.

Invited Lecture 2

Symbolic Computation of Conserved Densities for Systems of Nonlinear Evolution Equations

Willy Hereman

Dept. Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO 80401-1887
U.S.A.

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Outline Talk

- Purpose
- Motivation
- Other Programs
- Algorithm
- Implementation
- Examples
- Applications
- More Examples
- Scope and Limitations of Code **condens.m**
- Sample Data File and Output
- Conclusions and Future Work

- **Purpose**

  Design and implement an algorithm to compute polynomial-type
  conservation laws for nonlinear systems of evolution equations

- **Conservation Laws**

  Conservation law for a nonlinear PDE

  $$\rho_t + J_x = 0$$

  ρ is the density, J is the flux

  Consider a single nonlinear evolution equation

  $$u_t = \mathcal{F}(u, u_x, u_{xx}, ..., u_{nx})$$

  If ρ is a polynomial in u and its x derivatives, and does not depend explicitly on x and t, then ρ is called a polynomial conserved density

  If J is also polynomial in u and its x derivatives then this is called
  a polynomial conservation law

  Consequently

  $$P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}$$

  provided J vanishes at infinity

- **Motivation**
  - Conservation laws describe the conservation of fundamental physical quantities such as linear momentum and energy Compare with constants of motion (first integrals) in mechanics
  - or nonlinear PDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws assures complete integrability
  - Conservation laws provide a simple and efficient method to study both quantitative and qualitative properties of PDEs and their solutions, e.g. Hamiltonian structure(s)
  - Connection to generalized symmetries (Fokas, Stud. Appl. Math **77**, 1987)
  - Conservation laws can be used to test numerical integrators for PDEs

For KdV equation, u and u^2 are conserved quantities
Thus, a numerical scheme should preserves the quantities

$$\sum_j U_j^{n-1} = \sum_j U_j^n$$

and

$$\sum_j \left[U_j^{n-1}\right]^2 = \sum_j \left[U_j^n\right]^2$$

For two such schemes see Sanz-Serna, J. Comput. Phys. **47**, 1982

- **Conserved Densities Software**

  - Conserved densities programs **CONSD** and **SYMCD** by Ito and Kako (Reduce, 1985, 1994 & 1996)

  - Conserved densities in **DELiA** by Bocharov (Pascal, 1990)

  - Conserved densities and formal symmetries **FS** by Gerdt and Zharkov (Reduce, 1993)

  - Conserved densities by Roelofs, Sanders and Wang (Reduce 1994, Maple 1995, Form 1995, 1996)

  - Conserved densities **condens.m** by Hereman and Göktaş (Mathematica, 1995)
  - Conservation laws, based on **CRACK**, by Wolf (Reduce, 1995)

  - Conserved densities by Ahner *et al.* (Mathematica, 1995)

  Our program is available at ftp site: *mines.edu*
  in subdirectory
  pub/papers/math_cs_dept/software/condens

- **Example**

Consider the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{3x} = 0$$

Conserved densities

$$\begin{aligned}
\rho_1 &= u, \qquad (u)_t + (\frac{u^2}{2} + u_{2x})_x = 0 \\
\rho_2 &= u^2, \qquad (u^2)_t + (\frac{2u^3}{3} + 2uu_{2x} - {u_x}^2)_x = 0 \\
\rho_3 &= u^3 - 3{u_x}^2, \\
&\quad (u^3 - 3{u_x}^2)_t + \left(\frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x}\right)_x = 0 \\
&\ \vdots \\
\rho_6 &= u^6 - 60u^3{u_x}^2 - 30{u_x}^4 + 108u^2{u_{2x}}^2 \\
&\quad + \frac{720}{7}{u_{2x}}^3 - \frac{648}{7}u{u_{3x}}^2 + \frac{216}{7}{u_{4x}}^2, \qquad \text{...... long ......} \\
&\ \vdots
\end{aligned}$$

Note: KdV equation is invariant under the scaling symmetry

$$(x, t, u) \to (\lambda x, \lambda^3 t, \lambda^{-2} u)$$

u and t carry the weight of 2, resp. 3 derivatives with respect to x

$$u \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$$

- **Key Idea behind Construction of Densities**

Compute the building blocks of density with rank 6

(i) Take all the variables, except $(\frac{\partial}{\partial t})$, with positive weight
Here, only u with $w(u) = 2$

List all possible powers of u, up to rank 6

$$[u, u^2, u^3]$$

Introduce x derivatives to 'complete' the rank

u has weight 2, so introduce $\frac{\partial^4}{\partial x^4}$,

u^2 has weight 4, so introduce $\frac{\partial^2}{\partial x^2}$,

u^3 has weight 6, so no derivative needed

(ii) Apply the derivatives
Remove terms that are total derivatives with respect to x or total derivative up to terms kept earlier in the list

$$\begin{aligned} [u_{4x}] &\rightarrow [\,] \text{ empty list} \\ [{u_x}^2, uu_{2x}] &\rightarrow [{u_x}^2] \;\; (uu_{2x} = (uu_x)_x - {u_x}^2) \\ [u^3] &\rightarrow [u^3] \end{aligned}$$

Combine the ‘building blocks’

$$\rho = u^3 + c_1 {u_x}^2$$

the constant c_1 must be determined

(iii) Determine the unknown coefficients (c_1)

1. Compute $\dfrac{\partial \rho}{\partial t} = 3u^2 u_t + 2c_1 u_x u_{xt}$,
2. Replace u_t by $-(uu_x + u_{3x})$ and u_{xt} by $-(uu_x + u_{3x})_x$
3. Integrate the result with respect to x

   Carry out all integrations by parts

$$\frac{\partial \rho}{\partial t} = -[\frac{3}{4}u^4 + (c_1 - 3)uu_x^2 + 3u^2 u_{2x} - c_1 {u_{2x}}^2 + 2c_1 u_x u_{3x}]_x$$
$$-(c_1 + 3){u_x}^3,$$

4. The non-integrable (last) term must vanish. Thus, $c_1 = -3$

   Result:

$$\rho = u^3 - 3{u_x}^2$$

   Expression [. . .] yields

$$J = \frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x}$$

- **Algorithm and Implementation**

Consider a system of N nonlinear evolution equations

$$u_{i,t} + \mathcal{F}_i(u_j, u_j^{(1)}, \ldots, u_j^{(n)}) = 0 \qquad i, j = 1, 2, \ldots, N$$

where $u_{i,t} \stackrel{\text{def}}{=} \dfrac{\partial u_i}{\partial t}, \quad u_i^{(n)} \stackrel{\text{def}}{=} \dfrac{\partial^n (u_i)}{\partial x^n}$

All u_i depend on x and t

Algorithm consists of three major steps

1. Determine weights (scaling properties) of variables & parameters

2. Construct the form of the density (building blocks)

3. Determine the unknown numerical coefficients

- **Procedure to determine the weights (scaling properties)**

Define

weight of a variable: the number of partial derivatives with respect to x the variable carries

rank of a term: the total weight of that term in terms of partial derivatives with respect to x

