Symbolic Computation of Recursion Operators of Nonlinear Differential-Difference Equations

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Outline

- Motivation
- Definitions
  - Systems of Differential-difference equations
  - Dilation Invariance
  - Up-shift and Down-shift Operators
  - Conservation Law
  - Generalized (Higher Order) Symmetry
  - Recursion Operator
- Algorithm
- Computer Demonstration

Motivation

Differential-difference Equations (DDEs)

- arise in key branches of physics (classical, quantum, particle, statistical and plasma physics)
- model wave phenomena in nonlinear optics
- arise in bio-sciences

An integrable DDE

- is linearizable or solvable via IST
- has infinite sequence of conservation laws
- has infinite sequence of generalized (higher order) symmetries
- has recursion operator(s) linking these symmetries

Systems of nonlinear DDEs

Consider a system of nonlinear DDEs of first order,

$$\boxed{\dot{\mathbf{u}}_n = \mathbf{F}(\mathbf{u}_{n-\ell}, ..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ..., \mathbf{u}_{n+m})}$$

where $\mathbf{u}_n$ and $\mathbf{F}$ are vector-valued functions with N components, and $\mathbf{F}$ is nonlinear function.

- DDEs with *one* discrete variable, denoted by integer n, which often corresponds to the discretization of a space variable
- dot stands for differentiation with respect to the continuous variable (time t)
- each component of $\mathbf{F}$ is assumed to be a polynomial with constant coefficients

Leading Example: The Toda Lattice

One of the earliest and most famous examples of completely integrable DDEs is the Toda lattice:

$$\ddot{y}_n = \exp\left(y_{n-1} - y_n\right) - \exp\left(y_n - y_{n+1}\right),$$

where y_n is the displacement from equilibrium of the nth particle with unit mass under an exponential decaying interaction force between nearest neighbors.

In new variables (u_n, v_n), defined by $u_n = \dot{y}_n,\ v_n = \exp\left(y_n - y_{n+1}\right)$, lattice can be written in polynomial form

$$\boxed{\dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1})}$$

Dilation Invariance

A DDE is dilation invariant if it is invariant under a dilation (scaling) symmetry.

Example

Toda Lattice is invariant under scaling symmetry

$$\boxed{(t, u_n, v_n) \to (\lambda^{-1} t, \lambda^{1} u_n, \lambda^{2} v_n)}$$

Uniformity in Rank

Define the weight, w, of a variable as the exponent of the scaling parameter (λ) which multiplies that variable. Since λ can be selected at will, t can always be replaced by $\frac{t}{\lambda}$ and, thus, $w(\frac{\mathrm{d}}{\mathrm{dt}}) = w(\mathrm{D}_t) = 1$.

Weights of dependent variables are nonnegative, rational, and independent of n. For example, $w(u_{n-3}) = \cdots = w(u_n) = \cdots = w(u_{n+2})$.

The rank, R, of a monomial is defined as the total weight of the monomial. An expression is uniform in rank if all of its terms have the same rank.

Dilation symmetries, which are special Lie-point symmetries, are common to many DDEs.

Example

For the Toda lattice: $w(u_n) = 1$, and $w(v_n) = 2$

Requiring uniformity in rank for each equation in the Toda lattice allows one to compute the weights of the dependent variables (and, thus, the scaling symmetry) with simple linear algebra. Balancing the weights of the various terms,

$$w(u_n) + 1 = w(v_n), \quad w(v_n) + 1 = w(u_n) + w(v_n),$$

yields

$$\boxed{w(u_n) = 1, \quad w(v_n) = 2}$$

which confirms the dilation symmetry.

Up-Shift and Down-Shift Operator

Define the shift operator D by $\mathrm{D}\mathbf{u}_n = \mathbf{u}_{n+1}$.

The operator D is often called the up-shift operator or forward- or right-shift operator.

The inverse, D^{-1}, is the down-shift operator or backward- or left-shift operator, $\mathrm{D}^{-1}\mathbf{u}_n = \mathbf{u}_{n-1}$.

Shift operators apply to functions by acting on the arguments of the functions.

