# Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations in Multiple Space Dimensions

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# Outline

- Why compute conservation laws?
- The Korteweg-de Vries and Zakharov-Kuznetsov equations
- Demonstration of *ConservationLawsMD.m*
- Algorithm for computing conservation laws
- Tools from the calculus of variations
- Applicaton to the Zakharov-Kuznetsov equation
- Conclusions and future work

- Additional examples
  - Manakov-Santini system
  - Camassa-Holm equation
  - Khoklov-Zabolotskaya equation
  - Shallow water wave model for atmosphere
  - Kadomtsev-Petviashvili equation
  - Potential Kadomtsev-Petviashvili equation
  - Generalized Zakharov-Kuznetsov equation

# Conservation Laws for Nonlinear PDEs

- System of evolution equations of order $M$

$$\boxed{\mathbf{u}_t = \mathbf{F}(\mathbf{u}^{(M)}(\mathbf{x}))}$$

with $\mathbf{u} = (u, v, w, \ldots)$ and $\mathbf{x} = (x, y, z)$.

- Conservation law in (1+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \mathrm{D}_x J = 0}$$

evaluated on the PDE.

Conserved density $\rho$ and flux $J$.

- Conservation law in (2+1)-dimensions

$$\boxed{\mathrm{D}_t\rho + \nabla\cdot\mathbf{J} = \mathrm{D}_t\rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 = 0}$$

evaluated on the PDE.

Conserved density $\rho$ and flux $\mathbf{J} = (J_1, J_2)$.

- Conservation law in (3+1)-dimensions

$$\boxed{\mathrm{D}_t\rho + \nabla\cdot\mathbf{J} = \mathrm{D}_t\rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 + \mathrm{D}_z J_3 = 0}$$

evaluated on the PDE.

Conserved density $\rho$ and flux $\mathbf{J} = (J_1, J_2, J_3)$.

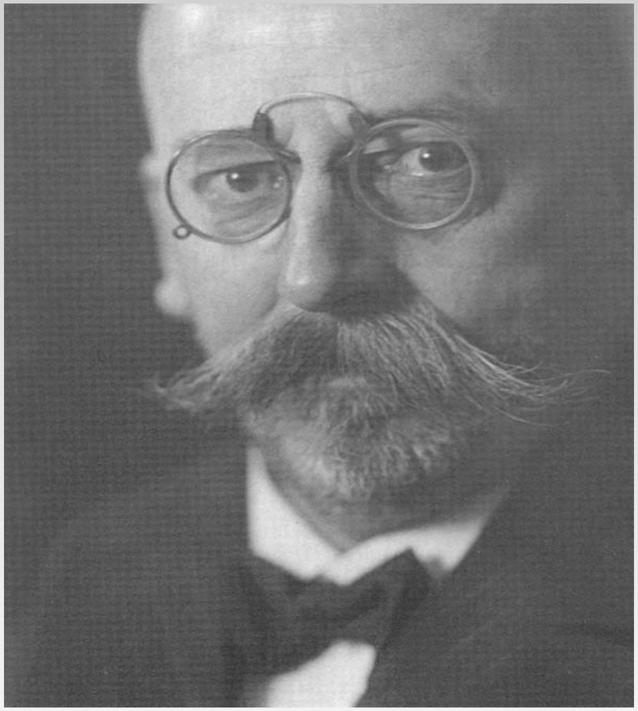
## Reasons for Computing Conservation Laws

- Conservation of physical quantities (linear momentum, mass, energy, electric charge, ... )
- Testing of complete integrability and application of Inverse Scattering Transform
- Testing of numerical integrators
- Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators, . . . )
- Verify the closure of a model

# Examples of PDEs with Conservation Laws

- Korteweg-de Vries (KdV) equation models shallow water waves, ion-acoustic waves in plasmas, etc.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \text{or} \quad \boxed{u_t + u u_x + u_{xxx} = 0}$$



Diederik Korteweg　　Gustav de Vries

- First six (of infinitely many) conservation laws:

$$\mathrm{D}_t\Big(u\Big) + \mathrm{D}_x\Big(\tfrac{1}{2}u^2 + u_{xx}\Big) = 0$$

$$\mathrm{D}_t\Big(u^2\Big) + \mathrm{D}_x\Big(\tfrac{2}{3}u^3 - {u_x}^2 + 2uu_{xx}\Big) = 0$$

$$\mathrm{D}_t\Big(u^3 - 3u_x^2\Big)$$

$$+\,\mathrm{D}_x\Big(\tfrac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{xx} + 3u_{xx}^2 - 6u_xu_{xxx}\Big) = 0$$

