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SYMBOLIC SOFTWARE
FOR
SOLITON THEORY

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I. **INTRODUCTION**

Symbolic Software

- Solitons via Hirota's method (Macsyma & Mathematica)
- Painlevé test for ODEs or PDEs (Macsyma)
- Conservation laws of PDEs (Mathematica)
- Lie symmetries for ODEs and PDEs (Macsyma)

Purpose of the programs

- Study of integrability of nonlinear PDEs
- Exact solutions as bench mark for numerical algorithms
- Classification of nonlinear PDEs
- Lie symmetries $\longrightarrow$ solutions via reductions

Collaborators

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II. FOUR SYMBOLIC PROGRAMS

Example 1 – Macsyma/Mathematica
Solitons – Hirota's Method

- Hirota's Direct Method
  allows to construct soliton solutions of
  - nonlinear evolution equations
  - wave equations
  - coupled systems
- Test conditions for existence of soliton solutions
- Examples:
  - Korteweg-de Vries equation (KdV)

$$u_t + 6uu_x + u_{3x} = 0$$

  - Kadomtsev-Petviashvili equation (KP)

$$(u_t + 6uu_x + u_{3x})_x + 3u_{2y} = 0$$

  - Sawada-Kotera equation (SK)

$$u_t + 45u^2u_x + 15u_xu_{2x} + 15uu_{3x} + u_{5x} = 0$$

Hirota's Method

Korteweg-de Vries equation

$$u_t + 6uu_x + u_{3x} = 0$$

Substitute

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

Integrate with respect to x

$$f f_{xt} - f_x f_t + f f_{4x} - 4 f_x f_{3x} + 3 f_{2x}^2 = 0$$

Bilinear form

$$B(f \cdot f) \stackrel{\text{def}}{=} \left(D_x D_t + D_x^4\right)(f \cdot f) = 0$$

Introduce the bilinear operator

$$D_x^m \, D_t^n (f \cdot g) = (\partial x - \partial x')^m (\partial t - \partial t')^n f(x,t)\, g(x',t')|_{x'=x, t'=t}$$

Use the expansion

$$f = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n$$

Substitute f into the bilinear equation

Collect powers in ϵ (book keeping parameter)

$$
\begin{aligned}
O(\epsilon^0) &: B(1\cdot 1) = 0 \\
O(\epsilon^1) &: B(1\cdot f_1 + f_1\cdot 1) = 0 \\
O(\epsilon^2) &: B(1\cdot f_2 + f_1\cdot f_1 + f_2\cdot 1) = 0 \\
O(\epsilon^3) &: B(1\cdot f_3 + f_1\cdot f_2 + f_2\cdot f_1 + f_3\cdot 1) = 0 \\
O(\epsilon^4) &: B(1\cdot f_4 + f_1\cdot f_3 + f_2\cdot f_2 + f_3\cdot f_1 + f_4\cdot 1) = 0 \\
O(\epsilon^n) &: B(\sum_{j=0}^{n} f_j\cdot f_{n-j}) = 0 \qquad \text{with } \; f_0 = 1
\end{aligned}
$$

Start with

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp\left(k_i\, x - \omega_i\, t + \delta_i\right)$$

k_i, ω_i and δ_i are constants
Dispersion law

$$\omega_i = k_i^3 \qquad (i = 1, 2, ..., N)$$

If the original PDE admits a N-soliton solution
then the expansion will truncate at level $n = N$

Consider the case N=3

Terms generated by $B(f_1, f_1)$ determine

$$\begin{aligned} f_2 &= a_{12}\,\exp(\theta_1+\theta_2) + a_{13}\,\exp(\theta_1+\theta_3) + a_{23}\,\exp(\theta_2+\theta_3) \\ &= a_{12}\,\exp\left[(k_1+k_2)\,x - (\omega_1+\omega_2)\,t + (\delta_1+\delta_2)\right] \\ &+ a_{13}\,\exp\left[(k_1+k_3)\,x - (\omega_1+\omega_3)\,t + (\delta_1+\delta_3)\right] \\ &+ a_{23}\,\exp\left[(k_2+k_3)\,x - (\omega_2+\omega_3)\,t + (\delta_2+\delta_3)\right] \end{aligned}$$

