THE PAINLEVÉ TEST FOR NONLINEAR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

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1 Introduction: Mathematical Background

- Roughly speaking, dynamical systems may be divided into two classes :

  1. Systems exhibiting chaotic behavior, i.e. their solutions depend sensitively on the initial data. Such systems are usually **not** explicitely **integrable** in terms of "elementary functions"
  2. Systems that are **algebraically completely integrable**

- Painlevé *et al* identified all second order ODEs of the form $f_{xx} = K(x, f, f_x)$, which are globally integrable in terms of elementary functions by quadratures or by linearization

  The restrictions on the function K, which is rational in f_x, algebraic in f, and analytic in x, arise from careful singularity analysis

- Integrability requires that the only **movable singularities** in the solution $f(x)$ are **poles**

- Singularities are **movable** if their location depends on the initial conditions

- **Critical points** (including logarithmic branch points and essential singularities) ought to be fixed to have integrability

- *Definition:* A simple equation or system has the **Painlevé Property (PP)** if its solution in the complex plane has no worse singularities than movable poles

- For ODEs :

  1. They admit a finite dimensional Hamiltonian formulation
  2. They have a finite number of first integrals

- For PDEs :

  1. Integrability became associated with the existence of a Lax representation which allows **linearization** of the given equation(s) (Inverse Scattering Transform)
  2. Ablowitz *et al* conjectured that every ODE, obtained by an exact reduction of a PDE solvable by IST, possesses the PP
  3. Similarity transformations reduce them into ODEs of Painlevé type, (e.g. in one of the six Painlevé transcendents)
  4. Special ones admit solitary travelling wave solutions (called **solitons** if they conserve their identity upon collision)
  5. They possess infinitely many conserved quantities and symmetries, nontrivial prolongation structures, associated Kac-Moody algebras, etc.

2 Algorithm

The algorithm below (Weiss *et al*) enables to verify if the ODE or PDE satisfies the **necessary criteria** to have the **PP**

- For the **PDE case**:

  The solution f, say in two independent variables (t, x), expressed as a Laurent series,

  $$f = g^{\alpha} \sum_{k=0}^{\infty} u_k g^k \tag{1}$$

  should only have movable poles

  In (1), $u_0(t, x) \neq 0$, α is a negative integer, and $u_k(t, x)$ are analytic functions in a neighborhood of the singular, non-characteristic manifold $g(t, x) = 0$, with $g_x(t, x) \neq 0$

- For the **ODE case**:

  x will be replaced by $g + x_0$ in (1); x_0 being the initial value for x

The **Painlevé test** is carried out in three steps:

- **Step 1**:
  1. Substitute the **leading order** term,
  $$f \propto u_0 \; g^{\alpha} \tag{2}$$
  into the given equation
  2. Determine the integer $\alpha < 0$ by balancing the minimal power terms
  3. Calculate u_0

- **Step 2**:
  1. Substitute the generic terms
  $$f \propto u_0 \; g^{\alpha} + u_r \; g^{\alpha+r} \tag{3}$$
  into the equation, only retaining its most singular terms
  2. Require that u_r is arbitrary
  3. Calculate the corresponding values of $r > 0$, called **resonances**

- **Step 3**:
  1. Substitute the truncated expansion
  $$f = g^{\alpha} \sum_{k=0}^{rmax} u_k \; g^k, \tag{4}$$
  where $rmax$ represents the largest resonance, into the complete equation
  2. Determine u_k unambiguously at the non-resonance levels
  3. Check whether or not the **compatibility condition** is satisfied at resonance levels

- An equation or system has the **Painlevé Property** and is conjectured to be integrable if :
  1. Step 1 thru 3 can be carried out consistently with $\alpha < 0$ and with positive resonances,
  2. The compatibility conditions are identically satisfied for all resonances

- For an equation to be integrable it is **necessary** but **not sufficient** that it passes the Painlevé test

- Equations for which $\alpha = 0$ deserve special attention

- For some equations, the resonances are complex conjugate, the compatibility being satisfied at real resonance levels

- Quite often the compatibility conditions impose conditions on the coefficients or parameters in the given equation

