

Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations

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Group30: XXXth International Colloquium on

Group Theoretical Methods in Physics

Ghent University, Belgium

Friday, July 18, 2014, 16:00

Acknowledgements

Douglas Poole (Chadron State College, Nebraska)

Ünal Gökteş (Turgut Özal University, Turkey)

Mark Hickman (Univ. of Canterbury, New Zealand)

Bernard Deconinck (Univ. of Washington, Seattle)

Several undergraduate and graduate students

Research supported in part by NSF
under Grant No. CCF-0830783

This presentation was made in TeXpower

Outline

- Why compute conservation laws?
- The Korteweg-de Vries and Zakharov-Kuznetsov equations
- Demonstration of *ConservationLawsMD.m*
- Algorithm for computing conservation laws
- Tools from the calculus of variations
- Application to the Zakharov-Kuznetsov equation
- Conclusions and future work

- Additional examples
 - ▶ Manakov-Santini system
 - ▶ Camassa-Holm equation in $(2+1)$ dimensions
 - ▶ Khoklov-Zabolotskaya equation
 - ▶ Shallow water wave model for atmosphere
 - ▶ Kadomtsev-Petviashvili equation
 - ▶ Potential Kadomtsev-Petviashvili equation
 - ▶ Generalized Zakharov-Kuznetsov equation

Notation – Computations on the Jet Space

- Independent variables: t (time), $\mathbf{x} = (x, y, z)$
- Dependent variables $\mathbf{u} = (u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(N)})$
In examples: $\mathbf{u} = (u, v, \theta, h, \dots)$

- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kxly} = \frac{\partial^{k+l} u}{\partial x^k \partial y^l}$, etc.

Examples: $u_{xxxxx} = u_{5x} = \frac{\partial^5 u}{\partial x^5}$

$$u_{xxyyyy} = u_{2x4y} = \frac{\partial^6 u}{\partial x^2 \partial y^4}$$

- *Differential functions*

Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

- *Total derivatives:* $D_t, D_x, D_y, \dots$

Example: Let $f = uvv_x + x^2u_x^3v_x + u_xv_{xx}$

Then

$$\begin{aligned}
 D_x f &= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{xx} \frac{\partial f}{\partial u_x} \\
 &\quad + v_x \frac{\partial f}{\partial v} + v_{xx} \frac{\partial f}{\partial v_x} + v_{xxx} \frac{\partial f}{\partial v_{xx}} \\
 &= 2xu_x^3v_x + u_x(vv_x) + u_{xx}(3x^2u_x^2v_x + v_{xx}) \\
 &\quad + v_x(uv_x) + v_{xx}(uv + x^2u_x^3) + v_{xxx}(u_x) \\
 &= 2xu_x^3v_x + vu_xv_x + 3x^2u_x^2v_xu_{xx} + u_{xx}v_{xx} \\
 &\quad + uv_x^2 + uvv_{xx} + x^2u_x^3v_{xx} + u_xv_{xxx}
 \end{aligned}$$

Conservation Laws for Nonlinear PDEs

- System of evolution equations of order M

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}^{(M)}(\mathbf{x}))$$

with $\mathbf{u} = (u, v, w, \dots)$ and $\mathbf{x} = (x, y, z)$.

- Conservation law in (1+1)-dimensions

$$D_t \rho + D_x J \doteq 0$$

where the dot means evaluated on the PDE.
Conserved density ρ and flux J .

$$P = \int_{-\infty}^{\infty} \rho dx = \text{constant in time}$$

if J vanishes at $\pm\infty$.

- Conservation law in (2+1)-dimensions

$$\boxed{D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 \doteq 0}$$

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$.

- Conservation law in (3+1)-dimensions

$$\boxed{D_t \rho + \nabla \cdot \mathbf{J} = D_t \rho + D_x J_1 + D_y J_2 + D_z J_3 \doteq 0}$$

Conserved density ρ and flux $\mathbf{J} = (J_1, J_2, J_3)$.

Reasons for Computing Conservation Laws

- Conservation of physical quantities (linear momentum, mass, energy, electric charge, ...).
- Testing of complete integrability and application of Inverse Scattering Transform.
- Testing of numerical integrators.
- Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators, ...).
- Verify the closure of a model.

Examples of PDEs with Conservation Laws

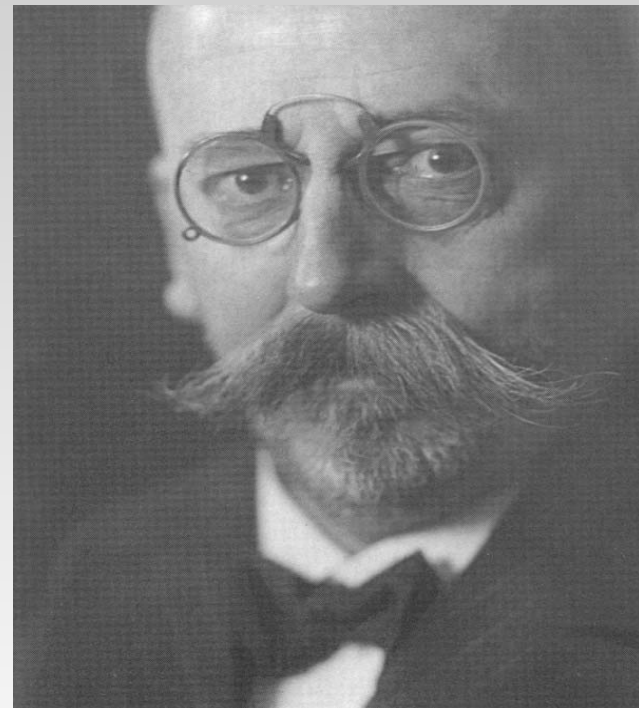
Example 1: KdV Equation

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \text{or} \quad \boxed{u_t + 6uu_x + u_{xxx} = 0}$$