For example:

$$\begin{aligned} u_t &\rightarrow r_{1,1} = w(u) + w(\frac{\partial}{\partial t}) \\ uu_x &\rightarrow r_{1,2} = 2w(u) + 1 \\ u_{3x} &\rightarrow r_{1,3} = w(u) + 3 \end{aligned}$$

where $r_{i,k}$ denotes the rank of the k^{th} term in the i^{th} equation
w denotes the weight of its argument
Uniformity in rank requires

$$r_{1,1} = r_{1,2} = r_{1,3}$$

Thus

$$w(u) = 2, \quad w(\frac{\partial}{\partial t}) = 3$$

Require that all terms in any particular equation have the same rank
(*uniformity in rank*)

Different equations in the same system may have different ranks

Introduce the following **notations**:

w returns the weight of its argument

g returns the degree of nonlinearity of its argument

d returns the number of partial derivatives with respect to its argument

$r_{i,k}$ denotes the rank of k^{th} term in the i^{th} equation

Pick

$$w(\frac{\partial}{\partial x}) = 1, \ \ldots, \ w(\frac{\partial^n}{\partial x^n}) = n$$

All weight are assumed nonnegative and rational

List of 'variables' that carry weights

$$\{\frac{\partial}{\partial t}, u_1, u_2, \ldots, u_N, p_1, p_2, \ldots, p_P\}$$

Step 1 Take the i^{th} equation with K_i terms

Step 2 For each of its terms compute the rank

$$r_{i,k} = d(x) + d(t)\ w(\frac{\partial}{\partial t}) + \sum_{j=1}^{N} g(u_j)\ w(u_j) + \sum_{j=1}^{P} g(p_j)\ w(p_j)$$
$$k = 1, 2, \ldots, K$$

If the variable u_j and/or the parameter p_j is missing then $g(u_j) = 0$ or $g(p_j) = 0$, or both

Step 3 Use uniformity in rank in the i^{th} equation
Form the linear system

$$A_i = \{r_{i,1} = r_{i,2} = \cdots = r_{i,K_i}\}$$

Step 4 Repeat steps (1) through (3) for all equations

Step 5 Gather the equations A_i
Form the global linear system

$$\mathcal{A} = \bigcup_{i=1}^{N} A_i$$

Step 6 Solve $\mathcal{A}$ for the $N + P + 1$ unknowns $w(u_j)$, $w(p_j)$ and $w(\frac{\partial}{\partial t})$

- **Example**

Consider the Boussinesq equation

$$u_{tt} - u_{2x} + u u_{2x} + {u_x}^2 + a\, u_{4x} = 0$$

with nonzero parameter a

Can be written as a system of evolution equations

$$\begin{aligned} u_{1,t} + {u_2}' &= 0 \\ u_{2,t} + {u_1}' - u_1 {u_1}' - a\, u_1^{(3)} &= 0 \end{aligned}$$

In the second equation

$${u_1}' \quad \text{and} \quad a\, u_1^{(3)}$$

do not allow for uniformity in rank

Introduce an auxiliary parameter b with weight and replace the system by

$$\begin{aligned} u_{1,t} + {u_2}' &= 0 \\ u_{2,t} + b\, {u_1}' - u_1 {u_1}' - a\, u_1^{(3)} &= 0 \end{aligned}$$

Determine ranks and weights

$$
\begin{aligned}
r_{1,1} &= 1\ w(\frac{\partial}{\partial t}) + 1\ w(u_1) \\
r_{1,2} &= 1 + 1\ w(u_2) \\
r_{2,1} &= 1\ w(\frac{\partial}{\partial t}) + 1\ w(u_2) \\
r_{2,2} &= 1 + 1\ w(u_1) + 1\ w(b) \\
r_{2,3} &= 1 + 2w(u_1) \\
r_{2,4} &= 3 + 1\ w(u_1)
\end{aligned}
$$

Uniformity in rank for each equation requires

$$
\begin{aligned}
A_1 &= \{r_{1,1} = r_{1,2}\} \\
A_2 &= \{r_{2,1} = r_{2,2} = r_{2,3} = r_{2,4}\} \\
\text{and} \quad \mathcal{A} &= A_1 \cup A_2
\end{aligned}
$$

Solve $\mathcal{A}$ for $w(u_1), w(u_2), w(\frac{\partial}{\partial t})$ and $w(b)$

$$
w(u_1) = 2,\ w(b) = 2,\ w(u_2) = 3 \ \text{ and } \ w(\frac{\partial}{\partial t}) = 2
$$

or

$$
u_1 \sim b \sim \frac{\partial^2}{\partial x^2}, \qquad u_2 \sim \frac{\partial^3}{\partial x^3}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2}
$$

- **Construct the Form of the Density**

Let $\mathcal{V} = \{v_1, v_2, \ldots, v_Q\}$ be the sorted list (descending weights) of all variables, including all parameters, but excluding $\dfrac{\partial}{\partial t}$

Step 1 Form all combinations of variables of rank R or less

Recursively, form sets consisting of ordered pairs

$$(T_{q,s}; W_{q,s})$$

where $T_{q,s}$ denotes a combination of different powers of the variables
and $W_{q,s}$ denotes the total weight of $T_{q,s}$

q refers to the variable v_q

s refers to the allowable power of v_q such that $W_{q,s} \leq R$

Set $\mathcal{B}_0 = \{(1; 0)\}$ and proceed as follows:

For $q = 1$ through Q **do**

For $m = 0$ through $M - 1$ **do**

Form $B_{q,m} = \bigcup_{s=0}^{b_{q,m}} \{(T_{q,s}; W_{q,s})\}$

M is the number of pairs in $\mathcal{B}_{q-1}$

$T_{q,s} = T_{q-1,m}\, {v_q}^s$

$W_{q,s} = W_{q-1,m} + s\, w(v_q)$

$(T_{q-1,m}; W_{q-1,m})$ is the $(m+1)^{st}$ ordered pair in $\mathcal{B}_{q-1}$

$b_{q,m} = \llbracket \frac{R - W_{q-1,m}}{w(v_q)} \rrbracket$ is the maximum allowable power of v_q

Set $\mathcal{B}_q = \bigcup_{m=0}^{M-1} B_{q,m}$

Step 2 Set $\mathcal{G} = \mathcal{B}_Q$

Note: $\mathcal{G}$ has all possible combinations of powers of variables that produce rank R or less

Step 3 Introduce partial derivatives with respect to x

For each pair $(T_{Q,s}; W_{Q,s})$ in $\mathcal{G}$, apply $\dfrac{\partial^{\ell}}{\partial x^{\ell}}$ to the term $T_{Q,s}$ provided $\ell = R - W_{Q,s}$ is integer

This introduces just enough partial derivatives with respect to x
so that all the pairs retain weight R

Gather in set $\mathcal{H}$ all the terms that result from computing $\dfrac{\partial^{\ell}(T_{Q,s})}{\partial x^{\ell}}$

Step 4 Remove the terms from $\mathcal{H}$ that can be written as a derivative with respect to x, or as a derivative up to terms kept prior in the set

Call the resulting set $\mathcal{I}$, which consists of the *building blocks* of the density ρ with desired rank R

