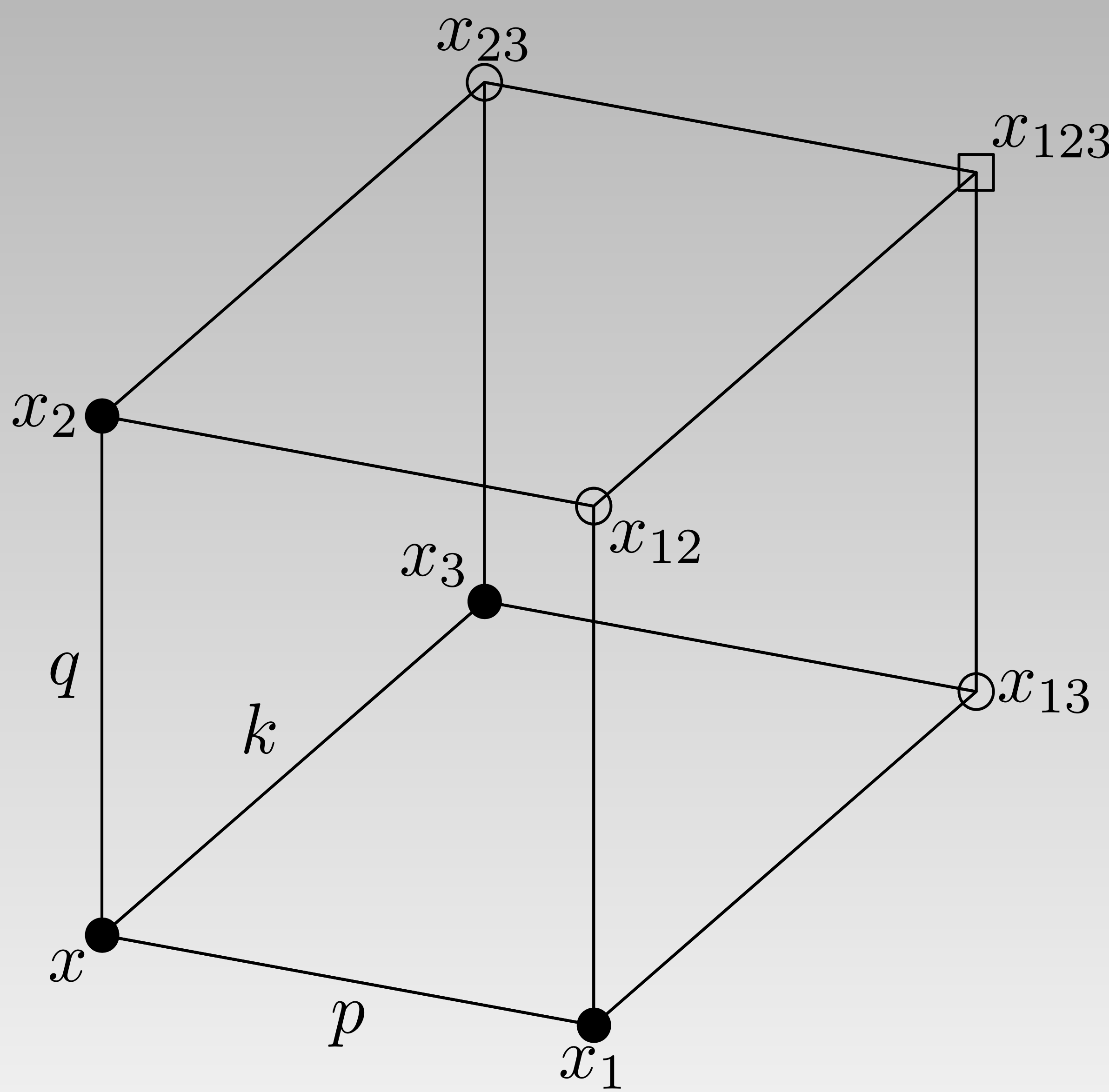
- Trick: Introduce a third lattice variable ℓ
- View v as dependent on three lattice points: n, m, ℓ. So, $x = v_{n,m} \longrightarrow x = v_{n,m,\ell}$
- Moves in three directions:

  $n \rightarrow n+1$ over distance p

  $m \rightarrow m+1$ over distance q

  $\ell \rightarrow \ell+1$ over distance k (spectral parameter)
- Require that the same lattice holds on front, bottom, and left face of the cube
- Require consistency for the computation of $x_{123} = v_{n+1,m+1,\ell+1}$


x_{23}
x_{123}
x_2
x_{12}
x_3
q
x_{13}
k
x
p
x_1

.

Verification of Consisteny Around the Cube

Example: discrete pKdV equation

⋆ Equation on front face of cube:

$$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

Solve for $x_{12} = x - \frac{p^2 - q^2}{x_1 - x_2}$

Compute x_{123} : $\quad x_{12} \longrightarrow x_{123} = x_3 - \frac{p^2 - q^2}{x_{13} - x_{23}}$

⋆ Equation on floor of cube:

$$(x - x_{13})(x_1 - x_3) - p^2 + k^2 = 0$$

Solve for $x_{13} = x - \frac{p^2 - k^2}{x_1 - x_3}$

Compute x_{123} : $\quad x_{13} \longrightarrow x_{123} = x_2 - \frac{p^2 - k^2}{x_{12} - x_{23}}$

⋆ Equation on left face of cube:

$$(x - x_{23})(x_3 - x_2) - k^2 + q^2 = 0$$

Solve for $x_{23} = x - \frac{q^2-k^2}{x_2-x_3}$

Compute x_{123} : $\quad x_{23} \longrightarrow x_{123} = x_1 - \frac{q^2-k^2}{x_{12}-x_{13}}$

⋆ Verify that all three coincide:

$$x_{123} = x_1 - \frac{q^2 - k^2}{x_{12} - x_{13}} = x_2 - \frac{p^2 - k^2}{x_{12} - x_{23}} = x_3 - \frac{p^2 - q^2}{x_{13} - x_{23}}$$

Upon substitution of x_{12}, x_{13}, and x_{23} :

$$x_{123} = \frac{p^2 x_1(x_2 - x_3) + q^2 x_2(x_3 - x_1) + k^2 x_3(x_1 - x_2)}{p^2(x_2 - x_3) + q^2(x_3 - x_1) + k^2(x_1 - x_2)}$$

Consistency around the cube is satisfied!