shallow water waves, ion-acoustic waves in plasmas



Diederik Korteweg



Gustav de Vries

- First six (of infinitely many) conservation laws:

$$D_t(u) + D_x(3u^2 + u_{xx}) \doteq 0$$

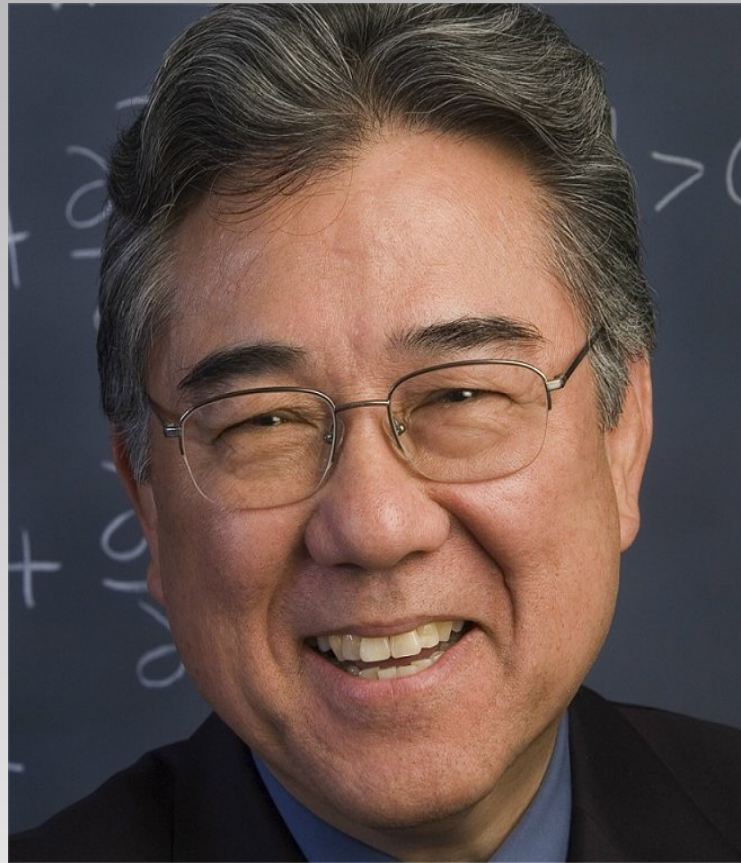
$$D_t(u^2) + D_x(4u^3 - u_x^2 + 2uu_{xx}) \doteq 0$$

$$D_t\left(u^3 - \frac{1}{2}u_x^2\right) + D_x\left(\frac{9}{2}u^4 - 6uu_x^2 + 3u^2u_{xx} + \frac{1}{2}u_{xx}^2 - u_xu_{xxx}\right) \doteq 0$$

$$D_t\left(u^4 - 2uu_x^2 + \frac{1}{5}u_{xx}^2\right) + D_x\left(\frac{24}{5}u^5 - 18uu_x^2 + 4u^3u_{xx} + 2u_x^2u_{xx} + \frac{16}{5}uu_{xx}^2 - 4uu_xu_{xxx} - \frac{1}{5}u_{xxx}^2 + \frac{2}{5}u_{xx}u_{4x}\right) \doteq 0$$

$$\begin{aligned}
& D_t \left(u^5 - 5 u^2 u_x^2 + u u_{xx}^2 - \frac{1}{14} u_{xxx}^2 \right) \\
& + D_x \left(5 u^6 - 40 u^3 u_x^2 - \dots - \frac{1}{7} u_{xxx} u_{5x} \right) \doteq 0 \\
& D_t \left(u^6 - 10 u^3 u_x^2 - \frac{5}{6} u_x^4 + 3 u^2 u_{xx}^2 \right. \\
& \quad \left. + \frac{10}{21} u_{xx}^3 - \frac{3}{7} u u_{xxx}^2 + \frac{1}{42} u_{4x}^2 \right) \\
& + D_x \left(\frac{36}{7} u^7 - 75 u^4 u_x^2 - \dots + \frac{1}{21} u_{4x} u_{6x} \right) \doteq 0
\end{aligned}$$

- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Seventh (sixth thru tenth): Robert Miura, 1966



Robert Miura

Dilation Symmetry

$$u_t + 6uu_x + u_{xxx} = 0$$

has dilation (scaling) symmetry $(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u)$
 λ is an arbitrary parameter.

Notion of **weight**: $W(x) = -1$, thus, $W(D_x) = 1$.

$$W(t) = -3, \text{ hence, } W(D_t) = 3.$$

$$W(u) = 2.$$

Notion of **rank** (total weight of a monomial).

$$\text{Examples: } \text{Rank}(u^3) = \text{Rank}(3u_x^2) = 6.$$

$$\text{Rank}(u^3 u_{xx}) = 10.$$

Key Observation: **Scaling Invariance**

Every term in a density has the same fixed rank.