Step 5 If $\mathcal{I}$ has I elements, then their linear combination will produce the polynomial density of rank R

$$\rho = \sum_{i=1}^{I} c_i \, \mathcal{I}(i)$$

$\mathcal{I}(i)$ denotes the i^{th} element in $\mathcal{I}$

c_i are numerical coefficients, still to be determined

- **Example**

Return to the Boussinesq equation, where

$w(u_1) = 2, w(b) = 2$, and $w(u_2) = 3$

For example, construct the density with rank $R = 6$

$\mathcal{V} = \{u_2, u_1, b\}$

Hence, $v_1 = u_2, v_2 = u_1, v_3 = b$ and $Q = 3$

Step 1 For $q = 1, m = 0$:

$b_{1,0} = [\![\frac{6}{3}]\!] = 2$

Thus, with $T_{1,s} = {u_2}^s$, and $W_{1,s} = 3s$, where $s = 0, 1, 2$

we obtain

$$\mathcal{B}_1 = B_{1,0} = \{(1; 0), (u_2; 3), ({u_2}^2; 6)\}$$

For $q = 2, m = 0$:

$b_{2,0} = [\![\frac{6-0}{2}]\!] = 3$

So, with $T_{2,s} = {u_1}^s$, and $W_{2,s} = 2s$, with $s = 0, 1, 2, 3$

we obtain

$$B_{2,0} = \{(1; 0), (u_1; 2), ({u_1}^2; 4), ({u_1}^3; 6)\}$$

For $q = 2, m = 1$:

$B_{2,1} = \{(u_2; 3), (u_1 u_2; 5)\}$ since $b_{2,1} = [\![\frac{6-3}{2}]\!] = 1$

$T_{2,s} = u_2 \, {u_1}^s$

and

$W_{2,s} = 3 + 2s, \quad \text{and} \quad s = 0, 1$

For $q = 2, m = 2$:

$b_{2,2} = [\![\frac{6-6}{2}]\!] = 0$

Therefore $B_{2,2} = \{({u_2}^2; 6)\}$

Hence,

$\mathcal{B}_2 = \{(1; 0), (u_1; 2), ({u_1}^2; 4), ({u_1}^3; 6), (u_2; 3), (u_1 u_2; 5), ({u_2}^2; 6)\}$

For $q = 3$: introduce possible powers of b

An analogous procedure leads to

$$
\begin{array}{rcl}
B_{3,0} & = & \{(1; 0), (b; 2), (b^2; 4), (b^3; 6)\} \\
B_{3,1} & = & \{(u_1; 2), (bu_1; 4), (b^2 u_1; 6)\} \\
B_{3,2} & = & \{({u_1}^2; 4), (b{u_1}^2; 6)\} \\
B_{3,3} & = & \{({u_1}^3; 6)\}
\end{array}
\qquad
\begin{array}{rcl}
B_{3,4} & = & \{(u_2; 3), (bu_2; 5)\} \\
B_{3,5} & = & \{(u_1 u_2; 5)\} \\
B_{3,6} & = & \{({u_2}^2; 6)\}
\end{array}
$$

Thus

$$
\begin{aligned}
\mathcal{B}_3 \ = \ & \{(1; 0), (b; 2), (b^2; 4), (b^3; 6), (u_1; 2), (bu_1; 4), (b^2 u_1; 6), ({u_1}^2; 4), \\
& (b{u_1}^2; 6), ({u_1}^3; 6), (u_2; 3), (bu_2; 5), (u_1 u_2; 5), ({u_2}^2; 6)\}
\end{aligned}
$$

Step 2 Set $\mathcal{G} = \mathcal{B}_3$

Step 3 Apply derivatives to the first components of the pairs in $\mathcal{G}$

Compute ℓ for each pair of $\mathcal{G}$:

$$\ell = 6, 4, 2, 0, 4, 2, 0, 2, 0, 0, 3, 1, 1, \text{ and } 0$$

Gather the terms after applying partial derivatives w.r.t. x

Hence

$$\begin{aligned} \mathcal{H} \ = \ & \{0, b^3, {u_1}^{(4)}, b{u_1}^{(2)}, b^2 u_1, ({u_1}')^2, u_1 {u_1}^{(2)}, \\ & b{u_1}^2, {u_1}^3, {u_2}^{(3)}, b{u_2}', u_1 {u_2}', {u_1}' u_2, {u_2}^2\} \end{aligned}$$

Step 4 Remove from $\mathcal{H}$ the terms
that can be written as a derivative with respect to x or
as a derivative up to terms retained earlier in that set

This gives

$$\mathcal{I} = \{b^2 u_1, b{u_1}^2, {u_1}^3, {u_2}^2, {u_1}' u_2, ({u_1}')^2\}$$

Step 5 Combine these building blocks and form ρ of rank 6

$$\rho = c_1 \, b^2 u_1 + c_2 \, b{u_1}^2 + c_3 \, {u_1}^3 + c_4 \, {u_2}^2 + c_5 \, {u_1}' u_2 + c_6 \, ({u_1}')^2$$

Calculus of Variations

provides a useful tool to verify if an expression is a derivative

Theorem

If

$$f = f(x, y_1, \ldots, y_1^{(n)}, \ldots, y_N, \ldots, y_N^{(n)})$$

then

$$\mathcal{L}_{\vec{y}}(f) \equiv \vec{0}$$

if and only if

$$f = \frac{\mathrm{d}}{dx} g$$

where

$$g = g(x, y_1, \ldots, y_1^{(n-1)}, \ldots, y_N, \ldots, y_N^{(n-1)})$$

Notations:

$$\vec{y} = [y_1, \ldots, y_N]^T$$

$$\mathcal{L}_{\vec{y}}(f) = [\mathcal{L}_{y_1}(f), \ldots, \mathcal{L}_{y_N}(f)]^T$$

$$\vec{0} = [0, \ldots, 0]^T$$

(T for transpose)

and **Euler Operator**:

$$\mathcal{L}_{y_i} = \frac{\partial}{\partial y_i} - \frac{d}{dx}(\frac{\partial}{\partial y_i{}'}) + \frac{d^2}{dx^2}(\frac{\partial}{\partial y_i{}''}) + \cdots + (-1)^n \frac{d^n}{dx^n}(\frac{\partial}{\partial y_i{}^{(n)}})$$

Proof: see Olver (1986, pp. 252)

- **Determine the Unknown Coefficients**

Step 1 Compute $\frac{\partial \rho}{\partial t}$

Replace all $(u_{i,t})^{(j)}, \ \ i, j = 0, 1, ...$ from the given system

Step 2 The resulting expression E must be the total derivative of some functional $(-J)$

Two options:

- Integrate by parts

  Isolate the non-integrable part

  Set it equal to zero

  The latter leads to a linear system for the coefficients c_i to be solved

- Use the Euler-Lagrange equations

  Apply the Euler operator

  $$\mathcal{L}_{u_i} = \frac{\partial}{\partial u_i} - \frac{d}{dx}(\frac{\partial}{\partial u_i{}'}) + \frac{d^2}{dx^2}(\frac{\partial}{\partial u_i{}''}) + \cdots + (-1)^n \frac{d^n}{dx^n}(\frac{\partial}{\partial u_i{}^{(n)}})$$

  to E

If E is completely integrable no terms will be left, i.e.

$$\mathcal{L}_{u_1}(E) \equiv 0, \ldots, \mathcal{L}_{u_N}(E) \equiv 0$$

otherwise set the remaining terms equal to zero

and form the linear system for the coefficients c_i

With either option, construct a linear system, denoted by $\mathcal{S}$

Step 3 Two cases may occur, depending on whether or not there are parameters in the system

