Head-on collisions of electrostatic solitons in multispecies plasmas

Frank Verheest[1,2], Manfred A Hellberg[2] and Willy A. Hereman[3]

[1] *Sterrenkundig Observatorium, Universiteit Gent, Belgium*
[2] *School of Chemistry and Physics, University of KwaZulu-Natal, Durban, SA*
[3] *Applied Mathematics and Statistics, Colorado School of Mines, Golden CO, USA*

1. Motivation and model

- KdV equations possess exact solutions for interactions between N solitons, with full nonlinearity during overtaking, but *only for solitons propagating in the direction that underlies the basic derivation* of the parent equation

- In contrast, *head-on collisions* between two electrostatic solitons can only be dealt with by *approximate methods*, which limit the range of validity but offer valuable insight

- Framework is based on Poincaré-Lighthill-Kuo formalism of strained coordinates, which yields here (m)KdV families of equations plus phase (time) shifts that occur in the interaction

- Plasma consists of number of cold (positive and negative) ion species and Boltzmann electrons

- *Continuity* and *momentum* equations for different ion species are coupled to *Poisson*'s equation

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x}(\rho_i u_i) = 0$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + z_i \frac{\partial \varphi}{\partial x} = 0$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \sum_i \rho_i z_i - \exp(\varphi) = 0$$

2. Basic formalism

- *Stretching* of independent variables reflects *propagation in opposite directions*

$$
\begin{aligned}
\xi &= \varepsilon(x-t) + \varepsilon^2 P(\xi,\eta,\tau) + \ldots \\
\eta &= \varepsilon(x+t) + \varepsilon^2 Q(\xi,\eta,\tau) + \ldots \\
\tau &= \varepsilon^3 t
\end{aligned}
$$

- Both space coordinates have to use same linear acoustic phase velocity in medium, here $c_a = 1$ with proper choice of normalization

- This is coupled to *expansions* for densities, velocities and electrostatic potential

$$
\begin{aligned}
\rho_i &= \rho_{i0} + \varepsilon\rho_{i1} + \varepsilon^2\rho_{i2} + \varepsilon^3\rho_{i3} + \varepsilon^4\rho_{i4} + \ldots \\
u_i &= \varepsilon u_{i1} + \varepsilon^2 u_{i2} + \varepsilon^3 u_{i3} + \varepsilon^4 u_{i4} + \ldots \\
\varphi &= \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \varepsilon^4\varphi_4 + \ldots
\end{aligned}
$$

- Steps in expansion are left general, so as to deal with generic and critical compositions in one coherent treatment

3. Lower order results and bifurcation

- To lowest order perturbations obey

$$\left(\frac{\partial \rho_{i1}}{\partial \eta} - \frac{\partial \rho_{i1}}{\partial \xi}\right) + \rho_{i0}\left(\frac{\partial u_{i1}}{\partial \xi} + \frac{\partial u_{i1}}{\partial \eta}\right) = 0$$

$$\left(\frac{\partial u_{i1}}{\partial \eta} - \frac{\partial u_{i1}}{\partial \xi}\right) u_{i1} + z_i\left(\frac{\partial \varphi_1}{\partial \xi} + \frac{\partial \varphi_1}{\partial \eta}\right) = 0$$

$$\sum_i \rho_{i1}\, z_i - \varphi_1 = 0$$

and lead to *separability* at linear level

$$\rho_{i1} = \rho_{i0}\, z_i \left(\varphi_{1\xi} + \varphi_{1\eta}\right) \qquad u_{i1} = z_i \left(\varphi_{1\xi} - \varphi_{1\eta}\right) \qquad \varphi_1 = \varphi_{1\xi} + \varphi_{1\eta}$$

- Next order leads to *bifurcation*

$$\left(3\sum_i \rho_{i0}\, z_i^3 - 1\right)\varphi_{1\xi}^2 = 0 \qquad \left(3\sum_i \rho_{i0}\, z_i^3 - 1\right)\varphi_{1\eta}^2 = 0$$

so that *in generic case* $\varphi_{1\xi} = \varphi_{1\eta} = 0$ or else, *at critical parameters* $\sum_i \rho_{i0}\, z_i^3 = \frac{1}{3}$