Every term in a flux has some other fixed rank.

The conservation law

$$\boxed{D_t \rho + D_x J \doteq 0}$$

is **uniform in rank**.

Hence,

$$\text{Rank}(\rho) + \text{Rank}(D_t) = \text{Rank}(J) + \text{Rank}(D_x)$$

- Conservation law explicitly dependent on t and x :

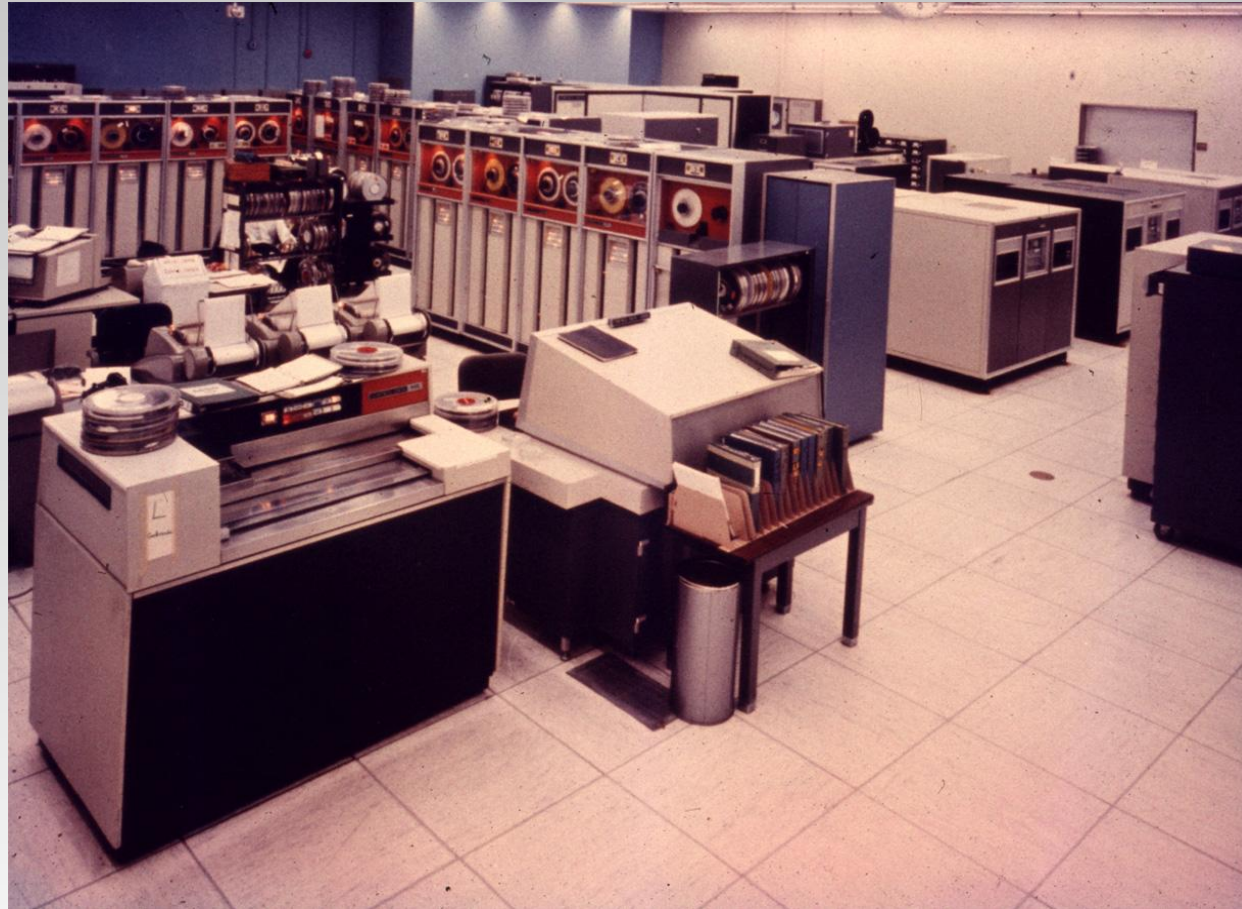
$$\begin{aligned} & D_t \left(tu^2 - \frac{1}{3}xu \right) \\ & + D_x \left(4tu^3 - xu^2 + \frac{1}{3}u_x - tu_x^2 + 2tuu_{xx} - \frac{1}{3}xu_{xx} \right) \doteq 0 \end{aligned}$$

- First five: IBM 7094 computer with FORMAC (1966) → storage space problem!



IBM 7094 Computer

- First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds)
→ large integers problem!



Control Data CDC-6600

Example 2:

The Zakharov-Kuznetsov Equation

$$u_t + \alpha u u_x + \beta (u_{xx} + u_{yy})_x = 0$$

models ion-sound solitons in a low pressure uniform magnetized plasma.