For example,

$$\begin{aligned}
&\mathrm{D}\mathbf{F}(\mathbf{u}_{n-\ell}, \cdots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \cdots, \mathbf{u}_{n+m}) \\
&= \mathbf{F}(\mathrm{D}\mathbf{u}_{n-\ell}, \cdots, \mathrm{D}\mathbf{u}_{n-1}, \mathrm{D}\mathbf{u}_n, \mathrm{D}\mathbf{u}_{n+1}, \ldots, \mathrm{D}\mathbf{u}_{n+m}) \\
&= \mathbf{F}(\mathbf{u}_{n-\ell+1}, \ldots, \mathbf{u}_n, \mathbf{u}_{n+1}, \mathbf{u}_{n+2}, \cdots, \mathbf{u}_{n+m+1}).
\end{aligned}$$

Conservation Law

A conservation law of a DDE system

$$\boxed{\mathsf{D}_t\, \rho + \Delta\, J = 0}$$

connects a conserved density ρ to an associated flux J, where both are scalar functions depending on $\mathbf{u}_n$ and its shifts.

- D_t is the total derivative with respect to time,
- $\Delta = \mathsf{D} - \mathrm{I}$ is the *forward difference operator*,
- I is the identity operator

For readability, the components of $\mathbf{u}_n$ will be denoted by $u_n, v_n, w_n,$ etc.

Example

The Toda lattice has infinitely many conservation laws

List of the densities of rank $1 \leq R \leq 4$:

$$
\begin{aligned}
\rho^{(1)} &= u_n \\
\rho^{(2)} &= \tfrac{1}{2}u_n^2 + v_n \\
\rho^{(3)} &= \tfrac{1}{3}u_n^3 + u_n(v_{n-1} + v_n) \\
\rho^{(4)} &= \tfrac{1}{4}u_n^4 + u_n^2(v_{n-1} + v_n) + u_n u_{n+1} v_n \\
&\quad + \tfrac{1}{2}v_n^2 + v_n v_{n+1}
\end{aligned}
$$

The first two density-flux pairs are easily computed by hand, and so is

$$\rho_n^{(0)} = \ln(v_n)$$

which is the only non-polynomial density (of rank 0).

Generalized Symmetry

A vector function $\mathbf{G}(\mathbf{u}_n)$ is called a generalized symmetry of DDE system if the infinitesimal transformation $\mathbf{u}_n \rightarrow \mathbf{u}_n + \epsilon\mathbf{G}$ leaves the DDE system invariant up to order ϵ. $\mathbf{G}$ must then satisfy

$$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}]$$

on solutions of the DDE system, where $\mathbf{F}'(\mathbf{u}_n)[\mathbf{G}]$ is the Fréchet derivative of $\mathbf{F}$ in the direction of $\mathbf{G}$.

In the vector case with, say, components u_n and v_n, the Fréchet derivative operator is a matrix operator:

$$\mathbf{F}'(\mathbf{u}_n) = \begin{pmatrix} \sum_k \frac{\partial F_1}{\partial u_{n+k}} \mathrm{D}^k & \sum_k \frac{\partial F_1}{\partial v_{n+k}} \mathrm{D}^k \\ \\ \sum_k \frac{\partial F_2}{\partial u_{n+k}} \mathrm{D}^k & \sum_k \frac{\partial F_2}{\partial v_{n+k}} \mathrm{D}^k \end{pmatrix}.$$

Applied to $\mathbf{G} = (G_1 \; G_2)^{\mathrm{T}}$, where T is transpose, one obtains

$$F_i{}'(\mathbf{u}_n)[\mathbf{G}] = \sum_k \frac{\partial F_i}{\partial u_{n+k}} \mathrm{D}^k G_1 + \sum_k \frac{\partial F_i}{\partial v_{n+k}} \mathrm{D}^k G_2,$$

with $i = 1, 2$.

Example

The first two non-trivial symmetries of the Toda lattice are

$$\mathbf{G}^{(1)} = \begin{pmatrix} v_n - v_{n-1} \\ v_n(u_{n+1} - u_n) \end{pmatrix},$$

$$\mathbf{G}^{(2)} = \begin{pmatrix} v_n(u_n + u_{n+1}) - v_{n-1}(u_{n-1} + u_n) \\ v_n(u_{n+1}^2 - u_n^2 + v_{n+1} - v_{n-1}) \end{pmatrix}.$$

Recursion Operator

A recursion operator $\mathcal{R}$ connects symmetries

$$\mathbf{G}^{(j+s)} = \mathcal{R}\mathbf{G}^{(j)},$$

where $j = 1, 2, \cdots$, and s is the gap length. The symmetries are linked consecutively if $s = 1$.

This happens in most (but not all) cases. For N-component systems, $\mathcal{R}$ is an $N \times N$ matrix operator.