Calculate the constants a_{12}, a_{13} and a_{23}

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \qquad i, j = 1, 2, 3$$

Terms from $B(f_1{\cdot}f_2 + f_2{\cdot}f_1)$ determine

$$\begin{aligned} f_3 &= b_{123}\exp(\theta_1+\theta_2+\theta_3) \\ &= b_{123}\exp\left[(k_1+k_2+k_3)x-(\omega_1+\omega_2+\omega_3)t+(\delta_1+\delta_2+\delta_3)\right] \end{aligned}$$

with

$$b_{123} = a_{12}\,a_{13}\,a_{23} = \frac{(k_1-k_2)^2\,(k_1-k_3)^2\,(k_2-k_3)^2}{(k_1+k_2)^2\,(k_1+k_3)^2\,(k_2+k_3)^2}$$

Subsequently, $f_i = 0$ for $i > 3$

Set $\epsilon = 1$

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 \\ &+ a_{12}\,\exp(\theta_1+\theta_2) + a_{13}\,\exp(\theta_1+\theta_3) + a_{23}\,\exp(\theta_2+\theta_3) \\ &+ b_{123}\,\exp(\theta_1+\theta_2+\theta_3) \end{aligned}$$

Return to the original $u(x,t)$

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

Single soliton solution

$$f = 1 + e^{\theta}\ , \qquad \theta = kx - \omega t + \delta$$

k, ω and δ are constants and $\omega = k^3$

Substituting f into

$$\begin{aligned} u(x,t) &= 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} \\ &= 2(\frac{f_{xx}f - f_x^2}{f^2}) \end{aligned}$$

Take $k = 2K$

$$u = 2K^2 \text{sech}^2 K(x - 4K^2 t + \delta)$$

Two-soliton solution

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$$

$$\theta_i = k_i x - \omega_i t + \delta_i$$

with $\omega_i = k_i^3$, $(i = 1, 2)$ and $a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$

Select

$$\begin{aligned} e^{\delta_i} &= \frac{c_i^2}{k_i} e^{k_i x - \omega_i t + \Delta_i} \\ \tilde{f} &= \frac{1}{4} f e^{-\frac{1}{2}(\tilde{\theta}_1 + \tilde{\theta}_2)} \\ \tilde{\theta}_i &= k_i x - \omega_i t + \Delta_i \\ c_i^2 &= \left(\frac{k_2 + k_1}{k_2 - k_1}\right) k_i \end{aligned}$$

Return to $u(x, t)$

$$\begin{aligned} u(x,t) &= \tilde{u}(x,t) = 2\frac{\partial^2 \ln \tilde{f}(x,t)}{\partial x^2} \\ &= \left(\frac{k_2^2 - k_1^2}{2}\right) \left(\frac{k_2^2 \mathrm{cosech}^2 \frac{\tilde{\theta}_2}{2} + k_1^2 \mathrm{sech}^2 \frac{\tilde{\theta}_1}{2}}{(k_2 \coth \frac{\tilde{\theta}_2}{2} - k_1 \tanh \frac{\tilde{\theta}_1}{2})^2}\right) \end{aligned}$$

HIROTA'S CONDITIONS

Single Bilinear equation

$$P(D_x, D_t)(f \cdot f) = 0$$

P is an arbitrary polynomial

Example: KdV equation

$$P(D_x, D_t) = D_x D_t + D_x^4$$

If P satisfies

$$\begin{aligned} P(D_x, D_t) &= P(-D_x, -D_t) \\ P(0,0) &= 0 \end{aligned}$$

then the equation always has a two-soliton solution

- For 2-soliton solution of KdV equation

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}$$

$$\theta_i = k_i x - \omega_i t + \delta_i$$

$$P(k_i, -\omega_i) = 0 \quad \text{or} \quad \omega_i = k_i^3 \ , \quad i = 1, 2$$

$$a_{12} = -\frac{P(k_1 - k_2, -\omega_1 + \omega_2)}{P(k_1 + k_2, -\omega_1 - \omega_2)} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