- Conservation laws:

$$D_t(u) + D_x\left(\frac{\alpha}{2}u^2 + \beta u_{xx}\right) + D_y(\beta u_{xy}) \doteq 0$$

$$\begin{aligned} D_t(u^2) + D_x\left(\frac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\right) \\ + D_y(-2\beta u_x u_y) \doteq 0 \end{aligned}$$

- More conservation laws (ZK equation):

$$\begin{aligned} & \mathbf{D}_t \left(u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + \mathbf{D}_x \left(3u^2 \left(\frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) \right. \\ & \quad \left. + \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{yy}^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \right) \\ & \quad + \mathbf{D}_y \left(3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0 \end{aligned}$$

$$\begin{aligned} & \mathbf{D}_t \left(tu^2 - \frac{2}{\alpha} xu \right) + \mathbf{D}_x \left(t \left(\frac{2\alpha}{3} u^3 - \beta (u_x^2 - u_y^2) + 2\beta u (u_{xx} + u_{yy}) \right) \right. \\ & \quad \left. - x \left(u^2 + \frac{2\beta}{\alpha} u_{xx} \right) + \frac{2\beta}{\alpha} u_x \right) + \mathbf{D}_y \left(-2\beta (tu_x u_y + \frac{1}{\alpha} x u_{xy}) \right) \doteq 0 \end{aligned}$$

Methods for Computing Conservation Laws

- Use the Lax pair L and A , satisfying $[L, A] = 0$.

If $L = D_x + U$, $A = D_t + V$ then $V_x - U_t + [U, V] = 0$.

$\hat{L} = TLT^{-1}$ gives the densities, $\hat{A} = TAT^{-1}$ gives the fluxes.

- Use Noether's theorem (Lagrangian formulation) to obtain conservation laws from symmetries (Ovsiannikov, Olver, etc.).
- Multiplier methods (Anderson, Bluman, Anco, Cheviakov, Wolf, etc.) require solving ODEs (or PDEs).

Proposed Algorithmic Method

- Density is linear combination of scaling invariant terms (in the jet space) with undetermined coefficients.
- Compute $D_t \rho$ with total derivative operator.
- Use variational derivative (Euler operator) to express exactness.
- Solve a (parametrized) linear system to find the undetermined coefficients.
- Use the homotopy operator to compute the flux (invert D_x or Div).

- Work with linearly independent pieces in finite dimensional spaces.
- Use linear algebra, calculus, and variational calculus (algorithmic).
- Implement the algorithm in [Mathematica](#).

Software Demonstration:

ConservationLawsMD.m

Software packages in *Mathematica*

Codes are available via the Internet:

<http://inside.mines.edu/~whereman/>

Tools from the Calculus of Variations

- Definition:

A differential function f is a **exact** iff $f = \text{Div } \mathbf{F}$.

Special case (1D): $f = D_x F$.

- Question: How can one test that $f = \text{Div } \mathbf{F}$?
- Theorem (exactness test):

$$f = \text{Div } \mathbf{F} \text{ iff } \mathcal{L}_{u^{(j)}(\mathbf{x})} f \equiv 0, \quad j = 1, 2, \dots, N.$$

N is the number of dependent variables.

The Euler operator annihilates divergences

- Euler operator in 1D (variable $u(x)$):

$$\begin{aligned}\mathcal{L}_{u(x)} &= \sum_{k=0}^M (-D_x)^k \frac{\partial}{\partial u_{kx}} \\ &= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \dots\end{aligned}$$

- Euler operator in 2D (variable $u(x, y)$):

$$\begin{aligned}\mathcal{L}_{u(x,y)} &= \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-D_x)^k (-D_y)^\ell \frac{\partial}{\partial u_{kx \ell y}} \\ &= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} \\ &\quad + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_y^2 \frac{\partial}{\partial u_{yy}} - D_x^3 \frac{\partial}{\partial u_{xxx}} \dots\end{aligned}$$

- Question: How can one compute $\mathbf{F} = \text{Div}^{-1} f$?
- Theorem (integration by parts):
 - In 1D: If f is exact then

$$F = D_x^{-1} f = \int f \, dx = \mathcal{H}_{\mathbf{u}(x)} f$$

- In 2D: If f is a divergence then

$$\mathbf{F} = \text{Div}^{-1} f = (\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f, \mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f)$$

The homotopy operator inverts total derivatives and divergences!

- Homotopy Operator in 1D (variable x):

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

with integrand

$$I_{u^{(j)}} f = \sum_{k=1}^{M_x^{(j)}} \left(\sum_{i=0}^{k-1} u_{ix}^{(j)} (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}^{(j)}}$$

$(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u} \rightarrow \lambda \mathbf{u}$, $\mathbf{u}_x \rightarrow \lambda \mathbf{u}_x$, *etc.*

General: $\mathbf{u} \rightarrow \lambda(\mathbf{u} - \mathbf{u}_0) + \mathbf{u}_0 = \lambda \mathbf{u} + (1 - \lambda)\mathbf{u}_0$

$\mathbf{u}_x \rightarrow \lambda(\mathbf{u}_x - \mathbf{u}_{x0}) + \mathbf{u}_{x0}$ *etc.*

- Homotopy Operator in 2D (variables x and y):

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(x)} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$\mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f = \int_0^1 \sum_{j=1}^N (I_{u^{(j)}}^{(y)} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where for dependent variable $u(x, y)$

$$\mathcal{I}_u^{(x)} f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix jy} \frac{\binom{i+j}{i} \binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} \right. \\ \left. (-D_x)^{k-i-1} (-D_y)^{\ell-j} \right) \frac{\partial f}{\partial u_{kx \ell y}}$$

Application: Zakharov-Kuznetsov Equation

$$u_t + \alpha u u_x + \beta(u_{xx} + u_{yy})_x = 0$$

- Step 1: Compute the dilation invariance

ZK equation is invariant under scaling symmetry

$$(t, x, y, u) \rightarrow \left(\frac{t}{\lambda^3}, \frac{x}{\lambda}, \frac{y}{\lambda}, \lambda^2 u\right) = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u})$$

λ is an arbitrary parameter.