- The above algorithm does not detect the existence of essential singularities

3 Spin-offs of the Painlevé Analysis

- Truncation of the Laurent series (1) at the constant level term leads to **auto-Bäcklund or Darboux transformations**
- The resulting Painlevé-Bäcklund equations, obtained by substitution of the truncated expansion and equating to zero powers of g, can be linearized to derive **Lax pairs** for various ODEs and PDEs
- As a consequence for ODEs, it is possible to construct **algebraic curves** and explicitly integrate the equations of motion
- For PDEs, Painlevé analysis determines the **speed of travelling wave** solutions (see Exs. 3 and 5)
- It provides insight in the construction of **soliton** solutions via direct methods (Hirota's formalism and its clones)
- The Painlevé test helps in identifying the infinite dimensional symmetry algebras for PDEs, which have the structure of subalgebras of **Kac-Moody and Virasoro algebras**

4 Scope and Limitations of the Program

4.1 Scope

- The program works for a single ODE or PDE
- The degree of nonlinearity in all the variables is unlimited
- The number of parameters in the equation is unlimited
- The number of independent variables is also unlimited
- ODEs and PDEs may have explicitly given time/space dependent coefficients of integer degree (see Ex. 4)
- PDEs may have arbitrary time/space dependent coefficients (see Ex. 4)
- Coefficients may be complex, although the usefulness of the Painlevé test is then debatable
- A selected positive or negative rational value of α, or $\alpha = 0$ can be supplied
- The time consuming calculation of the coefficients u_k and the verification of the compatibility conditions is optional
- It is possible to substitute an expansion of the form (4) with a selected number of terms, e.g. to carry on with the calculations beyond $rmax$
- The output provides vital information, including error messages and warnings, to remedy possible problems

4.2 Limitations

- Systems of equations are excluded
- The algorithm carries out the traditional Painlevé test based on the expansion (1), with at least rational α, hence general fractional expansions in g are excluded
- Transcendental terms are not allowed They can often be removed by a suitable transformation of the dependent variable (see Ex. 2)
- Arbitrary parameters in the powers of f and its derivatives are not allowed
- Neither are arbitrary (unspecified) functions of f and its derivatives
- Selective substitution of certain u_k is not possible. u_0, u_1, etc. are explicitly determined whenever possible, and their expressions are used in the calculation of the next u_k
- Nonlinear equations for u_0 are not solved. If they occur the program carries on with the undetermined coefficient u_0 (see Ex. 5)

- The program only checks if the compatibility condition is identically satisfied. It does not solve for arbitrary parameters (or functions) or for u_0 and its derivatives, should these occur (see Exs. 3, 4 and 5)
- Intermediate output is only possible by putting extra *print* statements in the program
- The expressions occuring in the output on the screen are not accessible for further interactive calculations

5 Using the Program

The program carries out the Painlevé test in **batch mode** without interaction by the user
The user only has to type in the LHS of the equation and possibly select some options

- **For ODEs:**
  1. The dependent variable f and independent variable x is mandatory
  2. A typical term in the ODE reads $fx[.](x)$, where within the brackets the order of derivation is inserted. The function without derivatives may be denoted by f itself
  3. The symbol *eq* denotes the LHS of the equation
  4. Ex.: To test the Fisher ODE, $f_{xx} + af_x - f^2 + f = 0$, one would enter

     $$eq : fx[2](x) + a * f[x](x) + f ** 2 - f;$$

     The program will then treat a as an arbitrary parameter
- **For PDEs:**
  1. A typical term reads $ftxyz[k, l, m, n](t, x, y, z)$, where the integers k, l, m, and n are the orders of derivation with respect to the variables t, x, y, and z
  2. Ex.: To test the KdV equation, $f_t + aff_x + f_{xxx} = 0$, one enters

     $$eq : ftx[1, 0](t, x) + a * f * ftx[0, 1](t, x) + ftx[0, 3](t, x);$$

     Again, the program will treat a as an arbitrary parameter

6 Examples

In the examples, a, b and c are arbitrary parameters, and a(t) is an arbitrary function

Example 1: The Korteweg-de Vries Equation

For the ubiquitous KdV equation,

$$f_t + f f_x + b f_{xxx} = 0, \tag{5}$$

the program provides the following output:

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PAINLEVE ANALYSIS OF EQUATION, $b f_{xxx} + f f_x + ft = 0$

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SUBSTITUTE $u_0 \; g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE $[2 \; alfa - 1, alfa - 3]$