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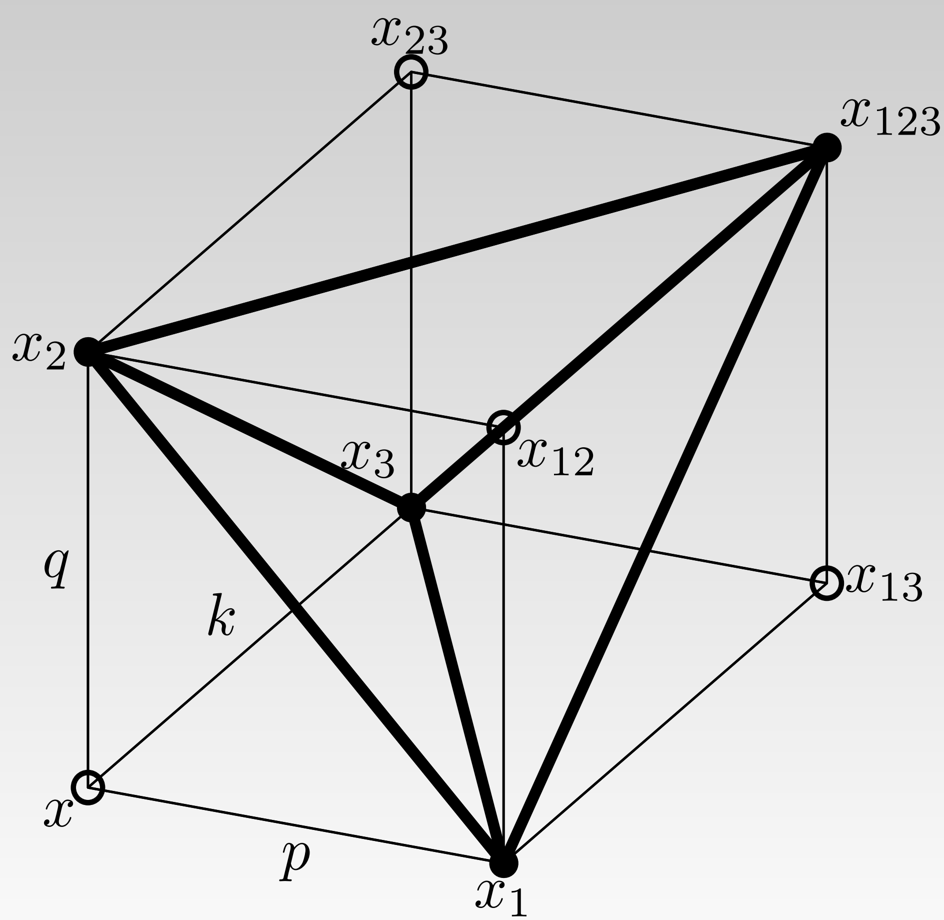
Tetrahedron property

$$x_{123} = \frac{p^2 x_1(x_2 - x_3) + q^2 x_2(x_3 - x_1) + k^2 x_3(x_1 - x_2)}{p^2(x_2 - x_3) + q^2(x_3 - x_1) + k^2(x_1 - x_2)}$$

is independent of x. Connects x_{123} to x_1, x_2 and x_3



Result of the ABS Classification

- List H
  - (H1) Equation

$$(x - x_{12})(x_1 - x_2) + q - p = 0$$

  - (H2) Equation

$$(x - x_{12})(x_1 - x_2) + (q - p)(x + x_1 + x_2 + x_{12}) + q^2 - p^2 = 0$$

  - (H3) Equation

$$p(xx_1 + x_2x_{12}) - q(xx_2 + xx_{12}) + \delta(p^2 - q^2) = 0$$

.

- List A
  - (A1) Equation

$$p(x+x_2)(x_1+x_{12})-q(x+x_1)(x_2+x_{12})-\delta^2 pq(p-q)=0$$

  - (A2) Equation

$$(q^2-p^2)(xx_1x_2x_{12}+1)+q(p^2-1)(xx_2+x_1x_{12})$$
$$-p(q^2-1)(xx_1+x_2x_{12})=0$$

.

- List Q
  - (Q1) Equation

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta^2 pq(p-q) = 0$$

  - (Q2) Equation

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta pq(p-q)$$
$$(x+x_1+x_2+x_{12})-\delta^2 pq(p-q)(p^2-pq+q^2)=0$$

  - (Q3) Equation

$$(q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12})$$
$$-p(q^2-1)(xx_2+x_1x_{12})-\frac{\delta^2}{4pq}(p^2-q^2)(p^2-1)(q^2-1)=0$$

.

- (Q4) Equation (mother)

$$
\begin{aligned}
&a_0 x x_1 x_2 x_{12} \\
&+a_1(x x_1 x_2 + x_1 x_2 x_{12} + x x_2 x_{12} + x x_1 x_{12}) \\
&+a_2(x x_{12} + x_1 x_2) + \bar{a}_2(x x_1 + x_2 x_{12}) \\
&+\tilde{a}_2(x x_2 + x_1 x_{12}) + a_3(x + x_1 + x_2 + x_{12}) + a_4 = 0
\end{aligned}
$$

the a_i depend on the lattice parameters (Weierstraß elliptic functions)

.

Lax Pair of Nonlinear PDEs

- Historical example: Korteweg-de Vries equation

$$\boxed{u_t + \alpha u u_x + u_{xxx} = 0}$$

- Lax equation: $\mathrm{L}_t + [\mathrm{L}, \mathrm{M}] = 0$ (on PDE)

  with commutator $[\mathrm{L}, \mathrm{M}] = \mathrm{LM} - \mathrm{ML}$

- Lax operators:

$$\boxed{L = \frac{\partial^2}{\partial x^2} + \frac{\alpha}{6} u}$$

$$\boxed{M = -4\frac{\partial^3}{\partial x^3} - \frac{\alpha}{2}\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x} u\right) + A(t)}$$

.

- Note: $\mathrm{L}_t\psi + [\mathrm{L},\mathrm{M}]\psi = \frac{\alpha}{6}\left(u_t + \alpha u u_x + u_{xxx}\right)\psi$
- Linear problem
  - $\star$ Sturm-Liouville equation: $\mathrm{L}\psi = \lambda\psi$

    For the KdV equation

$$\psi_{xx} + \left(\frac{\alpha}{6}u - \lambda\right)\psi = 0$$

  - $\star$ Time evolution of data: $\psi_t = \mathrm{M}\psi$
  - $\star$ Eigenvalues of L are constant: $\lambda_t = 0$

.