- Hence, the weights of the variables are

$$W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1.$$

- A conservation law is invariant under the scaling symmetry of the PDE.

$$W(u) = 2, \quad W(D_t) = 3, \quad W(D_x) = 1, \quad W(D_y) = 1.$$

For example,

$$\begin{aligned} & D_t \left(u^3 - \frac{3\beta}{\alpha} (u_x^2 + u_y^2) \right) + D_x \left(3u^2 \left(\frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u (u_x^2 + u_y^2) \right. \\ & \left. + \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{xy}^2) - \frac{6\beta^2}{\alpha} (u_x (u_{xxx} + u_{xyy}) + u_y (u_{xxy} + u_{yyy})) \right) \\ & + D_y \left(3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) \doteq 0 \end{aligned}$$

$$\text{Rank } (\rho) = 6, \quad \text{Rank } (J) = 8.$$

$$\text{Rank (conservation law)} = 9.$$

Compute the density of selected **rank**, say, 6.

- Step 2: Construct the candidate density

For example, construct a density of rank 6.

Make a list of all terms with rank 6:

$$\{u^3, u_x^2, uu_{xx}, u_y^2, uu_{yy}, u_x u_y, uu_{xy}, u_{4x}, u_{3xy}, u_{2x2y}, u_{x3y}, u_{4y}\}$$

Remove divergences and divergence-equivalent terms.

Candidate density of rank 6:

$$\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y$$

- Step 3: Compute the undetermined coefficients

Compute

$$\begin{aligned}
 D_t \rho &= \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \\
 &= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} \frac{\partial \rho}{\partial u_{kx \ell y}} D_x^k D_y^\ell u_t \\
 &= \left(3c_1 u^2 I + 2c_2 u_x D_x + 2c_3 u_y D_y + c_4 (u_y D_x + u_x D_y) \right) u_t
 \end{aligned}$$

Substitute $u_t = -\left(\alpha u u_x + \beta (u_{xx} + u_{yy})_x \right).$

$$\begin{aligned}
E = -\mathbf{D}_t \rho = & 3c_1 u^2 (\alpha u u_x + \beta (u_{xx} + u_{xy})_x) \\
& + 2c_2 u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x + 2c_3 u_y (\alpha u u_x \\
& + \beta (u_{xx} + u_{yy})_x)_y + c_4 (u_y (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_x \\
& + u_x (\alpha u u_x + \beta (u_{xx} + u_{yy})_x)_y)
\end{aligned}$$

Apply the **Euler operator** (variational derivative)

$$\begin{aligned}
\mathcal{L}_{u(x,y)} E &= \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-\mathbf{D}_x)^k (-\mathbf{D}_y)^\ell \frac{\partial E}{\partial u_{kx \ell y}} \\
&= -2 \left((3c_1 \beta + c_3 \alpha) u_x u_{yy} + 2(3c_1 \beta + c_3 \alpha) u_y u_{xy} \right. \\
&\quad \left. + 2c_4 \alpha u_x u_{xy} + c_4 \alpha u_y u_{xx} + 3(3c_1 \beta + c_2 \alpha) u_x u_{xx} \right) \\
&\equiv 0
\end{aligned}$$

Solve a parameterized **linear system** for the c_i :

$$3c_1\beta + c_3\alpha = 0, \quad c_4\alpha = 0, \quad 3c_1\beta + c_2\alpha = 0$$

Solution:

$$c_1 = 1, \quad c_2 = -\frac{3\beta}{\alpha}, \quad c_3 = -\frac{3\beta}{\alpha}, \quad c_4 = 0$$

Substitute the solution into the candidate density

$$\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y$$

Final density of rank 6:

$$\rho = u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2)$$

- Step 4: Compute the flux

Use the **homotopy operator** to invert **Div**:

$$\mathbf{J} = \text{Div}^{-1} E = \left(\mathcal{H}_{u(x,y)}^{(x)} E, \mathcal{H}_{u(x,y)}^{(y)} E \right)$$

where

$$\mathcal{H}_{u(x,y)}^{(x)} E = \int_0^1 (I_u^{(x)} E) [\lambda u] \frac{d\lambda}{\lambda}$$

with

$$\begin{aligned} \mathcal{I}_u^{(x)} E = & \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix jy} \frac{\binom{i+j}{i} \binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} \right. \\ & \left. (-D_x)^{k-i-1} (-D_y)^{\ell-j} \right) \frac{\partial E}{\partial u_{kx \ell y}} \end{aligned}$$

Similar formulas for $\mathcal{H}_{u(x,y)}^{(y)} E$ and $\mathcal{I}_u^{(y)} E$.