The defining equation for $\mathcal{R}$ is

$$\begin{aligned} &\mathrm{D}_t\mathcal{R} + [\mathcal{R}, \mathbf{F}'(\mathbf{u}_n)] \\ &= \frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[\mathbf{F}] + \mathcal{R} \circ \mathbf{F}'(\mathbf{u}_n) - \mathbf{F}'(\mathbf{u}_n) \circ \mathcal{R} = 0, \end{aligned}$$

where $[\ ,\ ]$ denotes the commutator and $\circ$ the composition of operators.

$\mathcal{R}'[\mathbf{F}]$ is the Fréchet derivative of $\mathcal{R}$ in the direction of $\mathbf{F}$. For the scalar case, the operator $\mathcal{R}$ is of the form

$$\mathcal{R} = U(u_n)\, \mathcal{O}((\mathrm{D}-\mathrm{I})^{-1}, \mathrm{D}^{-1}, \mathrm{I}, \mathrm{D})\, V(u_n),$$

and then

$$\mathcal{R}'[F] = \sum_k (\mathrm{D}^k F) \frac{\partial U}{\partial u_{n+k}} \mathcal{O} V + \sum_k U \mathcal{O} (\mathrm{D}^k F) \frac{\partial V}{\partial u_{n+k}}.$$

For the vector case, the elements of the $N \times N$ operator matrix $\mathcal{R}$ are of the form

$$\mathcal{R}_{ij} = U_{ij}(\mathbf{u}_n)\, \mathcal{O}_{ij}((\mathrm{D}-\mathrm{I})^{-1}, \mathrm{D}^{-1}, \mathrm{I}, \mathrm{D})\, V_{ij}(\mathbf{u}_n).$$

Hence, for the 2-component case

$$
\begin{aligned}
\mathcal{R}'[\mathbf{F}]_{ij} \;=\;& \sum_k (\mathrm{D}^k F_1)\,\frac{\partial U_{ij}}{\partial u_{n+k}}\,\mathcal{O}_{ij}\,V_{ij} \\
&+ \sum_k (\mathrm{D}^k F_2)\,\frac{\partial U_{ij}}{\partial v_{n+k}}\,\mathcal{O}_{ij}\,V_{ij} \\
&+ \sum_k U_{ij}\mathcal{O}_{ij}\,(\mathrm{D}^k F_1)\,\frac{\partial V_{ij}}{\partial u_{n+k}} \\
&+ \sum_k U_{ij}\mathcal{O}_{ij}\,(\mathrm{D}^k F_2)\,\frac{\partial V_{ij}}{\partial v_{n+k}}.
\end{aligned}
$$

Example

The recursion operator of the Toda lattice is

$$\mathcal{R} = \begin{pmatrix} u_n \mathrm{I} & \mathrm{D}^{-1} + \mathrm{I} + (v_n - v_{n-1})(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \\ \\ v_n \mathrm{I} + v_n \mathrm{D} & u_{n+1} \mathrm{I} + v_n (u_{n+1} - u_n)(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \end{pmatrix}.$$

It is straightforward to verify that $\mathcal{R} G^{(1)} = G^{(2)}$.

Algorithm for Recursion Operators

We will now construct the recursion operator the recursion operator for the Toda lattice.

Step 1: Determine the Rank of the Recursion Operator

The difference in the ranks of symmetries is used to compute the rank of the elements of the recursion operator. Recall that

$$\operatorname{rank} \mathbf{G}^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \operatorname{rank} \mathbf{G}^{(2)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Assume that $\mathcal{R}\,\mathbf{G}^{(1)} = \mathbf{G}^{(2)}$ and use the formula

$$\operatorname{rank} \mathcal{R}_{ij} = \operatorname{rank} G_i^{(k+1)} - \operatorname{rank} G_j^{(k)},$$

to compute a rank matrix associated to operator $\mathcal{R}$:

$$\operatorname{rank} \mathcal{R} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Step 2: Determine the Form of the Recursion Operator