$$\mathrm{D}_t\Big(u^4 - 12uu_x^2 + \tfrac{36}{5}u_{xx}^2\Big) + \mathrm{D}_x\Big(\tfrac{4}{5}u^5 - 18uu_x^2 + 4u^3u_{xx}$$

$$+12u_x^2u_{xx} + \tfrac{96}{5}uu_{xx}^2 - 24uu_xu_{xxx} - \tfrac{36}{5}u_{xxx}^2 + \tfrac{72}{5}u_{xx}u_{4x}\Big) = 0$$

$$\mathrm{D}_t\Big(u^5 - 30\,u^2u_x^2 + 36\,uu_{xx}^2 - \tfrac{108}{7}u_{xxx}^2\Big)$$

$$+\,\mathrm{D}_x\Big(\tfrac{5}{6}u^6 - 40u^3u_x^2 - \ \ldots\ - \tfrac{216}{7}u_{xxx}u_{5x}\Big) = 0$$

$$\mathrm{D}_t\Big(u^6 - 60\,u^3u_x^2 - 30\,u_x^4 + 108\,u^2u_{xx}^2$$

$$+\,\frac{720}{7}\,u_{xx}^3 - \frac{648}{7}\,uu_{xxx}^2 + \frac{216}{7}\,u_{4x}^2\Big)$$

$$+\,\mathrm{D}_x\Big(\tfrac{6}{7}u^7 - 75u^4u_x^2 - \ \ldots\ + \tfrac{432}{7}u_{4x}u_{6x}\Big) = 0$$

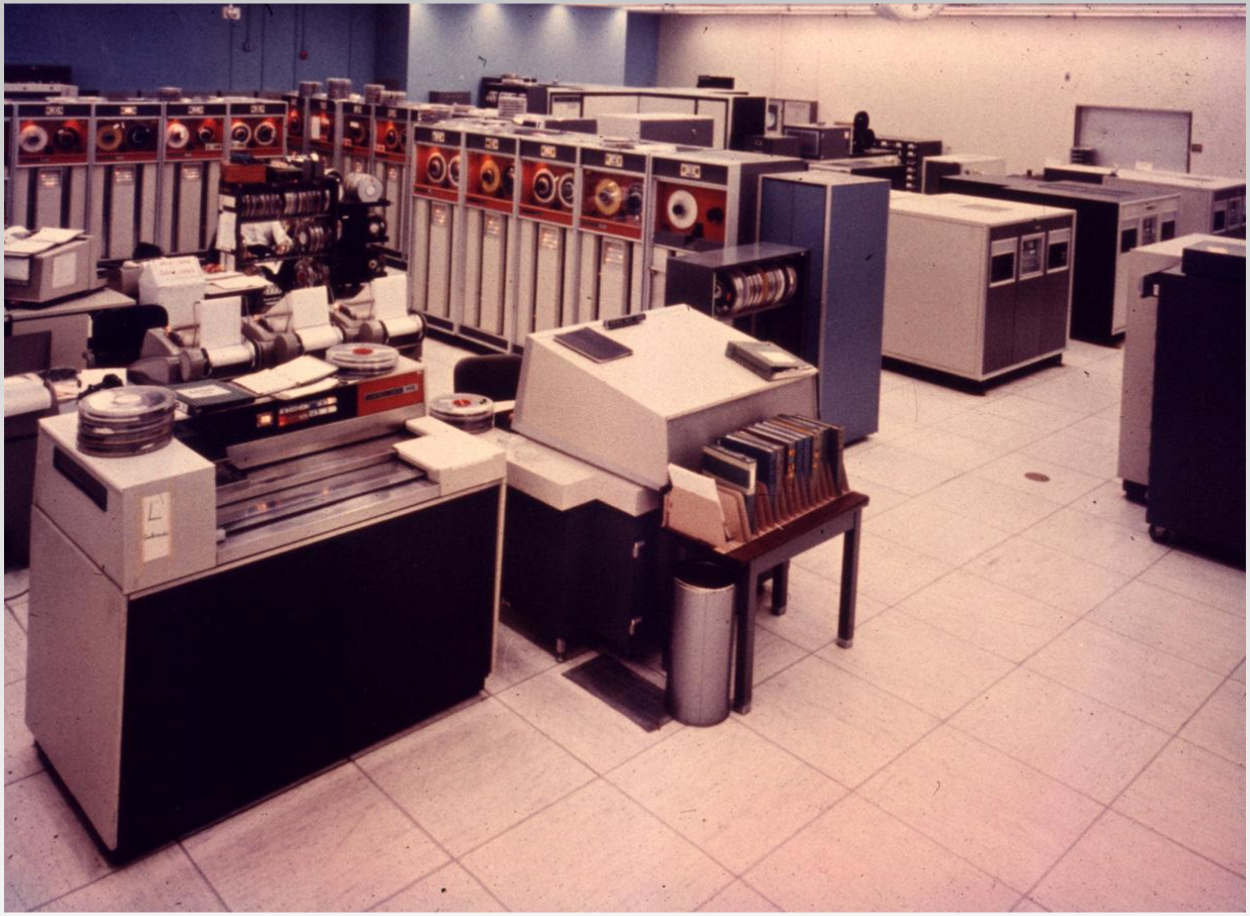
- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Seventh (sixth thru tenth): Robert Miura, 1966