4. Generic case: Korteweg-de Vries equations and phase shifts

- After much complicated algebra one arrives at *KdV equations and phase shifts*

$$\boxed{\begin{array}{lcl} \dfrac{\partial \varphi_{2\xi}}{\partial \tau} + A\varphi_{2\xi}\dfrac{\partial \varphi_{2\xi}}{\partial \xi} + \dfrac{1}{2}\dfrac{\partial^3 \varphi_{2\xi}}{\partial \xi^3} = 0 & \& & \dfrac{\partial P}{\partial \eta} = B\,\varphi_{2\eta} \\ \dfrac{\partial \varphi_{2\eta}}{\partial \tau} - A\varphi_{2\eta}\dfrac{\partial \varphi_{2\eta}}{\partial \eta} - \dfrac{1}{2}\dfrac{\partial^3 \varphi_{2\eta}}{\partial \eta^3} = 0 & \& & \dfrac{\partial Q}{\partial \xi} = B\,\varphi_{2\xi} \end{array}}$$

where

$$A = \frac{1}{2}\Big(3\sum_i \rho_{i0}\, z_i^3 - 1\Big) \gtrless 0 \qquad B = \frac{1}{4}\Big(\sum_i \rho_{i0}\, z_i^3 + 1\Big)$$

- One-soliton solutions for each KdV equation are

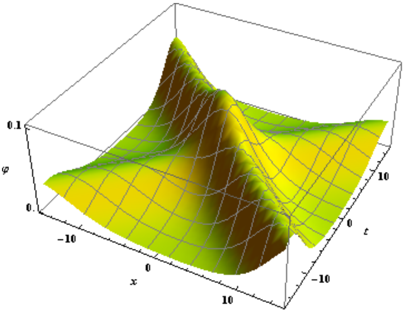
$$\varphi_{2\xi} = \frac{3v_\xi}{A}\,\mathrm{sech}^2\left[\sqrt{\frac{v_\xi}{2}}(\xi - v_\xi\tau)\right] \quad \& \quad P = \frac{3B\sqrt{2v_\eta}}{A}\left\{\tanh\left[\sqrt{\frac{v_\eta}{2}}(\eta + v_\eta\tau)\right] + 1\right\}$$

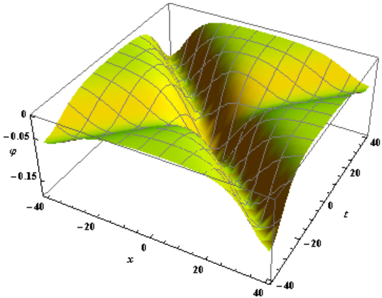
$$\varphi_{2\eta} = \frac{3v_\eta}{A}\,\mathrm{sech}^2\left[\sqrt{\frac{v_\eta}{2}}(\eta + v_\eta\tau)\right] \quad \& \quad Q = \frac{3B\sqrt{2v_\xi}}{A}\left\{\tanh\left[\sqrt{\frac{v_\xi}{2}}(\xi - v_\xi\tau)\right] - 1\right\}$$

- Only *same sign polarities (positive or negative)* are possible for both waves

5. Illustrations for two ion species: head-on collisions in generic case

- Normalization yields for two ion species that $\rho_{10} = \dfrac{1 - z_2}{z_1(z_1 - z_2)}$ and $\rho_{20} = \dfrac{z_1 - 1}{z_2(z_1 - z_2)}$





- Left figure: values $z_1 = 4$ and $z_2 = 1/4$, typical for H^+ and O^+, so that $A > 0$ (potential and density humps) and $B > 0$ (propagation delays due to interaction)

- Right figure: values $z_1 = 0.1$ and $z_2 = -0.1$, typical for a fullerene plasma with electron contamination, so that $A < 0$ (potential and density dips) but $B > 0$ (propagation delays due to interaction)