Case I:

If the only unknowns in $\mathcal{S}$ are c_i's, just solve $\mathcal{S}$ for c_i's

Substitute the nonempty solution into ρ to get its final form

Case II:

If in addition to the coefficients c_i's there are also parameters p_i in $\mathcal{S}$

Determine the conditions on the parameters so that ρ of the given form exists for at least some c_i's nonzero

These **compatibility conditions** assure that the system has other than trivial solutions

– Set $\mathcal{C} = \{c_1, c_2, \ldots, c_I\}$ and $i = 1$

– **While** $\mathcal{C} \neq \{\}$ **do**:

  For the building block $\mathcal{I}(i)$ with coefficient c_i to appear in ρ, one needs $c_i \neq 0$

  Therefore, set $c_i = 1$ and eliminate all the other c_j from $\mathcal{S}$

  This gives compatibility conditions consistent with the presence of the term $c_i \mathcal{I}(i)$ in ρ

If compatibility conditions require that some of the parameters are zero

then

c_i must be zero since parameters are assumed to be *nonzero*

Hence, set $\mathcal{C} = \mathcal{C} \backslash \{c_i\}$, and $i = i'$

where i' is the smallest index of the c_j that remain in $\mathcal{C}$

else

Solve the compatibility conditions and for each resulting branch

Solve the system $\mathcal{S}$ for c_j

Substitute the solution into ρ to obtain its final form

Collect those c_j which are zero for *all* of the branches into a set $\mathcal{Z}$

The c_i in $\mathcal{Z}$ might not have occurred in any density yet

Give them a chance to occur:
Set $\mathcal{C} = \mathcal{C} \cap \mathcal{Z}$, and $i = i'$

where i' is the smallest index of the c_j that are still in $\mathcal{C}$

- **Example**

Consider the parameterized coupled KdV equations (Hirota-Satsuma)

$$\begin{aligned} u_t - 6\alpha u u_x + 6 v v_x - \alpha u_{3x} &= 0 \\ v_t + 3 u v_x + v_{3x} &= 0 \end{aligned}$$

Here, $w(u) = w(u) = 2$ and the form of the density of rank 4 is

$$\rho = c_1\, u^2 + c_2\, uv + c_3\, v^2 = c_1\, {u_1}^2 + c_2\, u_1 u_2 + c_3\, {u_2}^2$$

Step 1 Compute ρ_t and replace all $(u_{i,t})^{(j)}$ to get

$$\begin{aligned} E \;=\; & 2c_1 u_1 \left(6\alpha u_1 {u_1}^{(1)} - 6 u_2 {u_2}^{(1)} + \alpha {u_1}^{(3)}\right) \\ & + c_2 u_2 \left(6\alpha u_1 {u_1}^{(1)} - 6 u_2 {u_2}^{(1)} + \alpha {u_1}^{(3)}\right) \\ & - c_2 u_1 \left(3 u_1 {u_2}^{(1)} + {u_2}^{(3)}\right) - 2 c_3 u_2 \left(3 u_1 {u_2}^{(1)} + {u_2}^{(3)}\right) \end{aligned}$$

Step 2 Either integrate by parts or apply the Euler operator

Get the linear system for the coefficients c_1, c_2 and c_3

$$\mathcal{S} = \{(1+\alpha)c_2 = 0,\ 2\, c_1 + c_3 = 0\}$$

Obviously, $\mathcal{C} = \{c_1, c_2, c_3\}$ with one parameter (α)

Step 3 Search for compatibility conditions

– Set $c_1 = 1$, which gives

$$\{c_1 = 1,\ c_2 = 0,\ c_3 = -2\}$$

without any constraint on the parameter α

Since only $c_2 = 0$, $\mathcal{Z} = \{c_2\}$ and $\mathcal{C} = \mathcal{C} \cap \mathcal{Z} = \{c_2\}$, with $i' = 2$

– Set $c_2 = 1$

This leads to the compatibility condition $\alpha = -1$, and

$$\{c_1 = -\frac{1}{2}\, c_3,\ c_2 = 1\}$$

Since $\mathcal{Z} = \{\}$ the procedure ends

One gets *two densities* of rank 4, one without any constraint on α, one with a constraint

In summary:

$$\rho = {u_1}^2 - 2\,{u_2}^2$$

and

$$\rho = -\frac{1}{2} c_3 {u_1}^2 + u_1 u_2 + c_3 {u_2}^2$$

with compatibility condition $\alpha = -1$

Search for densities of rank $R \leq 8$

Rank 2: No condition on α

One always has the trivial density $\rho = u$

Rank 4: At this level, two branches emerge

1. No condition on α

$$\rho = u^2 - 2v^2$$

2. For $\alpha = -1$

$$\rho = uv + c\,(v^2 - \frac{1}{2}u^2), \quad c \text{ is free}$$

Rank 6: No condition on α and

$$\rho = u^3 - \frac{3}{\alpha+1} u v^2 - \frac{1}{2} {u_x}^2 + \frac{3}{\alpha+1} {v_x}^2, \quad \alpha \neq -1$$

Rank 8: The system has conserved density

$$\rho = u^4 - \frac{12}{5} u^2 v^2 + \frac{12}{5} v^4 - 2u {u_x}^2 - \frac{24}{5} u {v_x}^2 - \frac{4}{5} v^2 u_{2x} + \frac{1}{5} {u_{2x}}^2 + \frac{8}{5} {v_{2x}}^2$$

provided that $\alpha = \frac{1}{2}$

Therefore, $\alpha = \frac{1}{2}$ (integrable case!) appears in the computation of density of rank 8

For $\alpha = \frac{1}{2}$, we also computed the density of **Rank 10**

$$\begin{aligned}\rho \ = \ & -\frac{7}{20} u^5 + u^3 v^2 - u v^4 + \frac{7}{4} u^2 {u_x}^2 + \frac{1}{2} v^2 {u_x}^2 + u^2 {v_x}^2 \\ & + 4 v^2 {v_x}^2 + u v^2 u_{2x} + {v_x}^2 u_{2x} - \frac{7}{20} u {u_{2x}}^2 - 2 u {v_{2x}}^2 + \frac{1}{40} {u_{3x}}^2 \\ & + \frac{2}{5} {v_{3x}}^2 + \frac{1}{10} v^2 u_{4x}\end{aligned}$$

- **Application 1**

A Class of Fifth-Order Evolution Equations

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

where α, β, γ are nonzero parameters

$$u \sim \frac{\partial^2}{\partial x^2}$$

Special cases:

$\alpha = 30$	$\beta = 20$	$\gamma = 10$	Lax
$\alpha = 5$	$\beta = 5$	$\gamma = 5$	Sawada – Kotera
			or Caudry–Dodd–Gibbon
$\alpha = 20$	$\beta = 25$	$\gamma = 10$	Kaup–Kupershmidt
$\alpha = 2$	$\beta = 6$	$\gamma = 3$	Ito

Under what conditions for the parameters α, β, and γ does this equation admit a density of fixed rank?