- Conservation law explicitly dependent on $t$ and $x$:

$$\mathrm{D}_t\Big(tu^2 - 2xu\Big) + \mathrm{D}_x\Big(\tfrac{2}{3}tu^3 - xu^2 + 2u_x - tu_x^2 + 2tuu_{xx} - 2xu_{xx}\Big) = 0$$

First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds)

$\longrightarrow$ large integers problem!



**Control Data CDC-6600**

- Zakharov-Kuznetsov (ZK) equation
  models ion-sound solitons in a low pressure uniform magnetized plasma

$$\boxed{u_t + \alpha u u_x + \beta(u_{xx} + u_{yy})_x = 0}$$

- Conservation laws:

$$\mathrm{D}_t\Big(u\Big) + \mathrm{D}_x\Big(\tfrac{\alpha}{2}u^2 + \beta u_{xx}\Big) + \mathrm{D}_y\Big(\beta u_{xy}\Big) = 0$$

$$\mathrm{D}_t\Big(u^2\Big) + \mathrm{D}_x\Big(\tfrac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\Big) - \mathrm{D}_y\Big(2\beta u_x u_y\Big) = 0$$

# More conservation laws (ZK equation):

$$\mathsf{D}_t\Big(u^3 - \tfrac{3\beta}{\alpha}(u_x^2 + u_y^2)\Big) + \mathsf{D}_x\Big(\tfrac{3\alpha}{4}u^4 + 3\beta u^2 u_{xx} - 6\beta u(u_x^2 + u_y^2)$$

$$+ \tfrac{3\beta^2}{\alpha}(u_{xx}^2 - u_{yy}^2) - \tfrac{6\beta^2}{\alpha}(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy}))\Big)$$

$$+ \mathsf{D}_y\Big(3\beta u^2 u_{xy} + \tfrac{6\beta^2}{\alpha}u_{xy}(u_{xx} + u_{yy})\Big) = 0$$

$$\mathsf{D}_t\Big(tu^2 - \tfrac{2}{\alpha}xu\Big) + \mathsf{D}_x\Big(t(\tfrac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy}))$$

$$- x(u^2 + \tfrac{2\beta}{\alpha}u_{xx}) + \tfrac{2\beta}{\alpha}u_x\Big) - \mathsf{D}_y\Big(2\beta(tu_x u_y + \tfrac{1}{\alpha}xu_{xy})\Big) = 0$$

# Methods for Computing Conservation Laws

- Use Noether's theorem (Lagrangian formulation): connection between symmetries and conservation laws (Olver, and many others)

- Integrating factor methods (Anderson, Bluman, Anco, Cheviakov, Wolf, etc.) require solving ODEs (or PDEs)

# Proposed Algorithmic Method

- Density is linear combination of scaling invariant terms with undetermined coefficients
- Compute $\mathrm{D}_t \rho$ with total derivative operator
- Use variational derivative (Euler operator) to express exactness
- Solve a (parametrized) linear system to find the undetermined coefficients
- Use the homotopy operator to compute the flux (invert $\mathrm{D}_x$ or $\mathrm{Div}$)

- Work with linearly independent pieces in finite dimensional spaces
- Use linear algebra, calculus, and variational calculus (algorithmic)
- Implement the algorithm in Mathematica

Software Demonstration

# Notation – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$
- Dependent variables $\mathbf{u} = (u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(N)})$
  In examples: $\mathbf{u} = (u, v, \theta, h, \ldots)$
- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kx\,ly} = \frac{\partial^{k+l} u}{\partial x^k y^l}$, etc.
  Examples: $u_{xxxxx} = u_{5x} = \frac{\partial^5 u}{\partial x^5}$
  $u_{xx\,yyyy} = u_{2x4y} = \frac{\partial^6 u}{\partial x^2 y^4}$
- $\mathbf{u}^{(M)}$ represents all components of $\mathbf{u}$ and all its partial derivatives up to order $M$.
- *Differential functions*
  Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

# Tools from the Calculus of Variations

- Definition:

  A differential function $f$ is a exact iff $f = \mathrm{Div}\mathbf{F}$.