- For the general N-soliton solution

$$f = \sum_{\mu=0,1} \exp\left[\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\theta_i\right]$$

$$a_{ij} = \exp A_{ij} = -\frac{P(k_i - k_j, -\omega_i + \omega_j)}{P(k_i + k_j, -\omega_i - \omega_j)}$$

Additional condition for $P(D_x, D_t)$

$$S[P, n] = \sum_{\sigma=\pm 1} P\left(\sum_{i=1}^{n} \sigma_i k_i, -\sum_{i=1}^{n} \sigma_i\omega_i\right)$$
$$\times \prod_{i<j}^{(n)} P(\sigma_i k_i - \sigma_j k_j, -\sigma_i\omega_i + \sigma_j\omega_j)\sigma_i\sigma_j = 0 \ ,$$
$$\text{for} \ \ n = 2, \ldots, N$$

Special Features of the Programs

- Expressions for Hirota's bilinear operators

$$D_x^n(f\cdot g) = \sum_{j=0}^{n} \frac{(-1)^{(n-j)} n!}{j!(n-j)!} \frac{\partial^j f}{\partial x^j} \frac{\partial^{n-j} g}{\partial x^{n-j}}$$

$$D_x^m D_t^n (f\cdot g) =$$

$$\sum_{j=0}^{m} \sum_{i=0}^{n} \frac{(-1)^{(m+n-j-i)} m! n!}{j!(m-j)! i!(n-i)!} \frac{\partial^{i+j} f}{\partial t^i \partial x^j} \frac{\partial^{n+m-i-j} g}{\partial t^{n-i} \partial x^{m-j}}$$

- Exponential functions $h(x,t) = \exp(kx - \omega t + \delta)$ are never introduced, they are defined via **gradef**

$$\frac{\partial h(x,t)}{\partial x} = k h(x,t)$$

$$\frac{\partial h(x,t)}{\partial t} = -\omega h(x,t)$$

- Test for existence of soliton solutions uses random numbers
- Symbolic test is available (may be slow)

Macsyma and Mathematica Programs

The user provides

- the bilinear operator B
- N for n-soliton solution
- number of random tests for 3-soliton solution
- symbolic test for 3-soliton solution (true or false)
- number of random tests for 4-soliton solution
- symbolic test for 4-soliton solution (true or false)

The symbolic programs calculates

- conditions for existence of up to a 4-soliton solution
- the one-, two- and three- soliton solutions
- a_{ij} and b_{123} in factored form
- the function f, so that $u(x,t)$ can be computed

Example 2 – Macsyma/Mathematica
Painlevé Integrability Test

- Painlevé test for 3rd order equations by Hajee (Reduce, 1982)
- Painlevé program (parts) by Hlavatý (Reduce, 1986)
- ODE_Painlevé by Winternitz & Rand (Macsyma, 1986)
- PDE_Painlevé by Hereman & Van den Bulck (Macsyma, 1987)
- Painlevé test by Conte & Musette (AMP, 1988)
- Painlevé analysis by Renner (Reduce, 1992)
- Painlevé test for simple systems by Hereman, Elmer and Göktaş (Macsyma, 1994-96, under development)

Example 3 - Mathematica
Conserved Densities

- **Purpose**

  Compute polynomial-type conservation laws of single PDEs and systems of PDEs

  Conservation law:

  $$\rho_t + J_x = 0$$

  both $\rho(u, u_x, u_{2x}, \ldots, u_{nx})$ and $J(u, u_x, u_{2x}, \ldots, u_{nx})$

  Consequently

  $$P = \int_{-\infty}^{+\infty} \rho dx = \text{constant}$$

  provided J vanishes at infinity

  Compare with constants of motions in classical mechanics

- **Example**

Consider the KdV equation

$$u_t + uu_x + u_{3x} = 0$$

Conserved densities:

$$\begin{aligned}
\rho_1 &= u \\
\rho_2 &= u^2 \\
\rho_3 &= u^3 - 3u_x^2 \\
&\vdots \\
\rho_6 &= u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \\
&+ \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2 \\
&\vdots
\end{aligned}$$