- **Rank 2:**
  No condition

$$\rho = u$$

- **Rank 4:**
  Condition: $\beta = 2\gamma$ (Lax and Ito cases)

$$\rho = u^2$$

– **Rank 6:**
Condition:
$$10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2$$
(Lax, SK, and KK cases)
$$\rho = u^3 + \frac{15}{(-2\beta+\gamma)} {u_x}^2$$

– **Rank 8:**

1. $\beta = 2\gamma$ (Lax and Ito cases)
$$\rho = u^4 - \frac{6\gamma}{\alpha} u {u_x}^2 + \frac{6}{\alpha} {u_{2x}}^2$$
2. $\alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45}$ (SK, KK and Ito cases)
$$\rho = u^4 - \frac{135}{2\beta+\gamma} u {u_x}^2 + \frac{675}{(2\beta+\gamma)^2} {u_{2x}}^2$$

– **Rank 10:**
Condition:
$$\beta = 2\gamma$$
and
$$10\alpha = 3\gamma^2$$
(Lax case)
$$\rho = u^5 - \frac{50}{\gamma} u^2 {u_x}^2 + \frac{100}{\gamma^2} u {u_{2x}}^2 - \frac{500}{7\gamma^3} {u_{3x}}^2.$$

What are the necessary conditions for the parameters α, β, and γ so that this equation could admit ∞ many polynomial conservation laws?

- If $\alpha = \frac{3}{10}\gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case)

- If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case)

- If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \frac{5}{2}\gamma$ then there is a sequence (with gaps!) of conserved densities (KK case)

- If
$$\alpha = -\frac{2\beta^2 - 7\beta\gamma + 4\gamma^2}{45}$$
or
$$\beta = 2\gamma$$
then there is a conserved density of rank 8

Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case)

• Application 2

A Class of Seventh-Order Evolution Equations

$$
\begin{aligned}
& u_t + a u^3 u_x + b {u_x}^3 + c u u_x u_{2x} + d u^2 u_{3x} + e u_{2x} u_{3x} \\
& + f u_x u_{4x} + g u u_{5x} + u_{7x} = 0
\end{aligned}
$$

where a, b, c, d, e, f, g are nonzero parameters

$$
u \sim \frac{\partial^2}{\partial x^2}
$$

Special cases:

SK – Ito Case	$a = 252,\ b = 63,\ c = 378,\ d = 126,$
	$e = 63,\ f = 42,\ g = 21,$
Lax Case	$a = 140,\ b = 70,\ c = 280,\ d = 70,$
	$e = 70,\ f = 42,\ g = 14$

What are the necessary conditions for the parameters so that this equation could admit ∞ many polynomial conservation laws?

Combine the conditions **Rank 2** through **Rank 8**:

- If $a = \frac{5}{98}g^3,\ b = \frac{5}{14}g^2,\ c = \frac{10}{7}g^2,\ d = \frac{5}{14}g^2,\ e = 5g,\ f = 3g$ then there is a sequence (without gaps!) of conserved densities
  (Lax case)
- If $a = \frac{4}{147}g^3,\ b = \frac{1}{7}g^2,\ c = \frac{6}{7}g^2, d = \frac{2}{7}g^2,\ e = 3g,\ f = 2g$ then there is a sequence (with gaps!) of conserved densities
  (SK-Ito case)
- What if $a = \frac{4}{147}g^3,\ b = \frac{5}{14}g^2,\ c = \frac{9}{7}g^2,\ d = \frac{2}{7}g^2,\ e = 6g,\ f = \frac{7}{2}g$?

  This case is not mentioned in the literature!

  With $g = 42$ first five densities

$$
\begin{aligned}
\rho_1 &= u, \\
\rho_2 &= -8u^3 + {u_x}^2, \\
&\vdots \\
\rho_5 &= -\frac{480}{53}u^7 + \frac{3780}{53}u^4{u_x}^2 + \frac{861}{106}u{u_x}^4 \\
&\quad -\frac{644}{53}u^3{u_{2x}}^2 - \frac{291}{212}{u_x}^2{u_{2x}}^2 - \frac{737}{318}u{u_{2x}}^3 \\
&\quad +u^2{u_{3x}}^2 + \frac{133}{636}u_{2x}{u_{3x}}^2 - \frac{2}{53}u{u_{4x}}^2 + \frac{1}{1908}{u_{5x}}^2
\end{aligned}
$$

  Extension of Kaup-Kupershmidt case? YES, proof by Sanders

- **More Examples**

- Nonlinear Schrödinger Equation

$$iq_t - q_{2x} + 2|q|^2 q = 0$$

Program can not handle complex equations
Replace by

$$\begin{aligned} u_t - v_{2x} + 2v(u^2 + v^2) &= 0 \\ v_t + u_{2x} - 2u(u^2 + v^2) &= 0 \end{aligned}$$

where $q = u + iv$

Scaling properties

$$u \sim v \sim \frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2}$$

First seven conserved densities:

$$\begin{aligned} \rho_1 &= u^2 + v^2 \\ \rho_2 &= v u_x \\ \rho_3 &= u^4 + 2u^2 v^2 + v^4 + {u_x}^2 + {v_x}^2 \\ \rho_4 &= u^2 v u_x + \frac{1}{3} v^3 u_x - \frac{1}{6} v u_{3x} \end{aligned}$$

$$
\begin{aligned}
\rho_5 \;=\; & -\frac{1}{2}u^6 - \frac{3}{2}u^4v^2 - \frac{3}{2}u^2v^4 - \frac{1}{2}v^6 - \frac{5}{2}u^2{u_x}^2 - \frac{1}{2}v^2{u_x}^2 \\
& -\frac{3}{2}u^2{v_x}^2 - \frac{5}{2}v^2{v_x}^2 + uv^2u_{2x} - \frac{1}{4}{u_{2x}}^2 - \frac{1}{4}{v_{2x}}^2
\end{aligned}
$$

$$
\begin{aligned}
\rho_6 \;=\; & -\frac{3}{4}u^4vu_x - \frac{1}{2}u^2v^3u_x - \frac{3}{20}v^5u_x + \frac{1}{4}v{u_x}^3 - \frac{1}{4}vu_x{v_x}^2 \\
& +uvu_xu_{2x} + \frac{1}{4}u^2vu_{3x} + \frac{1}{12}v^3u_{3x} - \frac{1}{40}vu_{5x}
\end{aligned}
$$

$$
\begin{aligned}
\rho_7 \;=\; & \frac{5}{4}u^8 + 5u^6v^2 + \frac{15}{2}u^4v^4 + 5u^2v^6 + \frac{5}{4}v^8 + \frac{35}{2}u^4{u_x}^2 \\
& -5u^2v^2{u_x}^2 + \frac{5}{2}v^4{u_x}^2 - \frac{7}{4}{u_x}^4 + \frac{15}{2}u^4{v_x}^2 + 25u^2v^2{v_x}^2 \\
& +\frac{35}{2}v^4{v_x}^2 - \frac{5}{2}{u_x}^2{v_x}^2 - \frac{7}{4}{v_x}^4 - 10u^3v^2u_{2x} - 5uv^4u_{2x} \\
& -5u{v_x}^2u_{2x} + \frac{7}{2}u^2{u_{2x}}^2 + \frac{1}{2}v^2{u_{2x}}^2 + \frac{5}{2}u^2{v_{2x}}^2 \\
& +\frac{7}{2}v^2{v_{2x}}^2 - v^2u_xu_{3x} + \frac{1}{4}{u_{3x}}^2 + \frac{1}{4}{v_{3x}}^2 + uv^2u_{4x}
\end{aligned}
$$