⋆ Compatibility of $\mathrm{L}\psi = \lambda\psi$ and $\psi_t = \mathrm{M}\psi$ gives

$$\mathrm{L}_t\psi + \mathrm{L}\psi_t = \lambda\psi_t$$

$$\begin{aligned} \mathrm{L}_t\psi + \mathrm{LM}\psi &= \lambda\mathrm{M}\psi \\ &= \mathrm{M}\lambda\psi \\ &= \mathrm{ML}\psi \end{aligned}$$

Thus,

$$\mathrm{L}_t\psi + (\mathrm{LM} - \mathrm{ML})\psi = \mathrm{O}$$

Lax equation:

$$\boxed{\mathrm{L}_t + [\mathrm{L}, \mathrm{M}] = 0 \ \ (\text{on PDE})}$$

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Peter D. Lax (1926-)

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Reasons to compute a Lax pair

- Replace nonlinear PDE by linear scattering problem and apply the IST
- Describe the time evolution of the scattering data
- Confirm the complete integrability of the PDE
- Zero-curvature representation of the PDE
- Compute conservation laws of the PDE
- Discover families of completely integrable PDEs

Question: How to find a Lax pair of a completely integrable PDE?

Answer: There is no completely systematic method

.

Lax Pair of Nonlinear PΔEs

- Require that $\psi_1 = \mathrm{L}\psi, \quad \psi_2 = \mathrm{M}\psi$

  Here L and M are 2×2 matrices and $\psi = \begin{bmatrix} f \\ g \end{bmatrix}$

  So, $\psi_1 = \psi_{n \to n+1}, \quad \psi_2 = \psi_{m \to m+1}$
- Compatibility:

$$\psi_{12} = \mathrm{L}_2\psi_2 = \mathrm{L}_2\mathrm{M}\psi$$

$$\psi_{12} = \mathrm{M}_1\psi_1 = \mathrm{M}_1\mathrm{L}\psi$$

  Hence, $\mathrm{L}_2\mathrm{M}\psi - \mathrm{M}_1\mathrm{L}\psi = 0$
- Lax equation:

$$\boxed{\mathrm{L}_2\mathrm{M} - \mathrm{M}_1\mathrm{L} = 0 \ \ (\text{on P}\Delta\text{E})}$$

.

- Example 1: Discrete pKdV equation

$$\boxed{(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0}$$

H1 after $p^2 \to p$, $q^2 \to q$

- Lax operators:

$$\boxed{L = t \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix}}$$

$$\boxed{M = s \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix}}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{k^2 - p^2}}$ and $s = \frac{1}{\sqrt{k^2 - q^2}}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = 1$

.

- Note: $\mathrm{L}_2\mathrm{M} - \mathrm{M}_1\mathrm{L} = \left((x - x_{12})(x_1 - x_2) - p^2 + q^2 \right) \mathrm{N}$

  with

$$\boxed{L = \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix}}$$

$$\boxed{M = \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix}}$$

$$\boxed{N = \frac{1}{\sqrt{(p^2 - k^2)(q^2 - k^2)}} \begin{bmatrix} -1 & x_1 + x_2 \\ 0 & 1 \end{bmatrix}}$$

.

- Example 2: Discrete modified KdV equation

$$\boxed{p(xx_2 - x_1x_{12}) - q(xx_1 - x_2x_{12}) = 0}$$

H3 for $\delta = 0$ and $x \to -x$ or $x_{12} \to -x_{12}$

- Lax operators:

$$\boxed{L = t\begin{bmatrix} -px & kxx_1 \\ k & -px_1 \end{bmatrix}}$$

$$\boxed{M = s\begin{bmatrix} -qx & kxx_2 \\ k & -qx_2 \end{bmatrix}}$$

with $t = \frac{1}{x_1}$ and $s = \frac{1}{x_2}$, or $t = s = \frac{1}{x}$

or $t = \frac{1}{\sqrt{(p^2-k^2)xx_1}}$ and $s = \frac{1}{\sqrt{(q^2-k^2)xx_2}}$

Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{x\,x_1}{x\,x_2} = \frac{x_1}{x_2}$

.

