

# Application of the Simplified Hirota (Homogenization) Method to a $(3+1)$ -Dimensional Evolution Equation for Deriving Multiple Soliton Solutions

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# Outline

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- Conclusion

## Motivation

- Hirota's method can be used to find exact solutions of nonlinear PDEs, provided these equations can be brought into a bilinear form.
- Finding bilinear forms for nonlinear PDEs, if they exist at all, is highly nontrivial.
- To circumvent this difficulty, we introduce a simplified version of Hirota's method, and use it to construct solitary and soliton solutions.
- Without bilinear forms, exact solutions can still be constructed straightforwardly by solving a perturbation scheme on the computer, using a symbolic manipulation package.

## Simplified Version of Hirota's Method

To illustrate the homogenization method, the well known Korteweg-de Vries (KdV) equation will be used;

$$u_t + buu_x + u_{xxx} = 0, \quad (1)$$

where  $b$  is any real constant.

For the KdV equation, the transformation

$$u = \frac{12f_{xx}}{bf} - \frac{12f_x^2}{bf^2} = \frac{12}{b} \frac{\partial^2}{\partial x^2} \ln f. \quad (2)$$

will allow one to homogenize the equations, once the equation has been “homogenized” it can be readily solved.

First, integrate the equation with respect to  $x$  as many times as possible and ignore integration “constants” (i.e., arbitrary functions of  $t$ ). For the KdV equation only one integration is possible, yielding

$$\int^x u_t dx + \frac{b}{2}u^2 + u_{2x} = 0. \quad (3)$$

Second, change of dependent variable according to

$$u(x, t) = K \frac{\partial^2}{\partial x^2} \ln f(x, t) = K \frac{(f f_{2x} - f_x^2)}{f^2}, \quad (4)$$

to transform (1) into a homogeneous equation in  $f$  and its derivatives of as low a degree as possible. For the KdV equation, setting  $K = \frac{12}{b}$  two terms vanish which allows one to cancel a common factor  $f^2$ . Hence, one obtains a homogeneous equation of second degree,

$$f[f_{xt} + f_{4x}] - f_x f_t - 4f_x f_{3x} + 3f_{2x}^2 = 0. \quad (5)$$

The approach below works for “quadratic” equations of the form

$$f \mathcal{L}(f) + \mathcal{N}(f, f) = 0, \quad (6)$$

where  $\mathcal{L}$  denotes a linear differential operator and  $\mathcal{N}$  is a bilinear (quadratic) differential operator.

For the KdV equation,

$$\mathcal{L}(f) = f_{xt} + f_{4x}, \quad (7)$$

and

$$\mathcal{N}(f, g) = -f_x g_t - 4f_x g_{3x} + 3f_{2x} g_{2x}. \quad (8)$$

Third, seek a solution of the form

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x, t), \quad (9)$$

where  $\epsilon$  serves as a bookkeeping parameter (not a small quantity).

Substitute (9) into (6).



Use Cauchy's product formula,

$$\left( \sum_{r=1}^{\infty} \epsilon^r a_r \right) \left( \sum_{s=1}^{\infty} \epsilon^s b_s \right) = \sum_{n=2}^{\infty} \epsilon^n \sum_{j=1}^{n-1} a_{n-j} b_j, \quad (10)$$

to group the terms in powers of  $\epsilon$ . Equate to zero the different powers of  $\epsilon$  :

$$O(\epsilon^1) \quad : \quad \mathcal{L}(f^{(1)}) = 0, \quad (11)$$

$$O(\epsilon^2) \quad : \quad \mathcal{L}(f^{(2)}) = -\mathcal{N}(f^{(1)}, f^{(1)}), \quad (12)$$

$$\vdots$$

$O(\epsilon^n) :$

$$\mathcal{L}(f^{(n)}) = -\left[\sum_{j=2}^{n-1} f^{(n-j)} \mathcal{L}(f^{(j)}) + \sum_{j=1}^{n-1} \mathcal{N}(f^{(n-j)}, f^{(j)})\right],$$
$$n \geq 3. \tag{13}$$

The latter formula can be written succinctly as

$$\mathcal{L}(f^{(n)}) = -\sum_{j=1}^{n-1} \left[ f^{(n-j)} \mathcal{L}(f^{(j)}) + \mathcal{N}(f^{(n-j)}, f^{(j)}) \right], \quad n \geq 3,$$
$$\tag{14}$$

provided one uses  $\mathcal{L}(f^{(1)}) = 0$ .