• **The Ito system**

$$\begin{aligned} u_t - u_{3x} - 6uu_x - 2vv_x &= 0 \\ v_t - 2u_xv - 2uv_x &= 0 \end{aligned}$$

$$u \sim \frac{\partial^2}{\partial x^2} \qquad v \sim \frac{\partial^2}{\partial x^2}$$

$$\begin{aligned}
\rho_1 &= c_1 u + c_2 v \\
\rho_2 &= u^2 + v^2 \\
\rho_3 &= 2u^3 + 2uv^2 - {u_x}^2 \\
\rho_4 &= 5u^4 + 6u^2v^2 + v^4 - 10u{u_x}^2 + 2v^2u_{2x} + {u_{2x}}^2 \\
\rho_5 &= 14u^5 + 20u^3v^2 + 6uv^4 - 70u^2{u_x}^2 + 10v^2{u_x}^2 \\
&\quad -4v^2{v_x}^2 + 20uv^2u_{2x} + 14u{u_{2x}}^2 - {u_{3x}}^2 + 2v^2u_{4x}
\end{aligned}$$

and more conservation laws

- **The dispersiveless long-wave system**

$$
\begin{aligned}
u_t + vu_x + uv_x &= 0 \\
v_t + u_x + vv_x &= 0
\end{aligned}
$$

$$u \ \text{ free}, \quad \text{v free}, \quad \text{but} \ \ \text{u} \sim 2\text{v}$$

$$\text{choose} \quad \text{u} \sim \frac{\partial}{\partial \text{x}} \quad \text{and} \quad 2\text{v} \sim \frac{\partial}{\partial \text{x}}$$

$$
\begin{aligned}
\rho_1 &= v \\
\rho_2 &= u \\
\rho_3 &= uv \\
\rho_4 &= u^2 + uv^2 \\
\rho_5 &= 3u^2v + uv^3 \\
\rho_6 &= \frac{1}{3}u^3 + u^2v^2 + \frac{1}{6}uv^4 \\
\rho_7 &= u^3v + u^2v^3 + \frac{1}{10}uv^5 \\
\rho_8 &= \frac{1}{3}u^4 + 2u^3v^2 + u^2v^4 + \frac{1}{15}uv^6
\end{aligned}
$$

and more

Always the same set irrespective the choice of weights

- **A generalized Schamel equation**

$$n^2 u_t + (n+1)(n+2) u^{\frac{2}{n}} u_x + u_{3x} = 0$$

where n is a positive integer

$$\rho_1 \; = \; u, \qquad \rho_2 = u^2$$

$$\rho_3 \; = \; \frac{1}{2} {u_x}^2 - \frac{n^2}{2} u^{2+\frac{2}{n}}$$

For $n \neq 1, 2$ no further conservation laws

- Three-Component Korteweg-de Vries Equation

$$
\begin{aligned}
u_t - 6uu_x + 2vv_x + 2ww_x - u_{3x} &= 0 \\
v_t - 2vu_x - 2uv_x &= 0 \\
w_t - 2wu_x - 2uw_x &= 0
\end{aligned}
$$

Scaling properties

$$
u \sim v \sim w \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}
$$

First five densities:

$$
\begin{aligned}
\rho_1 &= c_1 u + c_2 v + c_3 w \\
\rho_2 &= u^2 - v^2 - w^2 \\
\rho_3 &= -2u^3 + 2uv^2 + 2uw^2 + {u_x}^2 \\
\\
\rho_4 &= -\frac{5}{2}u^4 + 3u^2v^2 - \frac{1}{2}v^4 + 3u^2w^2 - v^2w^2 - \frac{1}{2}w^4 \\
&\quad +5u{u_x}^2 + v^2u_{2x} + w^2u_{2x} - \frac{1}{2}{u_{2x}}^2 \\
\\
\rho_5 &= -\frac{7}{10}u^5 + u^3v^2 - \frac{3}{10}uv^4 + u^3w^2 - \frac{3}{5}uv^2w^2 - \frac{3}{10}uw^4 \\
&\quad +\frac{7}{2}u^2{u_x}^2 + \frac{1}{2}v^2{u_x}^2 + \frac{1}{2}w^2{u_x}^2 + \frac{1}{5}v^2{v_x}^2 \\
&\quad -\frac{1}{5}w^2{v_x}^2 + \frac{1}{5}w^2{w_x}^2 + uv^2u_{2x} + uw^2u_{2x} - \frac{7}{10}u{u_{2x}}^2 \\
&\quad -\frac{1}{5}vw^2v_{2x} + \frac{1}{20}{u_{3x}}^2 + \frac{1}{10}v^2u_{4x} + \frac{1}{10}w^2u_{4x}
\end{aligned}
$$

- **A modified vector derivative NLS equation**

$$\frac{\partial \mathbf{B}_\perp}{\partial t} + \frac{\partial}{\partial x}(B_\perp^2 \mathbf{B}_\perp) + \alpha \mathbf{B}_{\perp 0}\mathbf{B}_{\perp 0} \cdot \frac{\partial \mathbf{B}_\perp}{\partial x} + \mathbf{e}_x \times \frac{\partial^2 \mathbf{B}_\perp}{\partial x^2} = 0$$

Replace the vector equation by

$$u_t + \left(u(u^2+v^2) + \beta u - v_x\right)_x = 0$$
$$v_t + \left(v(u^2+v^2) + u_x\right)_x = 0$$

u and v denote the components of $\mathbf{B}_\perp$ parallel and perpendicular to $\mathbf{B}_{\perp 0}$ and $\beta = \alpha B_{\perp 0}^2$

$$u^2 \sim \frac{\partial}{\partial x}, \qquad v^2 \sim \frac{\partial}{\partial x}, \qquad \beta \sim \frac{\partial}{\partial x}$$

First 6 conserved densities

$$\rho_1 = c_1 u + c_2 v$$

$$\rho_2 = u^2 + v^2$$

$$\rho_3 = \frac{1}{2}(u^2+v^2)^2 - u v_x + u_x v + \beta u^2$$

$$\rho_4 = \frac{1}{4}(u^2+v^2)^3 + \frac{1}{2}({u_x}^2 + {v_x}^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4)$$

$$
\begin{aligned}
\rho_5 \;=\;& \frac{1}{4}(u^2+v^2)^4 - \frac{2}{5}(u_x v_{2x} - u_{2x} v_x) + \frac{4}{5}(u u_x + v v_x)^2 \\
& + \frac{6}{5}(u^2+v^2)({u_x}^2 + {v_x}^2) - (u^2+v^2)^2(u v_x - u_x v) \\
& + \frac{\beta}{5}(2{u_x}^2 - 4u^3 v_x + 2u^6 + 3u^4 v^2 - v^6) + \frac{\beta^2}{5} u^4 \\
\rho_6 \;=\;& \frac{7}{16}(u^2+v^2)^5 + \frac{1}{2}(u_{2x}^2 + v_{2x}^2) \\
-\;& \frac{5}{2}(u^2+v^2)(u_x v_{2x} - u_{2x} v_x) + 5(u^2+v^2)(u u_x + v v_x)^2 \\
+\;& \frac{15}{4}(u^2+v^2)^2({u_x}^2 + {v_x}^2) - \frac{35}{16}(u^2+v^2)^3(u v_x - u_x v) \\
+\;& \frac{\beta}{8}(5u^8 + 10u^6 v^2 - 10u^2 v^6 - 5v^8 + 20u^2 {u_x}^2 \\
-\;& 12u^5 v_x + 60 u v^4 v_x - 20 v^2 {v_x}^2) \\
+\;& \frac{\beta^2}{4}(u^6 + v^6)
\end{aligned}
$$