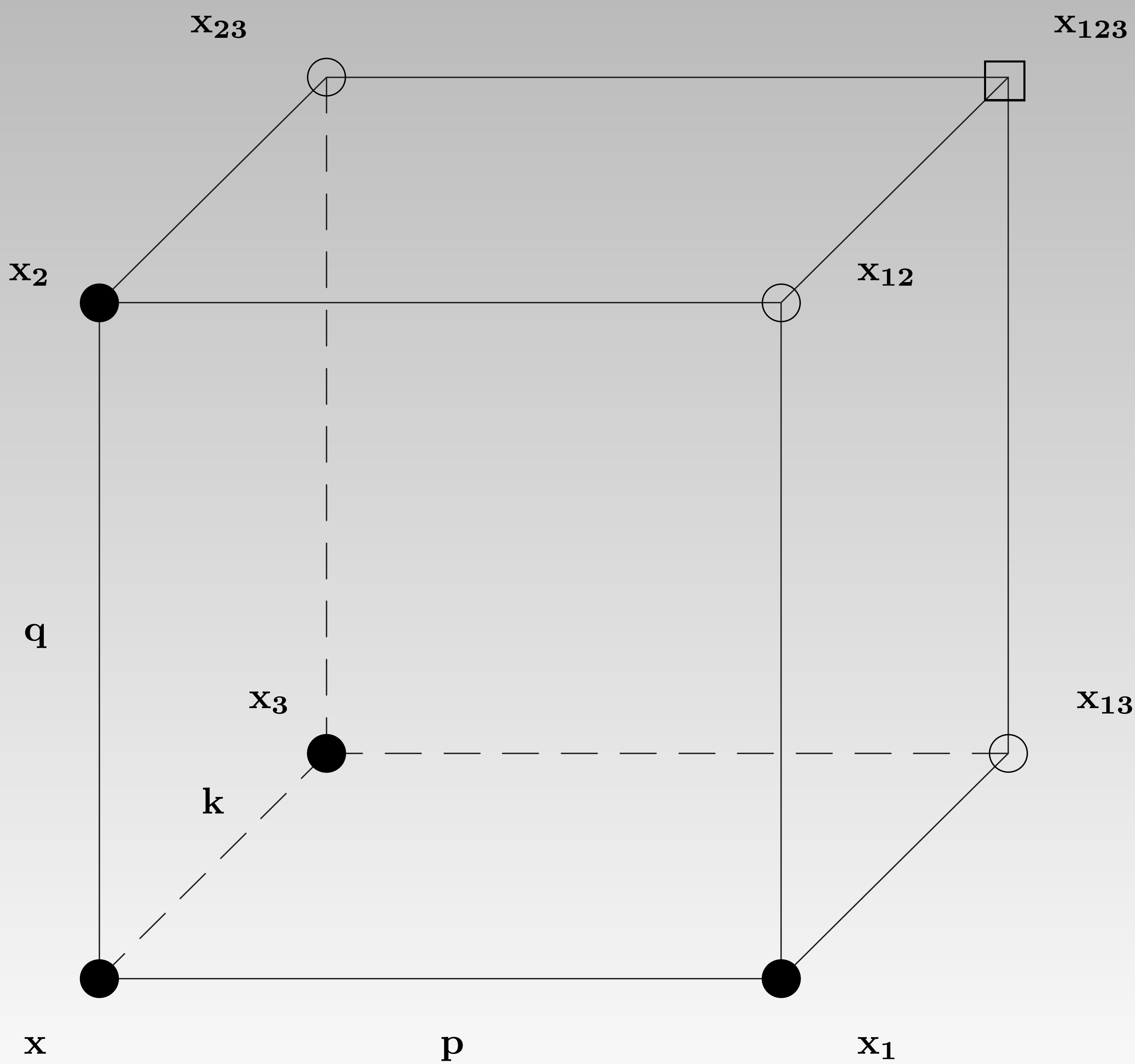
Algorithm to Compute a Lax Pair

(Nijhoff 2001, Bobenko & Suris 2001)

Applies to equations that are consistent on cube

Example: Discrete pKdV equation

- Step 1: Verify the consistency around the cube


x23
x123
x2
x12
q
x3
x13
k
x
p
x1

.

⋆ Equation on front face of cube:

$$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

Solve for $x_{12} = x - \frac{p^2 - q^2}{x_1 - x_2}$

Compute $x_{123} = x_3 - \frac{p^2 - q^2}{x_{13} - x_{23}}$

⋆ Equation on floor of cube:

$$(x - x_{13})(x_1 - x_3) - p^2 + k^2 = 0$$

Solve for $x_{13} = x - \frac{p^2 - k^2}{x_1 - x_3}$

Compute $x_{123} = x_2 - \frac{p^2 - k^2}{x_{12} - x_{23}}$

⋆ Equation on left face of cube:

$$(x - x_{23})(x_3 - x_2) - k^2 + q^2 = 0$$

Solve for $x_{23} = x - \frac{q^2 - k^2}{x_2 - x_3}$

Compute $x_{123} = x_1 - \frac{q^2 - k^2}{x_{12} - x_{13}}$

After substitution of x_{12}, x_{13}, and x_{23}

$$x_{123} = \frac{p^2 x_1(x_2 - x_3) + q^2 x_2(x_3 - x_1) + k^2 x_3(x_1 - x_2)}{p^2(x_2 - x_3) + q^2(x_3 - x_1) + k^2(x_1 - x_2)}$$

unique and independent of x (tetrahedron property)

Consistency around the cube is satisfied!

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- Step 2: Homogenization

Numerator and denominator of

$x_{13} = \frac{x_3 x - x x_1 + p^2 - k^2}{x_3 - x_1}$ and $x_{23} = \frac{x_3 x - x x_2 + q^2 - k^2}{x_3 - x_2}$

are linear in x_3

Substitute $x_3 = \frac{f}{g} \longrightarrow x_{13} = \frac{f_1}{g_1},\ x_{23} = \frac{f_2}{g_2}$.

From x_{13} : $\frac{f_1}{g_1} = \frac{xf + (p^2 - k^2 - x x_1) g}{f - x_1 g}$

Hence, $f_1 = t\left(xf + (p^2 - k^2 - x x_1) g\right)$ and $g_1 = t\,(f - x_1 g)$

or, in matrix form

$$\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = t \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

Matches $\psi_1 = \mathrm{L}\psi$ with $\psi = \begin{bmatrix} f \\ g \end{bmatrix}$

Similarly, from x_{23} :

$$\begin{bmatrix} f_2 \\ g_2 \end{bmatrix} = s \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

or $\psi_2 = \mathrm{M}\psi$. Therefore,

$$\boxed{L = t\, L_c = t \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix}}$$

$$\boxed{M = s\, M_c = s \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix}}$$

.

- Step 3: Determine t and s

$\star$ Substitute $L = tL_c, M = sM_c$ into $\mathrm{L_2 M - M_1 L} = 0$

$\longrightarrow t_2 s (L_c)_2 M_c - s_1 t (M_c)_1 L_c = 0$

$\star$ Solve the equation from the (2-1)-element for

$$\frac{t_2}{t}\,\frac{s}{s_1} = f(x, x_1, x_2, p, q, \ldots)$$

$\star$ If f factors as

$$f = \frac{\mathcal{F}(x, x_1, p, q, \ldots)\mathcal{G}(x, x_1, p, q, \ldots)}{\mathcal{F}(x, x_2, q, p, \ldots)\mathcal{G}(x, x_2, q, p, \ldots)}$$

then try $t = \frac{1}{\mathcal{F}(x, x_1, \ldots)}$ or $\frac{1}{\mathcal{G}(x, x_1, \ldots)}$ and $s = \frac{1}{\mathcal{F}(x, x_2, \ldots)}$ or $\frac{1}{\mathcal{G}(x, x_2, \ldots)}$

$\longrightarrow$ No square roots needed!

.

Works for the following lattices: mKdV, H3 with $\delta = 0$, Q1, Q3 with $\delta = 0$, (α, β)-equation.

Does not work for A1 and A2 equations!

Needs further investigation!

$\star$ If f does not factor, apply determinant to get

$$\frac{t_2}{t}\,\frac{s}{s_1} = \sqrt{\frac{\det L_c}{\det\,(L_c)_2}}\,\sqrt{\frac{\det\,(M_c)_1}{\det M_c}}$$

$\star$ A solution: $t = \frac{1}{\sqrt{\det L_c}}, \quad s = \frac{1}{\sqrt{\det M_c}}$

$\longrightarrow$ Introduces square roots!

.

How to avoid square roots?