For example, application of (13) with  $n = 3$  gives

$$\mathcal{L}(f^{(3)}) = - \left[ f^{(1)} \mathcal{L}(f^{(2)}) + \mathcal{N}(f^{(2)}, f^{(1)}) + \mathcal{N}(f^{(1)}, f^{(2)}) \right]. \quad (15)$$

The N-soliton solution of the KdV is then generated from

$$f^{(1)} = \sum_{i=1}^N \exp(\theta_i) = \sum_{i=1}^N \exp(k_i x - \omega_i t + \delta_i), \quad (16)$$

where  $k_i, \omega_i$  and  $\delta_i$  are constants. Substitution of (16) into (11) yields the dispersion relation  $P(k_i, \omega_i) = 0$ .

For the KdV equation,

$$P(k_i, \omega_i) = k_i(k_i^3 - \omega_i), \quad i = 1, 2, \dots, N. \quad (17)$$

The dispersion law is thus  $\omega_i = k_i^3$ . Using (16), we compute the right hand side of (12):

$$\begin{aligned} & - \sum_{i,j=1}^N 3k_i k_j^2 (k_i - k_j) \exp(\theta_i + \theta_j) = \\ & \sum_{1 \leq i < j \leq N} 3k_i k_j (k_i - k_j)^2 \exp(\theta_i + \theta_j). \end{aligned} \quad (18)$$

Observe that the terms in  $\exp(2\theta_i)$  are missing, which is typical for quadratic equations in  $f$  which admit solitons.

Furthermore, (18) determines the form of  $f^{(2)}$ :

$$f^{(2)} = \sum_{1 \leq i < j \leq N} a_{ij} \exp(\theta_i + \theta_j) = \sum_{1 \leq i < j \leq N} a_{ij} \exp[(k_i + k_j)x - (\omega_i + \omega_j)t + (\delta_i + \delta_j)]. \quad (19)$$

With (7), (17), and (19), the left hand side of (12) is readily computed

$$\begin{aligned} \mathcal{L}(f^{(2)}) &= \sum_{1 \leq i < j \leq N} P(k_i + k_j, \omega_i + \omega_j) a_{ij} \exp(\theta_i + \theta_j) \\ &= \sum_{1 \leq i < j \leq N} 3k_i k_j (k_i + k_j)^2 a_{ij} \exp(\theta_i + \theta_j). \end{aligned} \quad (20)$$

Equating (18) and (20), we have

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i < j \leq N. \quad (21)$$

Proceeding in a similar way with (15) leads to the explicit form of  $f^{(3)}$ .

For example, for  $N = 3$  we have

$$\begin{aligned} f^{(3)} &= b_{123} \exp(\theta_1 + \theta_2 + \theta_3) \\ &= b_{123} \exp[(k_1 + k_2 + k_3)x - (\omega_1 + \omega_2 + \omega_3)t \\ &\quad + (\delta_1 + \delta_2 + \delta_3)], \end{aligned} \quad (22)$$

with

$$b_{123} = a_{12} a_{13} a_{23} = \frac{(k_1 - k_2)^2 (k_1 - k_3)^2 (k_2 - k_3)^2}{(k_1 + k_2)^2 (k_1 + k_3)^2 (k_2 + k_3)^2}. \quad (23)$$

For  $N = 3$ , one can show by the computation of the rest of the scheme that  $f^{(n)} = 0$  for  $n > 3$ . Therefore the expansion in (9) truncates, and

$$\begin{aligned} f = & 1 + \exp \theta_1 + \exp \theta_2 + \exp \theta_3 + a_{12} \exp(\theta_1 + \theta_2) \\ & + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) \\ & + b_{123} \exp(\theta_1 + \theta_2 + \theta_3), \end{aligned} \quad (24)$$

where we set  $\epsilon = 1$ .

Note that  $f$  has no terms in  $\exp(2\theta_1)$ ,  $\exp(2\theta_2)$ ,  $\exp(2\theta_1 + \theta_2)$  and  $\exp(2\theta_2 + \theta_1)$ ,  $\dots$

Upon substitution of (24) into (4) the well-known three-soliton solution of the KdV equation follows.



## Conclusion

Hirota's direct method for solving completely integrable PDEs is algorithmic and implementable in the language of any computer algebra system.

The main advantages of the simplified version of Hirota's method is that the bilinear representation for the equation becomes superfluous.

Therefore, the method can be applied to equations for which the bilinear form is not known or does not exist.

# Software packages in *Mathematica*

Codes are available via the Internet:

URL: <http://inside.mines.edu/~whereman/>

**Thank You**