Assume that $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$. $\mathcal{R}_0$ is a sum of terms involving $\mathrm{D}^{-1}, \mathrm{I}$, and D. The coefficients of these terms are admissible power combinations of $u_n, u_{n+1}, v_n,$ and v_{n-1} (which come from the terms on the right hand sides of the Toda lattice), so that all the terms have the correct rank. The maximum up-shift and down-shift operator that should be included can be determined by comparing two consecutive symmetries. For the Toda lattice,

$$\mathcal{R}_0 = \begin{pmatrix} (\mathcal{R}_0)_{11} & (\mathcal{R}_0)_{12} \\ (\mathcal{R}_0)_{21} & (\mathcal{R}_0)_{22} \end{pmatrix},$$

with

$$
\begin{aligned}
(\mathcal{R}_0)_{11} &= (c_1 u_n + c_2 u_{n+1})\,\mathrm{I}, \\
(\mathcal{R}_0)_{12} &= c_3 \mathrm{D}^{-1} + c_4 \mathrm{I}, \\
(\mathcal{R}_0)_{21} &= (c_5 u_n^2 + c_6 u_n u_{n+1} + c_7 u_{n+1}^2 + c_8 v_{n-1} + c_9 v_n)\,\mathrm{I} \\
&\quad + (c_{10} u_n^2 + c_{11} u_n u_{n+1} + c_{12} u_{n+1}^2 + c_{13} v_{n-1} \\
&\quad + c_{14} v_n)\,\mathrm{D}, \\
(\mathcal{R}_0)_{22} &= (c_{15} u_n + c_{16} u_{n+1})\,\mathrm{I}.
\end{aligned}
$$

$\mathcal{R}_1$ is a linear combination (with undetermined coefficients $\tilde{c}_{jk}$) of all suitable products of symmetries and covariants, i.e. Fréchet derivatives of densities, sandwiching $(\mathrm{D}-\mathrm{I})^{-1}$. Hence,

$$
\sum_j \sum_k \tilde{c}_{jk} \mathbf{G}^{(j)} (\mathrm{D}-\mathrm{I})^{-1} \otimes \rho_n^{(k)\prime},
$$

where $\otimes$ denotes the matrix outer product, defined as

$$\begin{pmatrix} G_1^{(j)} \\ G_2^{(j)} \end{pmatrix} (\mathrm{D}-\mathrm{I})^{-1} \otimes \left(\rho_{n,1}^{(k)\prime} \;\; \rho_{n,2}^{(k)\prime} \right) =$$

$$\begin{pmatrix} G_1^{(j)}(\mathrm{D}-\mathrm{I})^{-1}\rho_{n,1}^{(k)\prime} & G_1^{(j)}(\mathrm{D}-\mathrm{I})^{-1}\rho_{n,2}^{(k)\prime} \\ G_2^{(j)}(\mathrm{D}-\mathrm{I})^{-1}\rho_{n,1}^{(k)\prime} & G_2^{(j)}(\mathrm{D}-\mathrm{I})^{-1}\rho_{n,2}^{(k)\prime} \end{pmatrix}.$$

Only the pair $(\mathbf{G}^{(1)}, \rho_n^{(0)\prime})$ can be used, otherwise the ranks in the recursion operator would be exceeded. Computing

$$\rho_n^{(0)\prime} = \begin{pmatrix} 0 & \frac{1}{v_n}\mathrm{I} \end{pmatrix},$$

After renaming $\tilde{c}_{10}$ to c_{17}, obtain

$$\mathcal{R}_1 = \begin{pmatrix} 0 & c_{17}(v_{n-1} - v_n)(\mathrm{D} - \mathrm{I})^{-1}\, \frac{1}{v_n}\, \mathrm{I} \\ 0 & c_{17} v_n (u_n - u_{n+1})(\mathrm{D} - \mathrm{I})^{-1}\, \frac{1}{v_n}\, \mathrm{I} \end{pmatrix}.$$

Then,

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1.$$

Step 3: Determine the unknown coefficients

Compute all the terms in the defining equation to find the c_i. We refer to our papers for the details of the computation, resulting in

$$c_2 = c_5 = c_6 = c_7 = c_8 = c_{10} = c_{11} = c_{12} = c_{13} = c_{15} = 0, \quad c_1 = c_3 = c_4 = c_9 = c_{14} = c_{16} = 1,$$ and $c_{17} = -1.$

Substitute the constants into the candidate recursion operator to get its final form:

$$\mathcal{R} = \begin{pmatrix} u_n \mathrm{I} & \mathrm{D}^{-1} + \mathrm{I} + (v_n - v_{n-1})(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \\ & \\ v_n \mathrm{I} + v_n \mathrm{D} & u_{n+1} \mathrm{I} + v_n (u_{n+1} - u_n)(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \end{pmatrix}.$$

Software Demonstration

Software packages in *Mathematica*

Codes are available via the Internet:

URL: http://inside.mines.edu/∼whereman/

Thank You