Remedy: Apply a change of variables $x = F(X)$

$$\frac{t_2}{t}\frac{s}{s_1} = f(F(X), F(X_1), F(X_2), p, q, \dots)$$

$$= \frac{\mathcal{F}(X, X_1, p, q, \dots)\mathcal{G}(X, X_1, p, q, \dots)}{\mathcal{F}(X, X_2, q, p, \dots)\mathcal{G}(X, X_2, q, p, \dots)}$$

Example 1: Q2 lattice

$$\frac{t_2}{t}\frac{s}{s_1} = \frac{q\left((x-x_1)^2 - 2\delta p^2(x+x_1) + \delta^2 p^4\right)}{p\left((x-x_2)^2 - 2\delta q^2(x+x_2) + \delta^2 q^4\right)}$$

$$= \frac{q\left((X+X_1)^2 - \delta p^2\right)\left((X-X_1)^2 - \delta p^2\right)}{p\left((X+X_2)^2 - \delta q^2\right)\left((X-X_2)^2 - \delta q^2\right)}$$

after setting $x = F(X) = X^2$

.

Example 2: Q3 lattice

$$\frac{t_2}{t}\frac{s}{s_1}=\frac{q(q^2-1)\left(4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x^2+x_2^2)-4q(1+q^2)xx_2+\delta^2(1-q^2)^2\right)}$$

Set $x=F(X)=\delta\cosh(X)$ then

$$\begin{aligned}
&4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2\\
&=(p-\mathrm{e}^{X+X_1})(p-\mathrm{e}^{-(X+X_1)})(p-\mathrm{e}^{X-X_1})(p-\mathrm{e}^{-(X-X_1)})\\
&=(p-\cosh(X+X_1)+\sinh(X+X_1))\\
&\quad(p-\cosh(X+X_1)-\sinh(X+X_1))\\
&\quad(p-\cosh(X-X_1)+\sinh(X-X_1))\\
&\quad(p-\cosh(X-X_1)-\sinh(X+X_1))
\end{aligned}$$

.

Equivalence under Gauge Transformations

Lax pairs are equivalent under a gauge transformation

If L and M form a Lax pair then so do

$\mathcal{L} = \mathrm{G}_1 \mathrm{L} \mathrm{G}^{-1}$ and $\mathcal{M} = \mathrm{G}_2 \mathrm{M} \mathrm{G}^{-1}$

where G is non-singular diagonal matrix (or scalar factor)

and $\phi = \mathrm{G}\psi$

Proof: Trivial verification that

$(\mathcal{L}_2 \mathcal{M} - \mathcal{M}_1 \mathcal{L})\,\phi = 0 \leftrightarrow (\mathrm{L}_2 \mathrm{M} - \mathrm{M}_1 \mathrm{L})\,\psi = 0$

Software Demonstration

.

Additional Examples

- Example 3: H1 equation (ABS classification)

$$\boxed{(x - x_{12})(x_1 - x_2) + q - p = 0}$$

- Lax operators:

$$\boxed{L = t \begin{bmatrix} x & p - k - x x_1 \\ 1 & -x_1 \end{bmatrix}}$$

$$\boxed{M = s \begin{bmatrix} x & q - k - x x_2 \\ 1 & -x_2 \end{bmatrix}}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{k-p}}$ and $s = \frac{1}{\sqrt{k-q}}$

Note: $\frac{t_2}{t}\frac{s}{s_1} = 1$

.

- Example 4: H2 equation (ABS 2003)

$$\boxed{(x-x_{12})(x_1-x_2)+(q-p)(x+x_1+x_2+x_{12})+q^2-p^2=0}$$

- Lax operators:

$$\boxed{L = t \begin{bmatrix} p-k+x & p^2-k^2+(p-k)(x+x_1)-xx_1 \\ 1 & -(p-k+x_1) \end{bmatrix}}$$

$$\boxed{M = s \begin{bmatrix} q-k+x & q^2-k^2+(q-k)(x+x_2)-xx_2 \\ 1 & -(q-k+x_2) \end{bmatrix}}$$

with $t = \frac{1}{\sqrt{2(k-p)(p+x+x_1)}}$ and $s = \frac{1}{\sqrt{2(k-q)(q+x+x_2)}}$

Note: $\frac{t_2}{t}\frac{s}{s_1} = \frac{p+x+x_1}{q+x+x_2}$

.

- Example 5: H3 equation (ABS 2003)

$$\boxed{p(xx_1 + x_2x_{12}) - q(xx_2 + xx_{12}) + \delta(p^2 - q^2) = 0}$$

- Lax operators:

$$\boxed{L = t\begin{bmatrix} kx & -\left(\delta(p^2 - k^2) + pxx_1\right) \\ p & -kx_1 \end{bmatrix}}$$

$$\boxed{M = s\begin{bmatrix} kx & -\left(\delta(q^2 - k^2) + qxx_2\right) \\ q & -kx_2 \end{bmatrix}}$$

with $t = \frac{1}{\sqrt{(p^2-k^2)(\delta p + xx_1)}}$ and $s = \frac{1}{\sqrt{(q^2-k^2)(\delta q + xx_2)}}$

Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{\delta p + xx_1}{\delta q + xx_2}$

.

- Example 6: H3 equation with $\delta = 0$ (ABS 2003)

$$\boxed{p(xx_1 + x_2x_{12}) - q(xx_2 + xx_{12}) = 0}$$

- Lax operators:

$$\boxed{L = t \begin{bmatrix} kx & -pxx_1 \\ p & -kx_1 \end{bmatrix}}$$

$$\boxed{M = s \begin{bmatrix} kx & -qxx_2 \\ q & -kx_2 \end{bmatrix}}$$

with $t = s = \frac{1}{x}$ or $t = \frac{1}{x_1}$ and $s = \frac{1}{x_2}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{x\, x_1}{x\, x_2} = \frac{x_1}{x_2}$

- Example 7: Q1 equation (ABS 2003)

$$\boxed{p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta^2 pq(p-q)=0}$$

- Lax operators:

$$\boxed{L=t\begin{bmatrix}(p-k)x_1+kx & -p\left(\delta^2k(p-k)+xx_1\right)\\ p & -\left((p-k)x+kx_1\right)\end{bmatrix}}$$

$$\boxed{M=s\begin{bmatrix}(q-k)x_2+kx & -q\left(\delta^2k(q-k)+xx_2\right)\\ q & -\left((q-k)x+kx_2\right)\end{bmatrix}}$$

with $t=\frac{1}{\delta p\pm(x-x_1)}$ and $s=\frac{1}{\delta q\pm(x-x_2)}$,
or $t=\frac{1}{\sqrt{k(p-k)\left((\delta p+x-x_1)(\delta p-x+x_1)\right)}}$ and

$$s=\frac{1}{\sqrt{k(q-k)\left((\delta q+x-x_2)(\delta q-x+x_2)\right)}}$$

Note: $\frac{t_2}{t}\,\frac{s}{s_1}=\frac{q\left(\delta p+(x-x_1)\right)\left(\delta p-(x-x_1)\right)}{p\left(\delta q+(x-x_2)\right)\left(\delta q-(x-x_2)\right)}$

