# SOLITARY WAVE SOLUTIONS OF COUPLED NONLINEAR EVOLUTION EQUATIONS USING MACSYMA

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## 1. INTRODUCTION

- Construct solitary wave solutions by a direct method
- Applicable to:

    Single nonlinear evolution and wave equations
    Systems of nonlinear PDEs
    Nonlinear ODEs

- Goal: Exact solutions

    Single solitary wave or soliton solutions
    N-solitons
    Implicit solutions

- Method:

    Hirota's direct method
    Rosales' perturbation method
    Trace method
    Hereman et al real exponential approach

- Requirements :

    Based on physical principles
    Simple and straightforward
    Programmable in MACSYMA, REDUCE,
    MATHEMATICA, SCRATCHPAD II

## 2. EXAMPLES

- Korteweg-de Vries equation and generalizations

$$u_t + au^n u_x + u_{xxx} = 0, \quad n \in \mathbb{N}$$

$$u(x,t) = \left\{ \frac{c(n+1)(n+2)}{2a} \operatorname{sech}^2 \left[ \frac{n}{2} \sqrt{c}(x - ct) + \delta \right] \right\}^{\frac{1}{n}}$$

- Burgers equation

$$u_t + auu_x - u_{xx} = 0$$

$$u(x,t) = \frac{c}{a} \left\{ 1 - \tanh \left[ \frac{c}{2}(x - ct) + \delta \right] \right\}$$

- Fisher equation and generalizations

$$u_t - u_{xx} - u(1 - u^n) = 0, \quad n \in \mathbb{N}$$

$$u(x,t) = \left\{ \frac{1}{2} \left[ 1 - \tanh \left[ \frac{n}{2\sqrt{2n+4}} (x - \frac{(n+4)}{\sqrt{2n+4}} t) + \delta \right] \right] \right\}^{\frac{2}{n}}$$

- Fitzhugh-Nagumo equation

$$u_t - u_{xx} + u(1-u)(a-u) = 0$$

$$u(x,t) = \frac{a}{2}\left\{1 + \tanh\left[\frac{a}{2\sqrt{2}}(x - \frac{(2-a)}{\sqrt{2}}t) + \delta\right]\right\}$$

- Kuramoto-Sivashinski equation

$$u_t + uu_x + au_{xx} + bu_{xxxx} = 0$$

$$\begin{aligned} u(x,t) &= c + \frac{165ak}{19}\left\{\tanh^3\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \\ &- \frac{135ak}{19}\left\{\tanh\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \end{aligned}$$

with $k = \sqrt{\frac{11a}{19b}}$

$$\begin{aligned} u(x,t) &= c - \frac{15ak}{19}\left\{\tanh^3\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \\ &+ \frac{45ak}{19}\left\{\tanh\left[\frac{k(x-ct)}{2} + \delta\right]\right\} \end{aligned}$$

with $k = \sqrt{\frac{-a}{19b}}$

- Harry Dym equation

$$u_t + (1-u)^3 u_{xxx} = 0$$

$$\begin{aligned} u(x,t) &= \operatorname{sech}^2\left[\frac{1}{2}\sqrt{c}\,[x-ct+\delta(x,t)]\right] \\ \delta(x,t) &= \frac{2}{\sqrt{c}}\tanh\left[\frac{\sqrt{c}}{2}[x-ct+\delta(x,t)]\right] \end{aligned}$$

- sine-Gordon equation

$$u_{tt} - u_{xx} - \sin u = 0$$

$$u(x,t) = 4\arctan\left\{\exp\left[\frac{1}{\sqrt{-c}}(x-ct)+\delta\right]\right\}$$

- Coupled Korteweg-de Vries equations

$$u_t - a(6uu_x + u_{xxx}) - 2b\ vv_x = 0,$$
$$v_t + 3uv_x + v_{xxx} = 0$$

$$\begin{aligned} u(x,t) &= 2\ c\ \mathrm{sech}^2\left[\sqrt{c}(x-ct)+\delta\right], \\ v(x,t) &= \pm c\sqrt{\frac{-2(4a+1)}{b}}\ \mathrm{sech}\left[\sqrt{c}(x-ct)+\delta\right], \end{aligned}$$

$$\begin{aligned} u(x,t) &= c\ \mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \\ v(x,t) &= \frac{3}{\sqrt{6|b|}}\ u(x,t) = \frac{3\ c}{\sqrt{6|b|}}\ \mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \end{aligned}$$

## 3. THE ALGORITHM

- Step 1: System of two coupled nonlinear PDEs

$$\begin{aligned} \mathcal{F}(u, v, u_t, u_x, v_t, v_x, u_{tx}, ..., u_{mx}, v_{nx}) &= 0, \\ \mathcal{G}(u, v, u_t, u_x, v_t, v_x, u_{tx}, ..., u_{px}, v_{qx}) &= 0, \quad (m, n, p, q \in \mathbb{N}) \end{aligned}$$

  where $\mathcal{F}$ and $\mathcal{G}$ are polynomials in their arguments and

$$u_t = \frac{\partial u}{\partial t}, \quad u_{nx} = \frac{\partial^n u}{\partial x^n}$$