- **Scope and Limitations**
  - Systems must be polynomial in dependent variables
  - Only two independent variables (x and t) are allowed
  - No terms should *explicitly* depend on x and t
  - Program only computes polynomial-type conserved densities
    only polynomials in the dependent variables and their derivatives
    no explicit dependencies on x and t
  - No limit on the number of evolution equations
    In practice: time and memory constraints
  - Input systems may have (nonzero) parameters
    Program computes the conditions for parameters such that densities (of a given rank) might exist
  - Systems can also have parameters with (unknown) weight
    Allows one to test systems with non-uniform rank
  - For systems where one or more of the weights are free
    Program prompts the user to enter values for the free weights
  - Negative weights are not allowed
  - Fractional weights are permitted
  - Form of ρ can be given (testing purposes)

- **Sample Data and Output**

Data file for the Hirota-Satsuma system

$$\begin{aligned} u_t - 6\alpha u u_x + 6 v v_x - \alpha u_{3x} &= 0 \\ v_t + 3 u v_x + v_{3x} &= 0 \end{aligned}$$

```
(* start of data file d_phrsat.m *)

debug = False;

(* Hirota-Satsuma System *)

eq[1][x,t] = D[u[1][x,t],t]-aa*D[u[1][x,t],{x,3}]-
             6*aa*u[1][x,t]*D[u[1][x,t],x]+
             6*u[2][x,t]*D[u[2][x,t],x];

eq[2][x,t] = D[u[2][x,t],t]+D[u[2][x,t],{x,3}]+
             3*u[1][x,t]*D[u[2][x,t],x];

noeqs = 2;
name = "Hirota-Satsuma System (parameterized)";
parameters = {aa};
weightpars = {};

formrho[x,t] = {};

(* end of data file d_phrsat.m *)
```

Explanation of the lines in the data file

```
debug = False;
```

No trace of output. Set it *True* to see a detailed trace

```
eq[k][x,t] = ...;
```

Give the k^{th} equation of the system in *Mathematica* notation
Note that there is no `== 0` at the end

```
noeqs = 2;
```

Specifies the number of equations in the system

```
name = "Hirota-Satsuma System (parameterized)";
```

Pick a short name for the system. The quotes are essential

```
parameters = {aa};
```

Give a list of the parameters in the system
If there are none, set *parameters* = { };

```
weightpars = {};
```

Give a list of those parameters that are assumed to have weights
Weighted parameters are listed in *weightpars*, not in *parameters*
The latter is only a list of weightless parameters

```
formrho[x,t] = {};
```

Unless the form of ρ is given, program will automatically compute it
This option allows to test forms of ρ (from the literature)

Anything within `(*` and `*)` are comments (ignored by *Mathematica*)

For testing purposes, the form of the density can be given

For example:

```
formrho[x,t]={c[1]*u[1][x,t]^3+c[2]*D[u[1][x,t],x]^2};
```

Density ρ must be given in expanded form and with coefficients c[i]
The braces are essential

If ρ is given, the program skips both the computation of

scaling properties and the construction of ρ

Program continues with given ρ, and computes the c[i]

For search for densities of specific rank, set `formrho[x,t] = { };`

Menu Interface and Sample Output

Example: Compute ρ of rank 4 for Drinfel'd-Sokolov system

$$u_t + 3vv_x = 0$$
$$v_t + 2v_{3x} + 2uv_x + u_xv = 0$$

Start *Mathematica*

Type

```
In[1]:= <<condens.m
```

Menu interface: program prompts automatically for answers

```
  *** MENU INTERFACE ***  (page: 3)
-------------------------------------------
 21) Kaup-Broer System (d_broer.m)
 22) Drinfel'd-Sokolov System (d_soko.m)
 23) Dispersiveless Long Wave System (d_disper.m)
 24) 3-Component KdV System (d_3ckdv.m)
 25) 2-Component Nonlinear Schrodinger Eq.(d_2cnls.m)
 26) Boussinesq System (d_bous.m)
 nn) Next Page
 tt) Your System
 qq) Exit the Program
------------------------------------------
```

```
ENTER YOUR CHOICE: 22

Enter the rank of rho: 4

Use Variational Derivative instead of

Integration by Parts? (y/n): y

Enter the name of the output file: d_soko4.o
```

```
**************************************************
          WELCOME TO THE MATHEMATICA PROGRAM
           by UNAL GOKTAS and WILLY HEREMAN
     FOR THE COMPUTATION OF CONSERVED DENSITIES OF
               Drinfel'd-Sokolov System
      Version 2.2 released on February 29, 1996
                    Copyright 1996
**************************************************

                                  2
This is the density: u[2][x, t]

***************************************************

This is the flux:

                          2            (1,0)      2
2 u[1][x, t] u[2][x, t]  - 2 (u[2])      [x, t]  +

                            (2,0)
>    4 u[2][x, t] (u[2])      [x, t]

***************************************************

Result of explicit verification (rho_t + J_x) = 0

***************************************************
```

```
In[2]:=
```

At the end of computation, density and flux are available

To see these, type

```
In[2]:= rho[x,t]

                           2
Out[2]= u[2][x, t]

In[3]:= j[x,t]


                                                 2
Out[3]= 2 u[1][x, t] u[2][x, t]   -

                (1,0)      2
>     2 (u[2])      [x, t]   +

                              (2,0)
>     4 u[2][x, t] (u[2])      [x, t]
```

- **Conclusions & Further Research**
  - Comparison with other programs
    - Parameter analysis
    - Not restricted to uniform rank
    - Not restricted to evolution equations provided that one can write the equation(s) as a system of evolution equations
  - Usefulness
    - Testing models for integrability
    - Study of classes of nonlinear PDEs
    - Study of generalized symmetries
  - Future work
    - Generalization towards broader classes of equations
    - Generalization towards non-local conservation laws
    - Conservation laws with variable coefficients
    - Interface issues between Mathematica, Maple and Reduce programs