- Example 8: Q1 equation with $\delta = 0$ (ABS 2003)

$$\boxed{p(x - x_2)(x_1 - x_{12}) - q(x - x_1)(x_2 - x_{12}) = 0}$$

which is the cross-ratio equation

$$\frac{(x - x_1)(x_{12} - x_2)}{(x_1 - x_{12})(x_2 - x)} = \frac{p}{q}$$

- Lax operators:

$$\boxed{L = t \begin{bmatrix} (p-k)x_1 + kx & -pxx_1 \\ p & -((p-k)x + kx_1) \end{bmatrix}}$$

$$\boxed{M = s \begin{bmatrix} (q-k)x_2 + kx & -qxx_2 \\ q & -((q-k)x + kx_2) \end{bmatrix}}$$

Here, $\frac{t_2}{t}\frac{s}{s_1} = \frac{q(x-x_1)^2}{p(x-x_2)^2}$. So, $t = \frac{1}{x-x_1}$ and $s = \frac{1}{x-x_2}$ or $t = \frac{1}{\sqrt{k(k-p)(x-x_1)}}$ and $s = \frac{1}{\sqrt{k(k-q)(x-x_2)}}$

- Example 9: Q2 equation (ABS 2003)

$$\boxed{\begin{array}{l} p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta pq(p-q) \\ (x+x_1+x_2+x_{12})-\delta^2 pq(p-q)(p^2-pq+q^2)=0 \end{array}}$$

- Lax operators:

$$\boxed{L=t\begin{bmatrix} (k-p)(\delta kp-x_1)+kx & -p\left(\delta k(k-p)(\delta k^2-\delta kp+\delta p^2-x-x_1)+xx_1\right) \\ p & -\left((k-p)(\delta kp-x)+kx_1\right) \end{bmatrix}}$$

$$\boxed{M=s\begin{bmatrix} (k-q)(\delta kq-x_2)+kx & -q\left(\delta k(k-q)(\delta k^2-\delta kq+\delta q^2-x-x_2)+xx_2\right) \\ q & -\left((k-q)(\delta kq-x)+kx_2\right) \end{bmatrix}}$$

.

- with

$$t = \frac{1}{\sqrt{k(k-p)\left((x-x_1)^2-2\delta p^2(x+x_1)+\delta^2 p^4\right)}}$$

and

$$s = \frac{1}{\sqrt{k(k-q)\left((x-x_2)^2-2\delta q^2(x+x_2)+\delta^2 q^4\right)}}$$

Note:

$$\begin{aligned}
\frac{t_2}{t}\,\frac{s}{s_1} &= \frac{q\left((x-x_1)^2-2\delta p^2(x+x_1)+\delta^2 p^4\right)}{p\left((x-x_2)^2-2\delta q^2(x+x_2)+\delta^2 q^4\right)} \\
&= \frac{p\left((X+X_1)^2-\delta p^2\right)\left((X-X_1)^2-\delta p^2\right)}{q\left((X+X_2)^2-\delta q^2\right)\left((X-X_2)^2-\delta q^2\right)}
\end{aligned}$$

with $x = X^2$, and, consequently, $x_1 = X_1^2$, $x_2 = X_2^2$

- Example 10: Q3 equation (ABS 2003)

$$\boxed{\begin{aligned}&(q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12})\\&-p(q^2-1)(xx_2+x_1x_{12})-\frac{\delta^2}{4pq}(p^2-q^2)(p^2-1)(q^2-1)=0\end{aligned}}$$

- Lax operators:

$$\boxed{L=t\begin{bmatrix}-4kp\left(p(k^2-1)x+(p^2-k^2)x_1\right) & \begin{array}{r}-(p^2-1)(\delta k^2-\delta^2k^4-\delta^2p^2+\delta^2k^2p^2-4k^2pxx_1)\end{array}\\ -4k^2p(p^2-1) & 4kp\left(p(k^2-1)x_1+(p^2-k^2)x\right)\end{bmatrix}}$$

$$\boxed{M=s\begin{bmatrix}-4kq\left(q(k^2-1)x+(q^2-k^2)x_2\right) & \begin{array}{r}-(q^2-1)(\delta k^2-\delta^2k^4-\delta^2q^2+\delta^2k^2q^2-4k^2qxx_2)\end{array}\\ -4k^2q(q^2-1) & 4kq\left(q(k^2-1)x_2+(q^2-k^2)x\right)\end{bmatrix}}$$

.

- with

$$t = \frac{1}{2k\sqrt{p(k^2-1)(k^2-p^2)\left(4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2\right)}}$$

and

$$s = \frac{1}{2k\sqrt{q(k^2-1)(k^2-q^2)\left(4q^2(x^2+x_2^2)-4q(1+q^2)xx_2+\delta^2(1-q^2)^2\right)}}$$

Note:

$$
\begin{aligned}
&\frac{t_2}{t}\,\frac{s}{s_1}\\
&= \frac{q(q^2-1)\left(4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x^2+x_2^2)-4q(1+q^2)xx_2+\delta^2(1-q^2)^2\right)}\\
&= \frac{q(q^2-1)\left(4p^2(x-x_1)^2-4p(p-1)^2xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x-x_2)^2-4q(q-1)^2xx_2+\delta^2(1-q^2)^2\right)}\\
&= \frac{q(q^2-1)\left(4p^2(x+x_1)^2-4p(p+1)^2xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x+x_2)^2-4q(q+1)^2xx_2+\delta^2(1-q^2)^2\right)}
\end{aligned}
$$

where

$$
\begin{aligned}
&4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2\\
&=(p-\mathrm{e}^{X+X_1})(p-\mathrm{e}^{-(X+X_1)})(p-\mathrm{e}^{X-X_1})(p-\mathrm{e}^{-(X-X_1)})\\
&=(p-\cosh(X+X_1)+\sinh(X+X_1))\\
&\quad(p-\cosh(X+X_1)-\sinh(X+X_1))\\
&\quad(p-\cosh(X-X_1)+\sinh(X-X_1))\\
&\quad(p-\cosh(X-X_1)-\sinh(X+X_1))
\end{aligned}
$$

with $x = \cosh(X)$, and, consequently,

$x_1 = \cosh(X_1),\ x_2 = \cosh(X_2)$

.