Let $A = \alpha u u_x + \beta(u_{xxx} + u_{xyy})$ so that

$$E = 3u^2 A - \frac{6\beta}{\alpha} u_x A_x - \frac{6\beta}{\alpha} u_y A_y$$

Then,

$$\begin{aligned} \mathbf{J} &= \left(\mathcal{H}_{u(x,y)}^{(x)} E, \mathcal{H}_{u(x,y)}^{(y)} E \right) \\ &= \left(\frac{3\alpha}{4} u^4 + \beta u^2 (3u_{xx} + 2u_{yy}) - 2\beta u (3u_x^2 + u_y^2) \right. \\ &\quad + \frac{3\beta^2}{4\alpha} u (u_{2x2y} + u_{4y}) - \frac{\beta^2}{\alpha} u_x \left(\frac{7}{2} u_{xyy} + 6u_{xxx} \right) \\ &\quad - \frac{\beta^2}{\alpha} u_y \left(4u_{xxy} + \frac{3}{2} u_{yyy} \right) + \frac{\beta^2}{\alpha} \left(3u_{xx}^2 + \frac{5}{2} u_{xy}^2 + \frac{3}{4} u_{yy}^2 \right) \\ &\quad + \frac{5\beta^2}{4\alpha} u_{xx} u_{yy}, \quad \beta u^2 u_{xy} - 4\beta u u_x u_y \\ &\quad - \frac{3\beta^2}{4\alpha} u (u_x 3y + u_{3xy}) - \frac{\beta^2}{4\alpha} u_x (13u_{xxy} + 3u_{yyy}) \\ &\quad \left. - \frac{5\beta^2}{4\alpha} u_y (u_{xxx} + 3u_{xyy}) + \frac{9\beta^2}{4\alpha} u_{xy} (u_{xx} + u_{yy}) \right) \end{aligned}$$

However, $\text{Div}^{-1}E$ is not unique.

Indeed, $\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{K}$, where $\mathbf{K} = (D_y\theta, -D_x\theta)$ is a curl term.

For example,

$$\theta = 2\beta u^2 u_y + \frac{\beta^2}{4\alpha} \left(3u(u_{xxy} + u_{yyy}) + 10u_x u_{xy} + 5u_y(3u_{yy} + u_{xx}) \right)$$

Shorter flux:

$$\begin{aligned} \tilde{\mathbf{J}} &= \mathbf{J} - \mathbf{K} \\ &= \left(3u^2 \left(\frac{\alpha}{4} u^2 + \beta u_{xx} \right) - 6\beta u(u_x^2 + u_y^2) + \frac{3\beta^2}{\alpha} (u_{xx}^2 - u_{yy}^2) \right. \\ &\quad \left. - \frac{6\beta^2}{\alpha} (u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})), \right. \\ &\quad \left. 3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{xx} + u_{yy}) \right) \end{aligned}$$

Additional Examples

- Manakov-Santini system

$$u_{tx} + u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y = 0$$

$$v_{tx} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} = 0$$

- Conservation laws for Manakov-Santini system:

$$\begin{aligned} & \mathbb{D}_t \left(f u_x v_x \right) + \mathbb{D}_x \left(f (u u_x v_x - u_x v_x v_y - u_y v_y) \right. \\ & \quad \left. - f' y (u_t + u u_x - u_x v_y) \right) + \mathbb{D}_y \left(f (u_x v_y + u_y v_x + u_x v_x^2) \right. \\ & \quad \left. + f' (u - y u_y - y u_x v_x) \right) = 0 \end{aligned}$$

where $f = f(t)$ is arbitrary.

Conservation laws – continued:

$$\begin{aligned}
 & \mathbf{D}_t \left(f(2u + v_x^2 - yu_x v_x) \right) + \mathbf{D}_x \left(f(u^2 + uv_x^2 + u_y v \right. \\
 & \quad \left. - v_y^2 - v_x^2 v_y - y(uu_x v_x - u_x v_x v_y - u_y v_y)) \right. \\
 & \quad \left. - f' y(v_t + uv_x - v_x v_y) + (f' - 2fx)y^2(u_t + uu_x - u_x v_y) \right) \\
 & + \mathbf{D}_y \left(f(v_x^3 + 2v_x v_y - u_x v - y(u_x v_x^2 + u_x v_y + u_y v_x)) \right. \\
 & \quad \left. + f'(v - y(2u + v_y + v_x^2)) + (f'y^2 - 2fx)(u_x v_x + u_y) \right) = 0
 \end{aligned}$$

where $f = f(t)$ is arbitrary.

There are three additional conservation laws.

- (2+1)-dimensional Camassa-Holm equation

$$(\alpha u_t + \kappa u_x - u_{txx} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{yy} = 0$$

Interchange t with y

$$(\alpha u_y + \kappa u_x - u_{xyx} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{tt} = 0$$

Set $v = u_t$ to get

$$u_t = v$$

$$\begin{aligned} v_t = & -\alpha u_{xy} - \kappa u_{xx} + u_{3xy} - 3\beta u_x^2 - 3\beta u u_{xx} + 2u_{xx}^2 \\ & + 3u_x u_{xxx} + u u_{4x} \end{aligned}$$