6. Critical case: modified Korteweg-de Vries equations and phase shifts

- Algebra in critical case is even more involved but leads to *mKdV equations plus phase shifts*

$$\frac{\partial \varphi_{1\xi}}{\partial \tau} + C\,\varphi_{1\xi}^2 \frac{\partial \varphi_{1\xi}}{\partial \xi} + \frac{1}{2}\frac{\partial^3 \varphi_{1\xi}}{\partial \xi^3} = 0 \qquad \& \qquad \frac{\partial P}{\partial \eta} = D\,\varphi_{1\eta}^2$$

$$\frac{\partial \varphi_{1\eta}}{\partial \tau} - C\,\varphi_{1\eta}^2 \frac{\partial \varphi_{1\eta}}{\partial \eta} - \frac{1}{2}\frac{\partial^3 \varphi_{1\eta}}{\partial \eta^3} = 0 \qquad \& \qquad \frac{\partial Q}{\partial \xi} = D\,\varphi_{1\xi}^2$$

where

$$C = \frac{1}{4}\Big(15\sum_i \rho_{i0}\, z_i^4 - 1\Big) > \frac{1}{6} \qquad\qquad D = \frac{1}{8}\Big(1 - \sum_i \rho_{i0}\, z_i^4\Big) < \frac{1}{9}$$

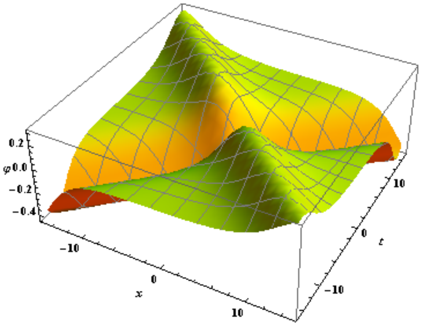
- Solutions can now be of either sign and thus allow for *collision between counterstreaming negative and positive polarity solitons*

$$\varphi_{1\xi} = \pm\sqrt{\frac{6v_\xi}{C}}\ \mathrm{sech}\Big[\sqrt{2v_\xi}(\xi - v_\xi\tau)\Big] \qquad \& \qquad P = \frac{3D}{C}\sqrt{2v_\eta}\ \Big\{\tanh\Big[\sqrt{2v_\eta}(\eta + v_\eta\tau)\Big] + 1\Big\}$$

$$\varphi_{1\eta} = \pm\sqrt{\frac{6v_\eta}{C}}\ \mathrm{sech}\Big[\sqrt{2v_\eta}(\eta + v_\eta\tau)\Big] \qquad \& \qquad Q = \frac{3D}{C}\sqrt{2v_\xi}\ \Big\{\tanh\Big[\sqrt{2v_\xi}(\xi - v_\xi\tau)\Big] - 1\Big\}$$

7. Illustrations for two ion species: head-on collisions in critical case

- At criticality, one also requires $$z_2 = \frac{z_1 - 1/3}{z_1 - 1} < 0$$



- Values $z_1 = 0.739$ and $z_2 = -1.552$, typical for Ar^+ and F^- plasma experiments at critical densities
- Here $C > 0$ and $D > 0$, but negative polarity soliton propagates to the right and positive polarity soliton to the left, with positive phase shifts (propagation delays)

8. Summary and remarks

- *In generic case KdV equations* govern collisions between left- and right-propagating solitons, with corresponding phase shifts
- *At critical plasma composition **modified** KdV equations* are needed, with corresponding phase shifts, *a case not addressed before*
- When all ion species are positive, *$A \geqslant 1$ and $B \geqslant 1/2$ (no critical compositions)*: polarities and phase shifts are positive (equivalent to delays compared to single-soliton trajectory)
- Negative polarities in general require at least one negative ion species, in addition to necessary positive ions, hence $A < 0$
- Criticality also needs *at least one negative species to make $A = 0$*, but then always has $C > 0$
- Comparison with recent experimental observations of two counter-propagating solitons of equal amplitude in a monolayer strongly coupled dusty plasma indicates qualitative agreement regarding delays occurring after interaction, and general behaviour
- However, amplitude of overlapping solitons during collision was less than sum of initial soliton amplitudes, which cannot correctly be dealt with by available Poincaré-Lighthill-Kuo formalism
- Analytical treatment of more complicated plasma models is qualitatively analogous