- Step 2:
  - Introduce the variable $\xi = x - ct$, ($c$ is the constant velocity)
  - Integrate the system of ODEs for $\phi(\xi) \equiv u(x,t)$ and $\psi(\xi) \equiv v(x,t)$. with respect to $\xi$ to reduce the order
  - Ignore integration constants and assume that the solutions $\phi$ and $\psi$ and their derivatives vanish at $\xi = \pm\infty$

- Step 3:
  - Expand $\phi$ and $\psi$ in a power series

$$\phi = \sum_{n=1}^{\infty} a_n\, g^n, \quad \psi = \sum_{n=1}^{\infty} b_n\, g^n$$

  - $g(\xi) = \exp(-K(c)\xi)$ solves the linear part of one of the equations

- Consider the dispersion laws $K(c)$ of the linearized equations
- Substitute the expansions into the full nonlinear system
- Use Cauchy's rule for multiple series to rearrange the multiple sums
- Equate the coefficient of $g^n$ to get the coupled recursion relations for $a_n$ and $b_n$

* Step 4:
  - Assume that $a_n$ and $b_n$ are polynomials in $n$
  - Determine their degrees $\delta_1$ and $\delta_2$
  - Substitute

$$a_n = \sum_{j=0}^{\delta_1} A_j\, n^j, \quad b_n = \sum_{j=0}^{\delta_2} B_j\, n^j$$

    into the recursion relations

- Compute the sums by using the formulae for

$$S_k = \sum_{i=1}^{n} i^k, \quad (k = 0, 1, 2, ...)$$

- Examples:

$$S_0 = n, \quad S_1 = \frac{(n+1)n}{2},$$

$$S_2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{etc.}$$

- Equate to zero the different coefficients of the polynomial in $n$
- Solve the algebraic (nonlinear) equations for the constant coefficients $A_j$ and $B_j$

- Step 5:
  - Find the closed forms for

$$\phi \ = \ \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_1} A_j \ n^j \ g^n \equiv \sum_{j=0}^{\delta_1} A_j \ F_j(g),$$

$$\psi \ = \ \sum_{n=1}^{\infty} \sum_{j=0}^{\delta_2} B_j \ n^j \ g^n \equiv \sum_{j=0}^{\delta_2} B_j \ F_j(g)$$

  with

$$F_j(g) \ \equiv \ \sum_{n=1}^{\infty} n^j g^n$$

$$F_{j+1}(g) \ = \ g F_j'(g), \quad j = 0, 1, 2, ...$$

– Examples

$$F_0(g) = \frac{g}{1-g}, \quad F_1(g) = \frac{g}{(1-g)^2},$$

$$F_2(g) = \frac{g(1+g)}{(1-g)^3}, \quad \text{etc.}$$

– Return to the original variables $x$ and $t$ to obtain the travelling wave solution(s)

## 4. EXAMPLE: The Coupled KdV Equations

- Step 1: System of PDEs:

$$\begin{aligned} u_t - a(6uu_x + u_{3x}) - 2b\, vv_x &= 0, \\ v_t + 3uv_x + v_{3x} &= 0, \quad a, b \in \mathbb{R} \end{aligned}$$

- Step 2:
  - Introduce the variable $\xi = x - ct$, $c$ is the constant velocity
  - Integrate the system of ODEs for $\phi(\xi) \equiv u(x,t)$ and $\psi(\xi) \equiv v(x,t)$

$$\begin{aligned} c\phi + 3a\phi^2 + \alpha\phi_{2\xi} + b\psi^2 &= 0, \\ -c\psi_\xi + 3\phi\psi_\xi + \psi_{3\xi} &= 0 \end{aligned}$$

- Step 3:
  - Expand $\phi$ and $\psi$ in a power series

$$\phi = \frac{c}{3} \sum_{n=1}^{\infty} a_n\, g^n, \quad \psi = \frac{c}{\sqrt{3|b|}} \sum_{n=1}^{\infty} b_n\, g^n$$

  - $g(\xi) = \exp(-K(c)\xi)$ solves the linear part of one of the equations in the system
  - Consider the dispersion law $K(c) = \sqrt{c}$ of the second equation

– Substitute the expansions into the full nonlinear system

– Use Cauchy's rule for multiple series to rearrange the sums

– Equate the coefficient of $g^n$ to get the coupled recursion relations

$$
\begin{aligned}
(1 + a\, n^2)\; a_n + \sum_{l=1}^{n-1} (a\; a_l\; a_{n-l} + e\; b_l\; b_{n-l}) &= 0, \\
n\;(n^2 - 1)\; b_n + \sum_{l=1}^{n-1} l\; b_l\; a_{n-l} &= 0, \quad n \geq 2 \\
(1+a)a_1 &= 0,
\end{aligned}
$$

with $b_1$ arbitrary, and $e = \pm 1$ if $|b| = \pm b$

- CASE 1: $a \neq -1$ then $a_1 = 0$ thus, $a_{2n-1} = 0$, $b_{2n} = 0, \quad (n = 1, 2, ...)$