Integrable equations have ∞ conservation laws

- **Algorithm and Implementation**

Consider the scaling (weights) of the KdV

$$u \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$$

Compute building blocks of ρ_3

(i) Start with building block u^3

Divide by u and differentiate twice $(u^2)_{2x}$

Produces the list of terms

$[u_x^2, uu_{2x}] \longrightarrow [u_x^2]$

Second list: remove terms that are total derivative
with respect to x or total derivative
up to terms earlier in the list

Divide by u^2 and differentiate twice $(u)_{4x}$

Produces the list: $[u_{4x}] \longrightarrow [\,]$

$[\,]$ is the empty list
Gather the terms:

$$\rho_3 = u^3 + c[1]u_x^2$$

where the constant c_1 must be determined

(ii) Compute $\dfrac{\partial \rho_3}{\partial t} = 3u^2 u_t + 2c_1 u_x u_{xt}$

Replace u_t by $-(uu_x + u_{xxx})$ and u_{xt} by $-(uu_x + u_{xxx})_x$

(iii) Integrate the result with respect to x

Carry out all integrations by parts

$$\frac{\partial \rho_3}{\partial t} = -[\frac{3}{4}u^4 + (c_1 - 3)uu_x^2 + 3u^2 u_{xx} - c_1 u_{xx}^2 + 2c_1 u_x u_{xxx}]_x \\ -(c_1 + 3)u_x^3$$

The last non-integrable term must vanish

Thus, $c_1 = -3$

Result:

$$\rho_3 = u^3 - 3u_x^2$$

(iv) Expression [...] yields

$$J_3 = \frac{3}{4}u^4 - 6uu_x^2 + 3u^2 u_{xx} + 3u_{xx}^2 - 6u_x u_{xxx}$$

Computer building blocks of ρ_6

(i) Start with u^6

Divide by u and differentiate twice

$(u^5)_{2x}$ produces the list of terms

$[u^3u_x^2, u^4u_{2x}] \longrightarrow [u^3u_x^2]$

Next, divide u^6 by u^2, and compute $(u^4)_{4x}$

Corresponding list:

$[u_x^4, uu_x^2u_{2x}, u^2u_{2x}^2, u^2u_xu_{3x}, u^3u_{4x}] \longrightarrow [u_x^4, u^2u_{2x}^2]$

Proceed with $(\frac{u^6}{u^3})_{6x} = (u^3)_{6x}, (\frac{u^6}{u^4})_{8x} = (u^2)_{8x}$

and $(\frac{u^6}{u^5})_{10x} = (u)_{10x}$

Obtain the lists:

$[u_{2x}^3, u_xu_{2x}u_{3x}, uu_{3x}^2, u_x^2u_{4x}, uu_{2x}u_{4x}, uu_xu_{5x}, u^2u_{6x}] \longrightarrow$
$[u_{2x}^3, uu_{3x}^2]$

$[u_{4x}^2, u_{3x}u_{5x}, u_{2x}u_{6x}, u_xu_{7x}, uu_{8x}] \longrightarrow [u_{4x}^2]$

and $[u_{10x}] \longrightarrow [\,]$

Gather the terms:

$$\rho_6 = u^6 + c_1u^3u_x^2 + c_2u_x^4 + c_3u^2u_{2x}^2 + c_4u_{2x}^3 + c_5uu_{3x}^2 + c_6u_{4x}^2$$

where the constants c_i must be determined

(ii) Compute $\frac{\partial}{\partial t}\rho_6$

Replace $u_t, u_{xt}, \ldots, u_{nx,t}$ by $-(uu_x + u_{xxx}), \ldots$

(iii) Integrate the result with respect to x

Carry out all integrations by parts

Require that non-integrabe part vanishes

Set to zero all the coefficients of the independent combinations involving powers of u and its derivatives with respect to x