- Conservation laws for the Camassa-Holm equation

$$\begin{aligned} & \mathcal{D}_t \left(f u \right) + \mathcal{D}_x \left(\frac{1}{\alpha} f \left(\frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - u u_{xx} - u_{tx} \right) \right. \\ & \left. + \left(\frac{1}{2} f' y^2 - \frac{1}{\alpha} f x \right) (\alpha u_t + \kappa u_x + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx} \right. \right. \\ & \left. \left. - u_{txx} \right) \right) + \mathcal{D}_y \left(\left(\frac{1}{2} f' y^2 - \frac{1}{\alpha} f x \right) u_y - f' y u \right) = 0 \end{aligned}$$

$$\begin{aligned} & \mathcal{D}_t \left(f y u \right) + \mathcal{D}_x \left(\frac{1}{\alpha} f y \left(\frac{3\beta}{2} u^2 + \kappa u - \frac{1}{2} u_x^2 - u u_{xx} - u_{tx} \right) \right. \\ & \left. + y \left(\frac{1}{6} f' y^2 - \frac{1}{\alpha} f x \right) (\alpha u_t + \kappa u_x + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx} \right. \right. \\ & \left. \left. - u_{txx} \right) \right) + \mathcal{D}_y \left(y \left(\frac{1}{6} f' y^2 - \frac{1}{\alpha} f x \right) u_y + \left(\frac{1}{\alpha} f x - \frac{1}{2} f' y^2 \right) u \right) = 0 \end{aligned}$$

where $f = f(t)$ is an arbitrary function.

- Khoklov-Zabolotskaya equation

describes e.g., sound waves in nonlinear media

$$(u_t - uu_x)_x - u_{yy} - u_{zz} = 0$$

Conservation law:

$$\begin{aligned} & \mathbf{D}_t \left(fu \right) - \mathbf{D}_x \left(\frac{1}{2} fu^2 + (fx + g)(u_t - uu_x) \right) \\ & + \mathbf{D}_y \left((fx + g)u_y - (f_yx + g_y)u \right) \\ & + \mathbf{D}_z \left((fx + g)u_z - (f_zx + g_z)u \right) = 0 \end{aligned}$$

under the constraints $\Delta f = 0$ and $\Delta g = f_t$
where $f = f(t, y, z)$ and $g = g(t, y, z)$.

- Shallow water wave model (atmosphere)

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2 \boldsymbol{\Omega} \times \mathbf{u} + \nabla(\theta h) - \frac{1}{2} h \nabla \theta = \mathbf{0}$$

$$\theta_t + \mathbf{u} \cdot (\nabla \theta) = 0$$

$$h_t + \nabla \cdot (\mathbf{u} h) = 0$$

where $\mathbf{u}(x, y, t)$, $\theta(x, y, t)$ and $h(x, y, t)$.

- In components:

$$u_t + uu_x + vu_y - 2\Omega v + \frac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\Omega u + \frac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- First few conservation laws of SWW model:

$$\rho_{(1)} = h \qquad \mathbf{J}^{(1)} = h \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(2)} = h \theta \qquad \mathbf{J}^{(2)} = h \theta \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(3)} = h \theta^2 \qquad \mathbf{J}^{(3)} = h \theta^2 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(4)} = h (u^2 + v^2 + h\theta) \qquad \mathbf{J}^{(4)} = h \begin{pmatrix} u (u^2 + v^2 + 2h\theta) \\ v (v^2 + u^2 + 2h\theta) \end{pmatrix}$$

$$\rho_{(5)} = \theta (2\Omega + v_x - u_y)$$

$$\mathbf{J}^{(5)} = \frac{1}{2} \theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}$$

- More general conservation laws for SWW model:

$$D_t \left(f(\theta) h \right) + D_x \left(f(\theta) h u \right) + D_y \left(f(\theta) h v \right) = 0$$

$$\begin{aligned} & D_t \left(g(\theta) (2\Omega + v_x - u_x) \right) \\ & + D_x \left(\frac{1}{2} g(\theta) (4\Omega u - 2u u_y + 2u v_x - h \theta_y) \right) \\ & + D_y \left(\frac{1}{2} g(\theta) (4\Omega v - 2u_y v + 2v v_x + h \theta_x) \right) = 0 \end{aligned}$$

for any functions $f(\theta)$ and $g(\theta)$.

- Kadomtsev-Petviashvili (KP) equation

$$(u_t + \alpha u u_x + u_{xxx})_x + \sigma^2 u_{yy} = 0$$

parameter $\alpha \in \mathbb{R}$ and $\sigma^2 = \pm 1$.

Equation be written as a conservation law

$$D_t(u_x) + D_x(\alpha u u_x + u_{xxx}) + D_y(\sigma^2 u_y) = 0.$$

Exchange y and t and set $u_t = v$

$$u_t = v$$

$$v_t = -\frac{1}{\sigma^2} (u_{xy} + \alpha u_x^2 + \alpha u u_{xx} + u_{xxx})$$

- Examples of conservation laws for KP equation
(explicitly dependent on t , x , and y)

$$D_t(xu_x) + D_x(3u^2 - u_{xx} - 6xu u_x + xu_{xxx}) + D_y(\alpha x u_y) = 0$$

$$D_t(yu_x) + D_x(y(\alpha u u_x + u_{xxx})) + D_y(\sigma^2(yu_y - u)) = 0$$

$$\begin{aligned} D_t(\sqrt{t}u) + D_x\left(\frac{\alpha}{2}\sqrt{t}u^2 + \sqrt{t}u_{xx} + \frac{\sigma^2 y^2}{4\sqrt{t}}u_t + \frac{\sigma^2 y^2}{4\sqrt{t}}u_{xxx} \right. \\ \left. + \frac{\alpha\sigma^2 y^2}{4\sqrt{t}}uu_x - x\sqrt{t}u_t - \alpha x\sqrt{t}uu_x - x\sqrt{t}u_{xxx}\right) \\ + D_y\left(x\sqrt{t}u_y + \frac{y^2 u_y}{4\sqrt{t}} - \frac{yu}{2\sqrt{t}}\right) = 0 \end{aligned}$$