$*$ COEFFICIENT OF $g^{2 \; alfa - 1}$ IS ${u_0}^2 \; alfa \; g_x$

$*$ COEFFICIENT OF $g^{alfa - 3}$ IS $u_0 \; (alfa - 2) \; (alfa - 1) \; alfa \; b \; (g_x)^3$

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FOR EXPONENTS $(2 \; alfa - 1)$ AND $(alfa - 3)$ OF g, DO

WITH $alfa = -2$, POWER OF g is $-5 \longrightarrow$ SOLVE FOR u_0

TERM $-2 \; u_0 \; g_x \; (12 \; b \; (g_x)^2 + u_0) \; \frac{1}{g^5}$ IS DOMINANT IN EQUATION.

WITH $u_0 = -12 \; b \; (g_x)^2 \longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0 \; g^{alfa} + u_r \; g^{r+alfa}$ FOR f IN EQUATION

TERM $b \; (g_x)^3 \; (r - 6) \; (r - 4) \; (r + 1) \; u_r \; g^{r-5}$ IS DOMINANT IN EQUATION.

THE 2 NON-NEGATIVE INTEGRAL ROOTS ARE $[r = 4, \; r = 6]$

WITH MAXIMUM RESONANCE $= 6 \longrightarrow$ CHECK RESONANCES.

SUBSTITUTE POWER SERIES $\sum_{k=0}^{6} g^{k-2} \; u_k$ FOR f IN EQUATION.

WITH $u_0 = -12 \; b \; (g_x)^2$

$*$ COEFFICIENT OF $\frac{1}{g^4}$ IS $6b(g_x)^2((-12b(g_x)^2)_x - 36bg_xg_{xx} + 5u_1g_x)$

$u_1 = 12bg_{xx}$

$*$ COEFFICIENT OF $\frac{1}{g^3}$ IS $24bg_x(4bg_xg_{xxx} - 3b(g_{xx})^2 + u_2(g_x)^2 + g_tg_x)$

$$u_2 = -\frac{4bg_xg_{xxx} - 3b(g_{xx})^2 + g_tg_x}{(g_x)^2}$$

$*$ COEFFICIENT OF $\frac{1}{g^2}$ IS

$$-12b(b(g_x)^2 g_{xxxx} - 4b g_x g_{xx} g_{xxx} + 3b(g_{xx})^3 - g_t g_x g_{xx} - u_3 (g_x)^4 + g_{tx}(g_x)^2)/g_x$$

$$u_3 = \frac{b(g_x)^2 g_{xxxx} - 4b g_x g_{xx} g_{xxx} + 3b(g_{xx})^3 - g_t g_x g_{xx} + g_{tx}(g_x)^2}{(g_x)^4}$$

$*$ COEFFICIENT OF $\frac{1}{g}$ IS 0

u_4 IS ARBITRARY !

COMPATIBILITY CONDITION IS SATISFIED !

$*$ COEFFICIENT OF 1 IS

$$-(b^2(g_x)^4 g_{xxxxxx} - 9b^2(g_x)^3 g_{xx} g_{xxxxx} - 17b^2(g_x)^3 g_{xxx} g_{xxxx} + 48b^2(g_x)^2(g_{xx})^2 g_{xxxx} - 2b g_t (g_x)^3 g_{xxxx} + 70b^2(g_x)^2 g_{xx}(g_{xxx})^2 - 174b^2 g_x (g_{xx})^3 g_{xxx} + 17b g_t (g_x)^2 g_{xx} g_{xxx} - 8b g_{tx}(g_x)^3 g_{xxx} + 81b^2(g_{xx})^5 - 21b g_t g_x (g_{xx})^3 + 21b g_{tx}(g_x)^2(g_{xx})^2 + 6u_4 b(g_x)^6 g_{xx} - 9b g_{txx}(g_x)^3 g_{xx} + (g_t)^2(g_x)^2 g_{xx} + 6u_5 b(g_x)^8 + 6(u_4)_x b(g_x)^7 + g_{tt}(g_x)^4 + 2b g_{txxx}(g_x)^4 - 2g_t g_{tx}(g_x)^3)/(g_x)^5$$