Solve the linear system for unknowns $c_1, c_2, \ldots, c_6$

Result:

$$\begin{aligned}\rho_6 &= u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \\ &+ \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2\end{aligned}$$

(iv) Flux J_6 can be computed by substituting the constants into the integrable part of ρ_6

- Further Examples

* Conservation laws of generalized Schamel equation

$$n^2 u_t + (n+1)(n+2) u^{\frac{2}{n}} u_x + u_{xxx} = 0$$

n positive integer

$$\begin{aligned} \rho_1 &= u \\ \rho_2 &= u^2 \\ \rho_3 &= \frac{1}{2} u_x^2 - \frac{n^2}{2} u^{2+\frac{2}{n}} \end{aligned}$$

no further conservation laws

* Conserved densities of modified vector derivative nonlinear Schrödinger equation

$$\frac{\partial \mathbf{B}_\perp}{\partial t} + \frac{\partial}{\partial x}(B_\perp^2 \mathbf{B}_\perp) + \alpha \mathbf{B}_{\perp 0} \mathbf{B}_{\perp 0} \cdot \frac{\partial \mathbf{B}_\perp}{\partial x} + \mathbf{e}_x \times \frac{\partial^2 \mathbf{B}_\perp}{\partial x^2} = 0$$

Replace vector equation by

$$\begin{aligned} u_t + \big(u(u^2+v^2) + \beta u - v_x\big)_x &= 0 \\ v_t + \big(v(u^2+v^2) + u_x\big)_x &= 0 \end{aligned}$$

u and v denote the components of $\mathbf{B}_\perp$ parallel and perpendicular to $\mathbf{B}_{\perp 0}$ and $\beta = \alpha B_{\perp 0}^2$

The first 5 conserved densities are:

$$\rho_1 = u^2 + v^2$$

$$\rho_2 = \frac{1}{2}(u^2 + v^2)^2 - uv_x + u_x v + \beta u^2$$

$$\rho_3 = \frac{1}{4}(u^2 + v^2)^3 + \frac{1}{2}(u_x^2 + v_x^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4)$$

$$\rho_4 = \frac{1}{4}(u^2 + v^2)^4 - \frac{2}{5}(u_x v_{xx} - u_{xx} v_x) + \frac{4}{5}(uu_x + vv_x)^2$$

$$+ \frac{6}{5}(u^2 + v^2)(u_x^2 + v_x^2) - (u^2 + v^2)^2(uv_x - u_x v)$$

$$+ \frac{\beta}{5}(2u_x^2 - 4u^3 v_x + 2u^6 + 3u^4 v^2 - v^6) + \frac{\beta^2}{5}u^4$$

$$
\begin{aligned}
\rho_5 &= \frac{7}{16}(u^2+v^2)^5 + \frac{1}{2}(u_{xx}^2+v_{xx}^2) \\
&- \frac{5}{2}(u^2+v^2)(u_x v_{xx} - u_{xx} v_x) + 5(u^2+v^2)(uu_x+vv_x)^2 \\
&+ \frac{15}{4}(u^2+v^2)^2(u_x^2+v_x^2)^2 - \frac{35}{16}(u^2+v^2)^3(uv_x - u_x v) \\
&+ \frac{\beta}{8}(5u^8 + 10u^6v^2 - 10u^2v^6 - 5v^8 + 20u^2u_x^2 \\
&- 12u^5v_x + 60uv^4v_x - 20v^2v_x^2) \\
&+ \frac{\beta^2}{4}(u^6+v^6)
\end{aligned}
$$

A Class of Fifth-order Evolution Equations

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

Special cases:

$\alpha = 30$	$\beta = 20$	$\gamma = 10$	Lax
$\alpha = 5$	$\beta = 5$	$\gamma = 5$	Sawada Kotera or Caudry–Dodd–Gibbon
$\alpha = 20$	$\beta = 25$	$\gamma = 10$	Kaup–Kuperschmidt
$\alpha = 2$	$\beta = 6$	$\gamma = 3$	Ito