Table 1: Conserved Densities for the Sawada-Kotera and Lax 5th-order equations		
Density	Sawada-Kotera equation	Lax equation
ρ_1	u	u
ρ_2	----	$\frac{1}{2}u^2$
ρ_3	$\frac{1}{3}u^3 - {u_x}^2$	$\frac{1}{3}u^3 - \frac{1}{6}{u_x}^2$
ρ_4	$\frac{1}{4}u^4 - \frac{9}{4}u{u_x}^2 + \frac{3}{4}{u_{2x}}^2$	$\frac{1}{4}u^4 - \frac{1}{2}u{u_x}^2 + \frac{1}{20}{u_{2x}}^2$
ρ_5	----	$\frac{1}{5}u^5 - u^2{u_x}^2 + \frac{1}{5}u{u_{2x}}^2 - \frac{1}{70}{u_{3x}}^2$
ρ_6	$\frac{1}{6}u^6 - \frac{25}{4}u^3{u_x}^2 - \frac{17}{8}{u_x}^4 + 6u^2{u_{2x}}^2$ $+2{u_{2x}}^3 - \frac{21}{8}u{u_{3x}}^2 + \frac{3}{8}{u_{4x}}^2$	$\frac{1}{6}u^6 - \frac{5}{3}u^3{u_x}^2 - \frac{5}{36}{u_x}^4 + \frac{1}{2}u^2{u_{2x}}^2$ $+\frac{5}{63}{u_{2x}}^3 - \frac{1}{14}u{u_{3x}}^2 + \frac{1}{252}{u_{4x}}^2$
ρ_7	$\frac{1}{7}u^7 - 9u^4{u_x}^2 - \frac{54}{5}u{u_x}^4 + \frac{57}{5}u^3{u_{2x}}^2$ $+\frac{648}{35}{u_x}^2{u_{2x}}^2 + \frac{489}{35}u{u_{2x}}^3 - \frac{261}{35}u^2{u_{3x}}^2$ $-\frac{288}{35}u_{2x}{u_{3x}}^2 + \frac{81}{35}u{u_{4x}}^2 - \frac{9}{35}{u_{5x}}^2$	$\frac{1}{7}u^7 - \frac{5}{2}u^4{u_x}^2 - \frac{5}{6}u{u_x}^4 + u^3{u_{2x}}^2$ $+\frac{1}{2}{u_x}^2{u_{2x}}^2 + \frac{10}{21}u{u_{2x}}^3 - \frac{3}{14}u^2{u_{3x}}^2$ $-\frac{5}{42}u_{2x}{u_{3x}}^2 + \frac{1}{42}u{u_{4x}}^2 - \frac{1}{924}{u_{5x}}^2$
ρ_8	----	$\frac{1}{8}u^8 - \frac{7}{2}u^5{u_x}^2 - \frac{35}{12}u^2{u_x}^4 + \frac{7}{4}u^4{u_{2x}}^2$ $+\frac{7}{2}u{u_x}^2{u_{2x}}^2 + \frac{5}{3}u^2{u_{2x}}^3 + \frac{7}{24}{u_{2x}}^4 + \frac{1}{2}u^3{u_{3x}}^2$ $-\frac{1}{4}{u_x}^2{u_{3x}}^2 - \frac{5}{6}u u_{2x}{u_{3x}}^2 + \frac{1}{12}u^2{u_{4x}}^2$ $+\frac{7}{132}u_{2x}{u_{4x}}^2 - \frac{1}{132}u{u_{5x}}^2 + \frac{1}{3432}{u_{6x}}^2$

Table 2: Conserved Densities for the Kaup-Kuperschmidt and Ito 5th-order equations		
Density	Kaup-Kuperschmidt equation	Ito equation
ρ_1	u	u
ρ_2	----	$\frac{u^2}{2}$
ρ_3	$\frac{u^3}{3} - \frac{1}{8}{u_x}^2$	----
ρ_4	$\frac{u^4}{4} - \frac{9}{16}u{u_x}^2 + \frac{3}{64}{u_{2x}}^2$	$\frac{u^4}{4} - \frac{9}{4}u{u_x}^2 + \frac{3}{4}{u_{2x}}^2$
ρ_5	----	----
ρ_6	$\frac{u^6}{6} - \frac{35}{16}u^3{u_x}^2 - \frac{31}{256}{u_x}^4 + \frac{51}{64}u^2{u_{2x}}^2$ $+\frac{37}{256}{u_{2x}}^3 - \frac{15}{128}u{u_{3x}}^2 + \frac{3}{512}{u_{4x}}^2$	----
ρ_7	$\frac{u^7}{7} - \frac{27}{8}u^4{u_x}^2 - \frac{369}{320}u{u_x}^4 + \frac{69}{40}u^3{u_{2x}}^2$ $+\frac{2619}{4480}{u_x}^2{u_{2x}}^2 + \frac{2211}{2240}u{u_{2x}}^3 - \frac{477}{1120}u^2{u_{3x}}^2$ $-\frac{171}{640}u_{2x}{u_{3x}}^2 + \frac{27}{560}u{u_{4x}}^2 - \frac{9}{4480}{u_{5x}}^2$	----
ρ_8	----	----

Table 3: Conserved Densities for the Sawada-Kotera-Ito and Lax 7th-order equations		
Density	Sawada-Kotera-Ito equation	Lax equation
ρ_1	u	u
ρ_2	----	u^2
ρ_3	$-u^3 + {u_x}^2$	$-2u^3 + {u_x}^2$
ρ_4	$3u^4 - 9u{u_x}^2 + {u_{2x}}^2$	$5u^4 - 10u{u_x}^2 + {u_{2x}}^2$
ρ_5	----	$-14u^5 + 70u^2{u_x}^2 - 14u{u_{2x}}^2 + {u_{3x}}^2$
ρ_6	$-\frac{12}{7}u^6 + \frac{150}{7}u^3{u_x}^2 + \frac{17}{7}{u_x}^4 - \frac{48}{7}u^2{u_{2x}}^2$ $-\frac{16}{21}{u_{2x}}^3 + u{u_{3x}}^2 - \frac{1}{21}{u_{4x}}^2$	$-\frac{7}{3}u^6 + \frac{70}{3}u^3{u_x}^2 + \frac{35}{18}{u_x}^4 - 7u^2{u_{2x}}^2$ $-\frac{10}{9}{u_{2x}}^3 + u{u_{3x}}^2 - \frac{1}{18}{u_{4x}}^2$
ρ_7	$5u^7 - 105u^4{u_x}^2 - 42u{u_x}^4 + \frac{133}{3}u^3{u_{2x}}^2$ $+24{u_x}^2{u_{2x}}^2 + \frac{163}{9}u{u_{2x}}^3 - \frac{29}{3}u^2{u_{3x}}^2$ $-\frac{32}{9}u_{2x}{u_{3x}}^2 + u{u_{4x}}^2 - \frac{1}{27}{u_{5x}}^2$	$-\frac{2}{3}u^7 + \frac{35}{3}u^4{u_x}^2 + \frac{35}{9}u{u_x}^4 - \frac{14}{3}u^3{u_{2x}}^2$ $-\frac{7}{3}{u_x}^2{u_{2x}}^2 - \frac{20}{9}u{u_{2x}}^3 + u^2{u_{3x}}^2$ $\frac{5}{9}u_{2x}{u_{3x}}^2 - \frac{1}{9}u{u_{4x}}^2 + \frac{1}{198}{u_{5x}}^2$
ρ_8	----	$\frac{3}{2}u^8 - 42u^5{u_x}^2 - 35u^2{u_x}^4 + 21u^4{u_{2x}}^2$ $+42u{u_x}^2{u_{2x}}^2 + 20u^2{u_{2x}}^3 + \frac{7}{2}{u_{2x}}^4 - 6u^3{u_{3x}}^2$ $-3{u_x}^2{u_{3x}}^2 - 10uu_{2x}{u_{3x}}^2 + u^2{u_{4x}}^2$ $+\frac{7}{11}u_{2x}{u_{4x}}^2 - \frac{1}{11}u{u_{5x}}^2 + \frac{1}{286}{u_{6x}}^2$