  – Shift the labels in the recurrence relations

$$
\begin{aligned}
(1 + 4a\; n^2)\; a_{2n} + \sum_{l=1}^{n-1} a\; a_{2l}\; a_{2(n-l)} + e \sum_{l=1}^{n} \; b_{2l-1}\; b_{2n-2l+1} &= 0, \\
4n\;(n-1)(2n-1)\; b_{2n-1} + \sum_{l=1}^{n-1} (2l-1)\; b_{2l-1}\; a_{2(n-l)} &= 0, \quad n
\end{aligned}
$$

- Step 4:
  - Assume that $a_{2n}$ and $b_{2n-1}$ polynomials in $n$ and determine their degrees $\delta_1 = 1$ and $\delta_2 = 0$
  - Substitute $a_{2n} = A_1 n + A_0;\ \ b_{2n-1} = B_0,\ (n = 1, 2, ...)$ into the recursion relations
  - Compute the sums by using the formulae for $S_k = \sum\limits_{i=1}^{n} i^k$
  - Equate to zero the different coefficients of the polynomial in $n$ of degree 3
  - Solve the algebraic (nonlinear) equations for the constant coefficients $A_1, A_0$ and $B_0$
  - Solution (with MACSYMA):

$$\begin{aligned} a_{2n} &= 24\, n\, (-1)^{n+1}\, {a_0}^n, \\ b_{2n-1} &= (-1)^{n-1}\, b_1\, {a_0}^{n-1}, \quad n = 1, 2, ... \end{aligned}$$

  with $a_0 = -e{b_1}^2/24(4a+1) > 0$
  - Remark: $b$ and $4a+1$ must have opposite signs

- Step 5:
  - Find the closed forms for $\phi$ and $\psi$
  - Use $F_0(g) = \dfrac{g}{1-g}$ and $F_1(g) = \dfrac{g}{(1-g)^2}$ to get

$$\phi \;=\; 8\,c \sum_{n=1}^{\infty} (-1)^{n+1}\, n\,(a_0\,g^2)^n = \frac{8\,c\,a_0\,g^2}{(1+a_0\,g^2)^2}$$

$$\psi \;=\; \frac{c}{\sqrt{3|b|}} \sum_{n=0}^{\infty} (-1)^n\, b_1\, {a_0}^n\, g^{2n+1} = \frac{c\,b_1\,g}{\sqrt{3|b|})(1+a_0\,g^2)}$$

  - Return to the variables $x$ and $t$

$$u(x,t) \;=\; 2\,c\,\mathrm{sech}^2[\sqrt{c}(x-ct)+\delta],$$
$$v(x,t) \;=\; \pm c\sqrt{\frac{-2(4a+1)}{b}}\,\mathrm{sech}[\sqrt{c}(x-ct)+\delta],$$

  with $\delta = \frac{1}{2}\ln|24(4a+1)/{b_1}^2|$

- CASE 2: $a=-1$ then $a_1$ and $b_1$ are arbitrary, take $e=1$
  - Solution of the recursion relations (with MACSYMA):

$$a_n \;=\; 12\,n\,(-1)^{n+1}\,{a_0}^n,$$
$${b_n}^2 \;=\; \frac{{a_n}^2}{2} = 72\,n^2\,{a_0}^{2n}, \quad n=1,2,...$$

  with $a_0 = a_1/12$.

– Return to the original variables

$$
\begin{aligned}
u(x,t) &= c\,\mathrm{sech}^2[\frac{1}{2}\sqrt{c}(x-ct)+\delta], \\
v(x,t) &= \frac{3}{\sqrt{6|b|}}\,u(x,t) = \frac{3\,c}{\sqrt{6|b|}}\,\mathrm{sech}^2[\frac{1}{2}\sqrt{c}(x-ct)+\delta],
\end{aligned}
$$

with $\delta = \frac{1}{2}\ln|12/a_1|$.

– Observe that for $v(x,t) = \frac{3}{\sqrt{6b}}\,u(x,t)$ both equations reduce to the KdV equation $u_t + 3uu_x + u_{3x} = 0$

- Sine-Gordon equation

$$u_{tt} - u_{xx} - \sin u = 0$$

$$u(x,t) = \arctan\left\{\exp\left[\frac{(1-c)}{2\sqrt{c}}(x - (\frac{c+1}{c-1})t) + \delta\right]\right\}$$

- Coupled Korteweg-de Vries equations

$$u_t - a(6uu_x + u_{xxx}) - 2b\, vv_x = 0,$$
$$v_t + 3uv_x + v_{xxx} = 0$$

$$\begin{aligned} u(x,t) &= 2\, c\, \mathrm{sech}^2\left[\sqrt{c}(x-ct)+\delta\right], \\ v(x,t) &= \pm c\sqrt{\frac{-2(4a+1)}{b}}\, \mathrm{sech}\left[\sqrt{c}(x-ct)+\delta\right], \end{aligned}$$

$$\begin{aligned} u(x,t) &= c\, \mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \\ v(x,t) &= \frac{3}{\sqrt{6|b|}}\, u(x,t) = \frac{3\, c}{\sqrt{6|b|}}\, \mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x-ct)+\delta\right] \end{aligned}$$