Table 1 Conserved Densities for Sawada-Kotera and Lax equations		
Density	Sawada-Kotera equation	Lax equation
ρ_1	u	u
ρ_2	----	$\frac{1}{2}u^2$
ρ_3	$\frac{1}{3}u^3 - u_x^2$	$\frac{1}{3}u^3 - \frac{1}{6}u_x^2$
ρ_4	$\frac{1}{4}u^4 - \frac{9}{4}uu_x^2 + \frac{3}{4}u_{2x}^2$	$\frac{1}{4}u^4 - \frac{1}{2}uu_x^2 + \frac{1}{20}u_{2x}^2$
ρ_6	----	$\frac{1}{5}u^5 - u^2u_x^2 + \frac{1}{5}uu_{2x}^2 - \frac{1}{70}u_{3x}^2$
ρ_6	$\frac{1}{6}u^6 - \frac{25}{4}u^3u_x^2 - \frac{17}{8}u_x^4 + 6u^2u_{2x}^2$ $+2u_{2x}^3 - \frac{21}{8}uu_{3x}^2 + \frac{3}{8}u_{4x}^2$	$\frac{1}{6}u^6 - \frac{5}{3}u^3u_x^2 - \frac{5}{36}u_x^4 + \frac{1}{2}u^2u_{2x}^2$ $+\frac{5}{63}u_{2x}^3 - \frac{1}{14}uu_{3x}^2 + \frac{1}{252}u_{4x}^2$
ρ_7	$\frac{1}{7}u^7 - 9u^4u_x^2 - \frac{54}{5}uu_x^4 + \frac{57}{5}u^3u_{2x}^2$ $+\frac{648}{35}u_x^2u_{2x}^2 + \frac{489}{35}uu_{2x}^3 - \frac{261}{35}u^2u_{3x}^2$ $-\frac{288}{35}u_{2x}u_{3x}^2 + \frac{81}{35}uu_{4x}^2 - \frac{9}{35}u_{5x}^2$	$\frac{1}{7}u^7 - \frac{5}{2}u^4u_x^2 - \frac{5}{6}uu_x^4 + u^3u_{2x}^2$ $+\frac{1}{2}u_x^2u_{2x}^2 + \frac{10}{21}uu_{2x}^3 - \frac{3}{14}u^2u_{3x}^2$ $-\frac{5}{42}u_{2x}u_{3x}^2 + \frac{1}{42}uu_{4x}^2 - \frac{1}{924}u_{5x}^2$
ρ_8	----	$\frac{1}{8}u^8 - \frac{7}{2}u^5u_x^2 - \frac{35}{12}u^2u_x^4 + \frac{7}{4}u^4u_{2x}^2$ $+\frac{7}{2}uu_x^2u_{2x}^2 + \frac{5}{3}u^2u_{2x}^3 + \frac{7}{24}u_{2x}^4 + \frac{1}{2}u^3u_{3x}^2$ $-\frac{1}{4}u_x^2u_{3x}^2 - \frac{5}{6}uu_{2x}u_{3x}^2 + \frac{1}{12}u^2u_{4x}^2$ $+\frac{7}{132}u_{2x}u_{4x}^2 - \frac{1}{132}uu_{5x}^2 + \frac{1}{3432}u_{6x}^2$