$$\begin{aligned} u_5 = {} & -(b^2(g_x)^4 g_{xxxxxx} - 9b^2(g_x)^3 g_{xx} g_{xxxxx} - 17b^2(g_x)^3 g_{xxx} g_{xxxx} \\ & + 48b^2(g_x)^2(g_{xx})^2 g_{xxxx} - 2b g_t (g_x)^3 g_{xxxx} + 70b^2(g_x)^2 g_{xx}(g_{xxx})^2 \\ & - 174b^2 g_x (g_{xx})^3 g_{xxx} + 17b g_t (g_x)^2 g_{xx} g_{xxx} - 8b g_{tx}(g_x)^3 g_{xxx} \\ & + 81b^2(g_{xx})^5 - 21b g_t g_x (g_{xx})^3 + 21b g_{tx}(g_x)^2(g_{xx})^2 + 6u_4 b(g_x)^6 g_{xx} \\ & - 9b g_{txx}(g_x)^3 g_{xx} + (g_t)^2(g_x)^2 g_{xx} + 6(u_4)_x b(g_x)^7 + g_{tt}(g_x)^4 \\ & + 2b g_{txxx}(g_x)^4 - 2g_t g_{tx}(g_x)^3)/(6b(g_x)^8) \end{aligned}$$

$*$ COEFFICIENT OF g IS 0

u_6 IS ARBITRARY !

COMPATIBILITY CONDITION IS SATISFIED !

Example 2: The sine-Gordon Equation

The transcendental term in the *sine-Gordon* equation, in light cone coordinates,

$$u_{tx} - sin(u) = 0, \tag{6}$$

can be removed by the simple substitution $f = exp(iu)$ to obtain an equivalent equation with polynomial terms:

$$-2f_t f_x + 2ff_{tx} - f^3 + f = 0. \tag{7}$$

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PAINLEVE ANALYSIS OF EQUATION, $-2f_t f_x + 2ff_{tx} - f^3 + f = 0$

$- -$

SUBSTITUTE $u_0\ g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE $[2alfa - 2,\ 3\ alfa,\ alfa]$

$*$ COEFFICIENT OF $g^{2\ alfa-2}$ IS $-2\ {u_0}^2\ alfa\ g_t\ g_x$

$*$ COEFFICIENT OF $g^{3\ alfa}$ IS $-{u_0}^3$

$*$ COEFFICIENT OF g^{alfa} IS u_0

$- -$

FOR EXPONENTS $2\ alfa - 2$ AND $3\ alfa$ OF g, DO

WITH $alfa = -2$, POWER OF g is 6 $\longrightarrow$ SOLVE FOR u_0

TERM ${u_0}^2\ (4g_t g_x - u_0)\frac{1}{g^6}$ IS DOMINANT IN EQUATION.

WITH $u_0 = 4g_t g_x$ $\longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0\ g^{alfa} + u_r\ g^{r+alfa}$ FOR f IN EQUATION

TERM $8(g_t)^2(g_x)^2(r-2)(r+1)\ u_r\ g^{r-6}$ IS DOMINANT
IN EQUATION.

THE ONLY NON-NEGATIVE INTEGRAL ROOT IS $[r = 2]$

WITH MAXIMUM RESONANCE = 2 $\longrightarrow$ CHECK RESONANCES.

SUBSTITUTE POWER SERIES $\sum_{k=0}^{2} g^{k-2} u_k$ FOR f IN EQUATION.

WITH $u_0 = 4g_t g_x$

$*$ COEFFICIENT OF $\frac{1}{g^5}$ IS $-16(g_t)^2(4g_{tx} + u_1)(g_x)^2$

$u_1 = -4\ g_{tx}$

$*$ COEFFICIENT OF $\frac{1}{g^4}$ IS 0

u_2 IS ARBITRARY !

COMPATIBILITY CONDITION IS SATISFIED !

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FOR EXPONENTS $(2\ alfa - 2)$ AND $(alfa)$ OF g, $alfa = 2$ IS
NON-NEGATIVE.

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FOR EXPONENTS (3 $alfa$) AND ($alfa$) OF g, $alfa = 0$ IS NON-NEGATIVE.