- Example 11: Q3 equation with $\delta = 0$ (ABS 2003)

$$\boxed{\begin{aligned}&(q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12})\\&-p(q^2-1)(xx_2+x_1x_{12})=0\end{aligned}}$$

- Lax operators:

$$\boxed{L=t\begin{bmatrix}(p^2-k^2)x_1+p(k^2-1)x & -k(p^2-1)xx_1\\(p^2-1)k & -\left((p^2-k^2)x+p(k^2-1)x_1\right)\end{bmatrix}}$$

$$\boxed{M=s\begin{bmatrix}(q^2-k^2)x_2+q(k^2-1)x & -k(q^2-1)xx_2\\(q^2-1)k & -\left((q^2-k^2)x+q(k^2-1)x_2\right)\end{bmatrix}}$$

.

- with $t = \frac{1}{px - x_1}$ and $s = \frac{1}{qx - x_2}$

  or $t = \frac{1}{px_1 - x}$ and $s = \frac{1}{qx_2 - x}$

  or $t = \frac{1}{\sqrt{(k^2-1)(p^2-k^2)(px-x_1)(px_1-x)}}$

  and $s = \frac{1}{\sqrt{(k^2-1)(q^2-k^2)(qx-x_2)(qx_2-x)}}$

  Note: $\frac{t_2}{t}\frac{s}{s_1} = \frac{(q^2-1)(px-x_1)(px_1-x)}{(p^2-1)(qx-x_2)(qx_2-x)}$

.

- Example 12: (α,β)-equation (Tran)

$$\boxed{\begin{aligned}&\big((p-\alpha)x-(p+\beta)x_1\big)\big((p-\beta)x_2-(p+\alpha)x_{12}\big)\\ &-\big((q-\alpha)x-(q+\beta)x_2\big)\big((q-\beta)x_1-(q+\alpha)x_{12}\big)=0\end{aligned}}$$

- Lax operators:

$$\boxed{L=t\begin{bmatrix}(p-\alpha)(p-\beta)x+(k^2-p^2)x_1 & -(k-\alpha)(k-\beta)xx_1\\ (k+\alpha)(k+\beta) & -\big((p+\alpha)(p+\beta)x_1+(k^2-p^2)x\big)\end{bmatrix}}$$

$$\boxed{M=s\begin{bmatrix}(q-\alpha)(q-\beta)x+(k^2-q^2)x_2 & -(k-\alpha)(k-\beta)xx_2\\ (k+\alpha)(k+\beta) & -\big((q+\alpha)(q+\beta)x_2+(k^2-q^2)x\big)\end{bmatrix}}$$

.

- with $t = \frac{1}{(\alpha-p)x+(\beta+p)x_1)}$ and $s = \frac{1}{(\alpha-q)x+(\beta+q)x_2)}$

  or $t = \frac{1}{(\beta-p)x+(\alpha+p)x_1)}$ and $s = \frac{1}{(\beta-q)x+(\alpha+q)x_2)}$

  or $t = \frac{1}{\sqrt{(p^2-k^2)\big((\beta-p)x+(\alpha+p)x_1\big)\big((\alpha-p)x+(\beta+p)x_1\big)}}$

  and $s = \frac{1}{\sqrt{(q^2-k^2)\big((\beta-q)x+(\alpha+q)x_2\big)\big((\alpha-q)x+(\beta+q)x_2\big)}}$

  Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{\big((\beta-p)x+(\alpha+p)x_1\big)\big((\alpha-p)x+(\beta+p)x_1\big)}{\big((\beta-q)x+(\alpha+q)x_2\big)\big((\alpha-q)x+(\beta+q)x_2\big)}$

- Example 13: A1 equation (ABS 2003)

$$\boxed{p(x+x_2)(x_1+x_{12})-q(x+x_1)(x_2+x_{12})-\delta^2 pq(p-q)=0}$$

Q1 if $x_1 \longrightarrow -x_1$ and $x_2 \longrightarrow -x_2$

- Lax operators:

$$\boxed{L = t\begin{bmatrix} (k-p)x_1 + kx & -p\left(\delta^2 k(k-p) + xx_1\right) \\ p & -\left((k-p)x + kx_1\right) \end{bmatrix}}$$

$$\boxed{M = s\begin{bmatrix} (k-q)x_2 + kx & -q\left(\delta^2 k(k-q) + xx_2\right) \\ q & -\left((k-q)x + kx_2\right) \end{bmatrix}}$$

.

- with $t = \frac{1}{\sqrt{k(k-p)\big((\delta p+x+x_1)(\delta p-x-x_1)\big)}}$ and $s = \frac{1}{\sqrt{k(k-q)\big((\delta q+x+x_2)(\delta q-x-x_2)\big)}}$

  Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{q\big(\delta p+(x+x_1)\big)\big(\delta p-(x+x_1)\big)}{p\big(\delta q+(x+x_2)\big)\big(\delta q-(x+x_2)\big)}$

  However, the choices $t = \frac{1}{\delta p \pm (x+x_1)}$ and $s = \frac{1}{\delta q \pm (x+x_2)}$ CANNOT be used.

  This lattice needs further investigation!

.

- Example 14: A2 equation (ABS 2003)

$$\boxed{\begin{aligned}&(q^2-p^2)(xx_1x_2x_{12}+1)+q(p^2-1)(xx_2+x_1x_{12})\\&-p(q^2-1)(xx_1+x_2x_{12})=0\end{aligned}}$$

Q3 with $\delta = 0$ via Möbius transformation:

$$x \longrightarrow x, x_1 \longrightarrow \frac{1}{x_1}, x_2 \longrightarrow \frac{1}{x_2}, x_{12} \longrightarrow x_{12}, p \longrightarrow p, q \longrightarrow q$$

- Lax operators:

$$\boxed{L = t\begin{bmatrix} k(p^2-1)x & -\left(p^2-k^2+p(k^2-1)xx_1\right) \\ p(k^2-1)+(p^2-k^2)xx_1 & -k(p^2-1)x_1 \end{bmatrix}}$$

$$\boxed{M = s\begin{bmatrix} k(q^2-1)x & -\left(q^2-k^2+q(k^2-1)xx_2\right) \\ q(k^2-1)+(q^2-k^2)xx_2 & -k(q^2-1)x_2 \end{bmatrix}}$$

.