- More general conservation laws for KP equation:

$$\begin{aligned} & D_t \left(f u \right) + D_x \left(f \left(\frac{\alpha}{2} u^2 + u_{xx} \right) \right. \\ & \left. + \left(\frac{\sigma^2}{2} f' y^2 - f x \right) (u_t + \alpha u u_x + u_{3x}) \right) \\ & + D_y \left(\left(\frac{1}{2} f' y^2 - \sigma^2 f x \right) u_y - f' y u \right) = 0 \end{aligned}$$

$$\begin{aligned} & D_t \left(f y u \right) + D_x \left(f y \left(\frac{\alpha}{2} u^2 + u_{xx} \right) \right. \\ & \left. + y \left(\frac{\sigma^2}{6} f' y^2 - f x \right) (u_t + \alpha u u_x + u_{3x}) \right) \\ & + D_y \left(y \left(\frac{1}{6} f' y^2 - \sigma^2 f x \right) u_y + \left(\sigma^2 f x - \frac{1}{2} f' y^2 \right) u \right) = 0 \end{aligned}$$

where $f(t)$ is arbitrary function.

- Potential KP equation

Replace u by u_x and integrate with respect to x .

$$u_{xt} + \alpha u_x u_{xx} + u_{xxxx} + \sigma^2 u_{yy} = 0$$

- Examples of conservation laws

(not explicitly dependent on x, y, t):

$$D_t(u_x) + D_x\left(\frac{\alpha}{2}u_x^2 + u_{xxx}\right) + D_y\left(\sigma^2 u_y\right) = 0$$

$$D_t(u_x^2) + D_x\left(\frac{2\alpha}{3}u_x^3 - u_{xx}^2 + 2u_x u_{xxx} - \sigma^2 u_{yy}\right) \\ + D_y\left(2\sigma^2 u_x u_y\right) = 0$$

Conservation laws for pKP equation – continued:

$$\begin{aligned} & D_t \left(u_x u_y \right) + D_x \left(\alpha u_x^2 u_y + u_t u_y + 2u_{xxx} u_y - 2u_{xx} u_{xy} \right) \\ & + D_y \left(\sigma^2 u_y^2 - \frac{1}{3} u_x^3 - u_t u_x + u_{xx}^2 \right) = 0 \end{aligned}$$

$$\begin{aligned} & D_t \left(2\alpha u u_x u_{xx} + 3u u_{4x} - 3\sigma^2 u_y^2 \right) + D_x \left(2\alpha u_t u_x^2 + 3u_t^2 \right. \\ & \quad \left. - 2\alpha u u_x u_{tx} - 3u_{tx} u_{xx} + 3u_t u_{xxx} + 3u_x u_{txx} - 3u u_{txxx} \right) \\ & + D_y \left(6\sigma^2 u_t u_y \right) = 0 \end{aligned}$$

Various generalizations exist.

- Generalized Zakharov-Kuznetsov equation

$$u_t + \alpha u^n u_x + \beta(u_{xx} + u_{yy})_x = 0$$

where n is rational, $n \neq 0$.

Conservation laws:

$$D_t(u) + D_x\left(\frac{\alpha}{n+1}u^{n+1} + \beta u_{xx}\right) + D_y(\beta u_{xy}) = 0$$

$$D_t(u^2) + D_x\left(\frac{2\alpha}{n+2}u^{n+2} - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\right) \\ + D_y(-2\beta u_x u_y) = 0$$

- Third conservation law for gZK equation:

$$\begin{aligned}
& \mathbf{D}_t \left(u^{n+2} - \frac{(n+1)(n+2)\beta}{2\alpha} (u_x^2 + u_y^2) \right) \\
& + \mathbf{D}_x \left(\frac{(n+2)\alpha}{2(n+1)} u^{2(n+1)} + (n+2)\beta u^{n+1} u_{xx} \right. \\
& - (n+1)(n+2)\beta u^n (u_x^2 + u_y^2) + \frac{(n+1)(n+2)\beta^2}{2\alpha} (u_{xx}^2 - u_{yy}^2) \\
& \left. - \frac{(n+1)(n+2)\beta^2}{\alpha} (u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})) \right) \\
& + \mathbf{D}_y \left((n+2)\beta u^{n+1} u_{xy} + \frac{(n+1)(n+2)\beta^2}{\alpha} u_{xy} (u_{xx} + u_{yy}) \right) = 0.
\end{aligned}$$

Conclusions and Future Work

- The power of Euler and homotopy operators:
 - ▶ Testing exactness
 - ▶ Integration by parts: D_x^{-1} and Div^{-1}

- Integration of non-exact expressions

Example: $f = u_x v + u v_x + u^2 u_{xx}$

$$\int f dx = uv + \int u^2 u_{xx} dx$$

- Use other homotopy formulas (moving terms amongst the components of the flux; prevent curl terms)

- Broader class of PDEs (beyond evolution type)

Example: short pulse equation (nonlinear optics)

$$u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2u_{xx}$$

with non-polynomial conservation law

$$D_t \left(\sqrt{1 + 6u_x^2} \right) - D_x \left(3u^2 \sqrt{1 + 6u_x^2} \right) = 0$$

- Continue the implementation in *Mathematica*
- Software: <http://inside.mines.edu/~whereman>

Thank You

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