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Example 3: The Fisher Equation

From rigorous analysis it follows that if the initial datum is given by $u(0,x) = 1 \quad (x \leq 0)$, $u(0,x) = 0$ $(x > 0)$, then the solution of the *Fisher* equation,

$$u_t - u_{xx} + u^2 - u = 0, \tag{8}$$

will converge to a travelling wave of speed $c = 2$. Furthermore, for every speed $c \geq 2$ there is a travelling wave with $u(t, -\infty) = 1, u(t, \infty) = 0$.
In 1979, an exact closed form solution of (8) was constructed:
$u(t,x) = U(x - x_0 - \frac{5t}{\sqrt{6}}) = U(\xi)$, where

$$U(\xi) = \frac{1}{4}\left(1 - \tanh\left(\frac{\xi}{2\sqrt{6}}\right)\right)^2, \tag{9}$$

with x_0 any constant.
The Painlevé analysis for (8), put into a travelling frame of reference, exactly determines this particular wave speed $c = \frac{5}{\sqrt{6}}$, which, indeed, is larger than 2.

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PAINLEVE ANALYSIS OF EQUATION, $f_{xx} + cf_x - f^2 + f = 0$

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SUBSTITUTE $u_0\ g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE [2 $alfa$, $alfa - 2$]

∗ COEFFICIENT OF $g^{2\,alfa}$ IS $-{u_0}^2$

∗ COEFFICIENT OF g^{alfa-2} IS $u_0\ (alfa - 1)\ alfa$

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FOR EXPONENTS (2 $alfa$) AND ($alfa - 2$) OF g, DO

WITH $alfa = -2$, POWER OF g is $-4 \longrightarrow$ SOLVE FOR u_0

TERM $-(u_0 - 6)\ u_0\ \frac{1}{g^4}$ IS DOMINANT IN EQUATION.

WITH $u_0 = 6 \longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0\ g^{alfa} + u_r\ g^{r+alfa}$ FOR f IN EQUATION

TERM $(r - 6)(r + 1)\ u_r\ g^{r-4}$ IS DOMINANT IN EQUATION.

THE ONLY NON-NEGATIVE INTEGRAL ROOT IS [$r = 6$]

WITH MAXIMUM RESONANCE = 6 $\longrightarrow$ CHECK RESONANCES.

SUBSTITUTE POWER SERIES $\sum_{k=0}^{6} g^{k-2}\ u_k$ FOR f IN EQUATION.

WITH $u_0 = 6$

$*$ COEFFICIENT OF $\frac{1}{g^3}$ IS $-2\,(6c+5u_1)$

$u_1 = -\dfrac{6c}{5}$

$*$ COEFFICIENT OF $\frac{1}{g^2}$ IS $-\dfrac{6(c^2+50u_2-25)}{25}$

$u_2 = -\dfrac{(c-5)(c+5)}{50}$

$*$ COEFFICIENT OF $\frac{1}{g}$ IS $-\dfrac{6(c^3+250u_3)}{125}$

$u_3 = -\dfrac{c^3}{250}$

$*$ COEFFICIENT OF 1 IS $-\dfrac{7c^4+5000u_4-125}{500}$

$u_4 = -\dfrac{7c^4-125}{5000}$

$*$ COEFFICIENT OF g IS $-\dfrac{79c^5-1375c+75000u_5}{12500}$

$u_5 = -\dfrac{c(79c^4-1375)}{75000}$

$*$ COEFFICIENT OF g^2 IS $-\dfrac{c^2(6c^2-25)(6c^2+25)}{6250} = 0$

u_6 IS ARBITRARY !

COMPATIBILITY CONDITION: $-\dfrac{c^2(6c^2-25)(6c^2+25)}{6250} = 0,$

$***$ CONDITION IS NOT SATISFIED .$***$

$***$ CHECK FOR FREE PARAMETERS OR PRESENCE OF u_0 $***$ −

Example 4: The cylindrical KDV Equation

The *cylindrical Korteweg-de Vries* equation,

$$\frac{f_x}{2t} + f_{xxxx} + 6ff_{xx} + 6(f_x)^2 + f_{tx} = 0, \tag{10}$$

has the Painlevé property. One easily determines the coefficient $\frac{1}{2t}$ in (10), by analyzing a cylindrical KdV equation with arbitrary coefficient $a(t)$ of f_x. Integration of the compatibility condition $a(t)_t + 2a(t)^2 = 0$, gives $a(t) = \frac{1}{2t}$.