- with $t = \frac{1}{\sqrt{(k^2-1)(k^2-p^2)(p-xx_1)(pxx_1-1)}}$

  and $s = \frac{1}{\sqrt{(k^2-1)(k^2-q^2)(q-xx_2)(qxx_2-1)}}$

  Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{(q^2-1)(p-xx_1)(pxx_1-1)}{(p^2-1)(q-xx_2)(qxx_2-1)}$

  However, the choices $t = \frac{1}{p-xx_1}$ and $s = \frac{1}{q-xx_2}$

  or $t = \frac{1}{pxx_1-1}$ and $s = \frac{1}{qxx_2-1}$

  CANNOT be used.

  This lattice needs further investigation!

.

- Example 15: Discrete sine-Gordon equation

$$\boxed{xx_1x_2x_{12} - pq(xx_{12} - x_1x_2) - 1 = 0}$$

H3 with $\delta = 0$ via extended Möbius transformation:

$x \longrightarrow x, x_1 \longrightarrow x_1, x_2 \longrightarrow \frac{1}{x_2}, x_{12} \longrightarrow -\frac{1}{x_{12}}, p \longrightarrow p, q \longrightarrow \frac{1}{q}$

Discrete sine-Gordon equation is NOT consistent around the cube, but has a Lax pair!

- Lax operators:

$$\boxed{L = \begin{bmatrix} p & -kx_1 \\ -\frac{k}{x} & \frac{px_1}{x} \end{bmatrix}}$$

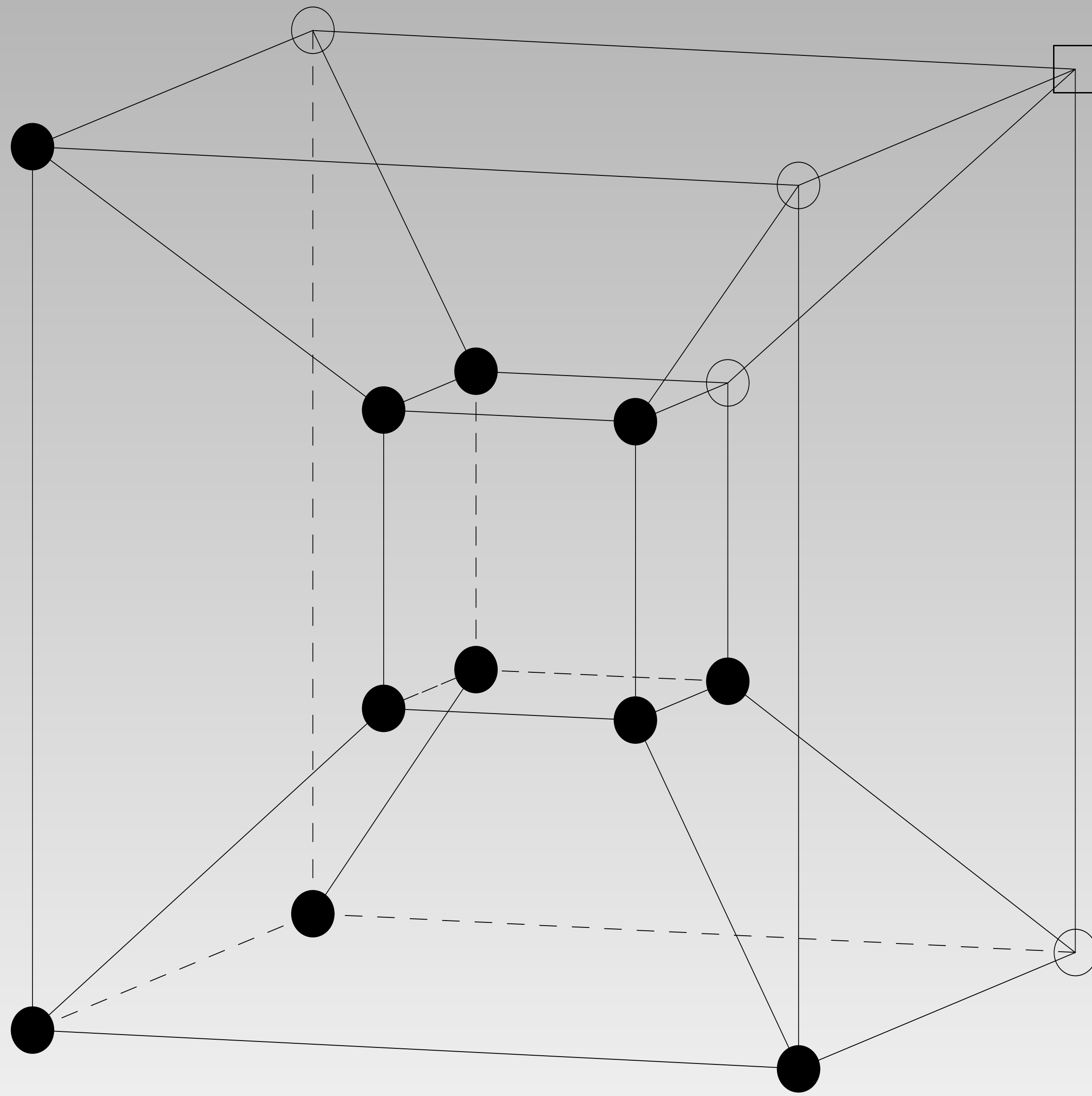
$$\boxed{M = \begin{bmatrix} \frac{qx_2}{x} & -\frac{1}{kx} \\ -\frac{x_2}{k} & q \end{bmatrix}}$$

Conclusions and Future Work

- *Mathematica* code works for $\mathbf{P\Delta E}s$ in 2D defined on quad-graphs (quadrilateral faces)
- Code can be used to test (i) consistency around the cube and compute or test (ii) Lax pairs
- Consistency around cube $\Longrightarrow$ $\mathrm{P\Delta E}$ has Lax pair
- $\mathrm{P\Delta E}$ has Lax pair $\not\Rightarrow$ consistency around cube. Indeed, there are $\mathrm{P\Delta Es}$ with a Lax pair that are not consistent around the cube.
  Example: discrete sine-Gordon equation

.

- Avoid the determinant method to avoid square roots! Factorization plays an essential role!
- Hard case: $Q4$ equation (elliptic curves, Weierstraß functions) (Nijhoff, 2001)
- $\mathbf{P\Delta}\mathrm{E}s$ in 3D: Lax pair will be expressed in terms of tensors. Consistency around a “hypercube”. Examples: discrete Kadomtsev-Petviashvili (KP) equations.



Thank You