Table 2 Conserved Densities for Kaup-Kuperschmidt and Ito equations

Density	Kaup-Kuperschmidt equation	Ito equation
ρ_1	u	u
ρ_2	----	$\frac{u^2}{2}$
ρ_3	$\frac{u^3}{3} - \frac{1}{8}u_x^2$	----
ρ_4	$\frac{u^4}{4} - \frac{9}{16}uu_x^2 + \frac{3}{64}u_{2x}^2$	$\frac{u^4}{4} - \frac{9}{4}uu_x^2 + \frac{3}{4}u_{2x}^2$
ρ_5	----	----
ρ_6	$\frac{u^6}{6} - \frac{35}{16}u^3u_x^2 - \frac{31}{256}u_x^4 + \frac{51}{64}u^2u_{2x}^2$ $+\frac{37}{256}u_{2x}^3 - \frac{15}{128}uu_{3x}^2 + \frac{3}{512}u_{4x}^2$	----
ρ_7	$\frac{u^7}{7} - \frac{27}{8}u^4u_x^2 - \frac{369}{320}uu_x^4 + \frac{69}{40}u^3u_{2x}^2$ $+\frac{2619}{4480}u_x^2u_{2x}^2 + \frac{2211}{2240}uu_{2x}^3 - \frac{477}{1120}u^2u_{3x}^2$ $-\frac{171}{640}u_{2x}u_{3x}^2 + \frac{27}{560}uu_{4x}^2 - \frac{9}{4480}u_{5x}^2$	----
ρ_8	----	----

Example 4 – Macsyma
Lie-point Symmetries

- System of m differential equations of order k

$$\Delta^i(x, u^{(k)}) = 0, \quad i = 1, 2, ..., m$$

with p independent and q dependent variables

$$\begin{aligned} x &= (x_1, x_2, ..., x_p) \in \mathbb{R}^p \\ u &= (u^1, u^2, ..., u^q) \in \mathbb{R}^q \end{aligned}$$

- The group transformations have the form

$$\tilde{x} = \Lambda_{group}(x, u), \quad \tilde{u} = \Omega_{group}(x, u)$$

where the functions Λ_{group} and Ω_{group} are to be determined

- Look for the Lie algebra $\mathcal{L}$ realized by the vector field

$$\alpha = \sum_{i=1}^{p} \eta^i(x, u) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x, u) \frac{\partial}{\partial u^l}$$

Procedure for finding the coefficients

- Construct the k^{th} prolongation $\text{pr}^{(k)}\alpha$ of the vector field α
- Apply it to the system of equations
- Request that the resulting expression vanishes on the solution set of the given system

$$\text{pr}^{(k)}\alpha\Delta^i \,|_{\Delta^j=0} \qquad i,j=1,...,m$$

- This results in a system of linear homogeneous PDEs for η^i and φ_l, with independent variables x and u (*determining equations*)
- Procedure thus consists of two major steps:

deriving the determining equations
solving the determining equations

Procedure for Computing the Determining Equations

- Use multi-index notation $J = (j_1, j_2, ..., j_p) \in \mathbb{N}^p$, to denote partial derivatives of u^l

$$u^l_J \equiv \frac{\partial^{|J|} u^l}{\partial x_1{}^{j_1} \partial x_2{}^{j_2} ... \partial x_p{}^{j_p}} \,,$$

  where $|J| = j_1 + j_2 + ... + j_p$

- $u^{(k)}$ denotes a vector whose components are all the partial derivatives of order 0 up to k of all the u^l

- Steps:

  (1) Construct the k^{th} prolongation of the vector field

$$\text{pr}^{(k)} \alpha = \alpha + \sum_{l=1}^{q} \sum_{J} \psi_l^J(x, u^{(k)}) \frac{\partial}{\partial u^l_J}, \quad 1 \leq |J| \leq k$$

  The coefficients ψ_l^J of the first prolongation are:

$$\psi_l^{J_i} = D_i \varphi_l(x, u) - \sum_{j=1}^{p} u^l_{J_j} D_i \eta^j(x, u),$$

  where J_i is a $p-$tuple with 1 on the i^{th} position and zeros elsewhere

  D_i is the total derivative operator

$$D_i = \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \sum_{J} u^l_{J+J_i} \frac{\partial}{\partial u^l_J}, \quad 0 \leq |J| \leq k$$

Higher order prolongations are defined recursively:

$$\psi_l^{J+J_i} = D_i \psi_l^J - \sum_{j=1}^{p} u^l_{J+J_j} D_i \eta^j(x,u), \quad |J| \geq 1$$

(2) Apply the prolonged operator $\mathrm{pr}^{(k)}\alpha$ to each equation $\Delta^i(x, u^{(k)}) = 0$

Require that $\mathrm{pr}^{(k)}\alpha$ vanishes on the solution set of the system

$$\mathrm{pr}^{(k)}\alpha \; \Delta^i \,|_{\Delta^j=0} \; = 0 \quad i, j = 1, ..., m$$

(3) Choose m components of the vector $u^{(k)}$, say $v^1, ..., v^m$, such that:

(a) Each v^i is equal to a derivative of a u^l $(l = 1, ..., q)$ with respect to at least one variable x_i $(i = 1, ..., p)$.