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PAINLEVE ANALYSIS OF EQUATION,

$$a(t)f_x + f_{xxxx} + 6ff_{xx} + 6(f_x)^2 + f_{tx} = 0$$

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SUBSTITUTE $u_0\ g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE $[2\ alfa - 2,\ alfa - 4]$

$*$ COEFFICIENT OF $g^{2\ alfa-2}$ IS $6{u_0}^2\ alfa\ (2alfa - 1)\ (g_x)^2$

$*$ COEFFICIENT OF g^{alfa-4} IS $u_0\ (alfa - 3)(alfa - 2)(alfa - 1)\ alfa\ (g_x)^4$

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FOR EXPONENTS $(2\ alfa - 2)$ AND $(alfa - 4)$ OF g, DO

WITH $alfa = -2$, POWER OF g is $-6 \longrightarrow$ SOLVE FOR u_0

TERM 60 $u_0\ (g_x)^2\ (2(g_x)^2 + u_0)\ \frac{1}{g^6}$ IS DOMINANT IN EQUATION.

WITH $u_0 = -2(g_x)^2 \longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0\ g^{alfa} + u_r\ g^{r+alfa}$ FOR f IN EQUATION

TERM $(g_x)^4(r - 6)(r - 5)(r - 4)(r + 1)u_r g^{r-6}$ IS DOMINANT IN EQUATION.

THE 3 NON-NEGATIVE INTEGRAL ROOTS ARE

$[r = 4,\ r = 5,\ r = 6]$

WITH MAXIMUM RESONANCE = 6 $\longrightarrow$ CHECK RESONANCES.

SUBSTITUTE POWER SERIES $\sum_{k=0}^{6} g^{k-2}u_k$ FOR f IN EQUATION.

WITH $u_0 = -2(g_x)^2$

$*$ COEFFICIENT OF $\frac{1}{g^5}$ IS $120(g_x)^4(2g_{xx} - u_1)$

$u_1 = 2g_{xx}$

$*$ COEFFICIENT OF $\frac{1}{g^4}$ IS

$-12(g_x)^2(4g_xg_{xxx} - 3(g_{xx})^2 + 6u_2(g_x)^2 + g_tg_x)$

$$u_2 = -\frac{4g_x g_{xxx} - 3(g_{xx})^2 + g_t g_x}{6(g_x)^2}$$

$*$ COEFFICIENT OF $\frac{1}{g^3}$ IS

$$4((g_x)^3 a(t) + (g_x)^2 g_{xxxx} - 4g_x g_{xx} g_{xxx} + 3(g_{xx})^2 - g_t g_x g_{xx} - 6u_3 (g_x)^4 + g_{tx}(g_x)^2)$$

$$u_3 = ((g_x)^3 a(t) + (g_x)^2 g_{xxxx} - 4g_x g_{xx} g_{xxx} + 3(g_{xx})^2 - g_t g_x g_{xx} + g_{tx}(g_x)^2)/(6(g_x)^4)$$

$*$ COEFFICIENT OF $\frac{1}{g^2}$ IS 0

u_4 IS ARBITRARY !

COMPATIBILITY CONDITION IS SATISFIED !

$*$ COEFFICIENT OF $\frac{1}{g}$ IS 0

u_5 IS ARBITRARY !

COMPATIBILITY CONDITION IS SATISFIED !

$*$ COEFFICIENT OF 1 IS $\dfrac{a(t)_t + 2a(t)^2}{6}$

u_6 IS ARBITRARY !

COMPATIBILITY CONDITION: $\dfrac{a(t)_t + 2a(t)^2}{6} = 0$,

$***$ CONDITION IS NOT SATISFIED .$***$

$***$ CHECK FOR FREE PARAMETERS OR PRESENCE OF u_0 $***$

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Example 5: The Fitz Hugh-Nagumo Equation

In 1975 it was found that the *Fitz Hugh-Nagumo* equation,

$$u_t - u_{xx} - u(1-u)(u-a) = 0, \tag{11}$$

has a closed form travelling wave solution, $u(t,x) = U(x - x_0 - ct) = U(\xi)$, where

$$U(\xi) = \left(1 + \exp(-\frac{\xi}{\sqrt{2}})\right)^{-1} = \frac{1}{2}\left(1 + \tanh\left(\frac{\xi}{2\sqrt{2}}\right)\right), \tag{12}$$

and $c = \frac{2a-1}{\sqrt{2}}$.

Motivated by the results of the Painlevé analysis, recently two more closed form solutions to the Fitz Hugh-Nagumo equation were found. Both take the form,

$$U(\xi) = \frac{1}{2}\left(A + B\,\tanh\left(\frac{C\xi}{2\sqrt{2}}\right)\right), \tag{13}$$

where

$$A = B = C = a \quad \text{for} \quad c = \frac{2-a}{\sqrt{2}},$$

and

$$A = 1 + a \quad \text{and} \quad B = C = a - 1 \quad \text{for} \quad c = \frac{-(a+1)}{\sqrt{2}}.$$

Carrying out the Painlevé test for (11), in a travelling frame, leads to a compatibility condition which for $u_0 = \sqrt{2}$ factors into

$$c\left(c - \frac{(2a-1)}{\sqrt{2}}\right)\left(c + \frac{(a+1)}{\sqrt{2}}\right)\left(c + \frac{(a-2)}{\sqrt{2}}\right) = 0. \tag{14}$$

The nonzero roots for c correspond with the wave speeds in (12) and (13). Remark that for $a = \frac{1}{2}$ the wave (12) is stationary $(c = 0)$

PAINLEVE ANALYSIS OF EQUATION, $f_{xx} + cf_x - f^3 + (a+1)f^2 - af = 0$

SUBSTITUTE $u_0\ g^{alfa}$ FOR f IN ORIGINAL EQUATION.

MINIMUM POWERS OF g ARE $[3alfa,\ 2alfa,\ alfa - 2]$

$*$ COEFFICIENT OF g^{3alfa} IS $-{u_0}^3$

$*$ COEFFICIENT OF g^{2alfa} IS ${u_0}^2\ (a+1)$

$*$ COEFFICIENT OF g^{alfa-2} IS $u_0\ (alfa - 1)\ alfa$

FOR EXPONENTS $(3alfa)$ AND $(2alfa)$ OF g, $alfa = 0$ IS NON-NEGATIVE.

FOR EXPONENTS $(3alfa)$ AND $(alfa-2)$ OF g, DO

WITH $alfa=-1$, POWER OF g is $-3 \longrightarrow$ SOLVE FOR u_0

TERM $-u_0({u_0}^2-2)\frac{1}{g^3}$ IS DOMINANT IN EQUATION.

WITH ${u_0}^2=2 \longrightarrow$ FIND RESONANCES

SUBSTITUTE $u_0\ g^{alfa}+u_r\ g^{r+alfa}$ FOR f IN EQUATION

TERM $(r-4)(r+1)\ u_r\ g^{r-3}$ IS DOMINANT
IN EQUATION.

THE ONLY NON-NEGATIVE INTEGRAL ROOT IS $[r=4]$

WITH MAXIMUM RESONANCE $=4 \longrightarrow$ CHECK RESONANCES.

SUBSTITUTE POWER SERIES $\sum_{k=0}^{4} u_k g^{k-1}$ FOR f IN EQUATION.

WITH ${u_0}^2=2$

$*$ COEFFICIENT OF $\frac{1}{g^2}$ IS $-(u_0c-2a+6u_1-2)$

$$u_1=-\frac{u_0c-2a-2}{6}$$

$*$ COEFFICIENT OF $\frac{1}{g}$ IS $-\frac{u_0c^2-2u_0a^2+2u_0a+36u_2-2u_0}{6}$

$$u_2=-\frac{u_0(c^2-2a^2+2a-2)}{36}$$

$*$ COEFFICIENT OF 1 IS

$$-\frac{2u_0c^3-3u_0a^2c+3u_0ac-3u_0c-2a^3+3a^2+3a+108u_3-2}{27}$$

$$u_3=-\frac{2u_0c^3-3u_0a^2c+3u_0ac-3u_0c-2a^3+3a^2+3a-2}{108}$$

$*$ COEFFICIENT OF g IS

$$-\frac{c(2u_0c^3-3u_0a^2c+3u_0ac-3u_0c-2a^3+3a^2+3a-2)}{27}$$

u_4 IS ARBITRARY !

COMPATIBILITY CONDITION:

$$-\frac{c(2u_0c^3-3u_0a^2c+3u_0ac-3u_0c-2a^3+3a^2+3a-2)}{27}=0,$$

$***$ CONDITION IS NOT SATISFIED. $***$

$***$ CHECK FOR FREE PARAMETERS OR PRESENCE OF u_0 $***$

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