(b) None of the v^i is the derivative of another one in the set.

(c) The system can be solved algebraically for the v^i in terms of the remaining components of $u^{(k)}$, which we denoted by w:

$$v^i = S^i(x, w), \quad i = 1, ..., m.$$

(d) The derivatives of v^i,

$$v^i_J = D_J S^i(x, w),$$

where $D_J \equiv D_1^{j_1} D_2^{j_2} ... D_p^{j_p}$, can all be expressed in terms of the components of w and their derivatives, without ever reintroducing the v^i or their derivatives.

For instance, for a system of evolution equations

$$u_t^i(x_1, ..., x_{p-1}, t) = F^i(x_1, ..., x_{p-1}, t, u^{(k)}), \quad i = 1, ..., m,$$

where $u^{(k)}$ involves derivatives with respect to the variables x_i but not t, choose $v^i = u_t^i$.

(4) Eliminate all v^i and their derivatives from the expression prolonged vector field, so that all the remaining variables are independent

(5) Obtain the determining equations for $\eta^i(x, u)$ and $\varphi_l(x, u)$ by equating to zero the coefficients of the remaining independent derivatives u_J^l.

III. OTHER SOFTWARE

Painlevé Integrability Test

- Painlevé test for 3rd order equations by Hajee (Reduce, 1982)
- Painlevé program (parts) by Hlavatý (Reduce, 1986)
- ODE_Painlevé by Winternitz & Rand (Macsyma, 1986)
- PDE_Painlevé by Hereman & Van den Bulck (Macsyma, 1987)
- Painlevé test by Conte & Musette (AMP, 1988)
- Painlevé analysis by Renner (Reduce, 1992)
- Painlevé test for simple systems by Hereman, Elmer and Göktaş (Macsyma, 1994-96, under development)

.

Conserved Densities, Lax Pairs & Bäcklund Transformations

- Lax pairs by Ito (Reduce, 1985)
- Conserved densities by Ito & Kako (Reduce, 1985)
- Lax pairs & Bäcklund Transformations by Conte & Musette (AMP, C++, 1991-1993)
- Conserved densities by Gerdt (Reduce, 1993)
- Conserved densities by Hereman, Verheest and Göktaş (Mathematica, 1993-1995)

.

Explicit Solitary Wave Solutions & Solitons

- Hirota operators by Ito (Reduce, 1988)
- Solitary wave solutions via truncated Laurent series by Hereman (Macsyma, 1989)
- Solitary wave solutions based on exponential method by Hereman (Macsyma, 1992)
- Classification of bilinear operators by Hietarinta (Reduce, 1989)
- Hirota's method by Hereman & Zhuang (Macsyma, 1990)
- Hirota's method by Hereman & Zhuang (Mathematica, 1995)
- Simplified version of Hirota's method by Hereman and Nuseir (Macsyma, 1995)

IV. **PLANS FOR THE FUTURE**

Extension of Symbolic Software Packages (Macsyma/Mathematica)

- Lie symmetries of differential-difference equations
- Solver for systems of linear, homogeneous PDEs (Hereman)
- Painlevé test for systems of PDEs (Elmer, Göktaş & Coffey)
- Solitons via Hirota's method for bilinear equations (Zhuang)
- Simplification of Hirota's method (Hereman & Nuseir)
- Conservation laws of PDEs with variable coefficients (Göktaş)
- Lax pairs, special solutions, ...

New Software

- Wavelets (prototype/educational tool)
- Other methods for Differential Equations