  Special case (1D): $f = \mathrm{D}_x\, F$.

- Question: How can one test that $f = \mathrm{Div}\mathbf{F}\,$?

- Theorem (exactness test):

  $f = \mathrm{Div}\,\mathbf{F}$ iff $\mathcal{L}_{u^{(j)}(\mathbf{x})} f \equiv 0, \quad j = 1, 2, \ldots, N.$

  $N$ is the number of dependent variables.

**The Euler operator annihilates divergences**

- Euler operator in 1D (variable $u(x)$):

$$\mathcal{L}_{u(x)} = \sum_{k=0}^{M} (-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}}$$

$$= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots$$

- Euler operator in 2D (variable $u(x,y)$):

$$\mathcal{L}_{u(x,y)} = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-\mathrm{D}_x)^k (-\mathrm{D}_y)^\ell \frac{\partial}{\partial u_{kx\,\ell y}}$$

$$= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} - \mathrm{D}_y \frac{\partial}{\partial u_y}$$

$$+ \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} + \mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u_{xy}} + \mathrm{D}_y^2 \frac{\partial}{\partial u_{yy}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{xxx}} \cdots$$

- Question: How can one compute $\mathbf{F} = \mathrm{Div}^{-1} f$ ?
- Theorem (integration by parts):
  - In 1D: If $f$ is exact then

$$F = \mathrm{D}_x^{-1} f = \int f\, dx = \mathcal{H}_{\mathbf{u}(x)} f$$

  - In 2D: If $f$ is a divergence then

$$\mathbf{F} = \mathrm{Div}^{-1} f = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f,\ \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f)$$

**The homotopy operator inverts total derivatives and divergences!**

- Homotopy Operator in 1D (variable $x$):

$$\boxed{\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^{N} (I_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}}$$

with integrand

$$I_{u^{(j)}} f = \sum_{k=1}^{M_x^{(j)}} \left( \sum_{i=0}^{k-1} u_{ix}^{(j)} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}^{(j)}}$$

$(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u} \rightarrow \lambda \mathbf{u}$, $\mathbf{u}_x \rightarrow \lambda \mathbf{u}_x$, *etc.*

More general: $\mathbf{u} \rightarrow \lambda(\mathbf{u} - \mathbf{u}_0) + \mathbf{u}_0$

$\mathbf{u}_x \rightarrow \lambda(\mathbf{u}_x - \mathbf{u}_{x0}) + \mathbf{u}_{x0}$ etc.

# Application to Zakharov-Kuznetsov Equation

$$\boxed{u_t + \alpha u u_x + \beta (u_{xx} + u_{yy})_x = 0}$$

- Step 1: Compute the dilation invariance

  ZK equation is invariant under scaling symmetry

  $$(t, x, y, u) \rightarrow (\frac{t}{\lambda^3}, \frac{x}{\lambda}, \frac{y}{\lambda}, \lambda^2 u) = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u})$$

  $\lambda$ is an arbitrary parameter.

- Assign weights to each variable

  $$W(u) = 2, \ \ W(\mathsf{D}_t) = 3, \ \ W(\mathsf{D}_x) = 1, \ \ W(\mathsf{D}_y) = 1.$$

- Rank of a monomial is the sum of the weights of the variables.

  Example: $\mathrm{Rank}(\alpha u u_x) = 2W(u) + W(\mathsf{D}_x) = 5.$

- Key observation: A conservation law is invariant under the scaling symmetry of the PDE

$$W(u) = 2, \;\; W(\mathsf{D}_t) = 3, \;\; W(\mathsf{D}_x) = 1, \;\; W(\mathsf{D}_y) = 1.$$

For example,

$$\mathsf{D}_t\Big(u^3 - \tfrac{3\beta}{\alpha}(u_x^2 + u_y^2)\Big) + \mathsf{D}_x\Big(\tfrac{3\alpha}{4}u^4 + 3\beta u^2 u_{xx} - 6\beta u(u_x^2 + u_y^2)$$
$$+ \tfrac{3\beta^2}{\alpha}(u_{xx}^2 - u_{xy}^2) - \tfrac{6\beta^2}{\alpha}(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy}))\Big)$$
$$+ \mathsf{D}_y\Big(3\beta u^2 u_{xy} + \tfrac{6\beta^2}{\alpha}u_{xy}(u_{xx} + u_{yy})\Big) = 0$$

$\mathrm{Rank}(\rho) = 6$, $\mathrm{Rank}(J) = 8$.

Rank (conservation law) $= 9$.

**Rank of the density needs to be selected!**

- Step 2: Construct the candidate density

For example, construct a density of rank 6.

Make a list of all terms with rank 6:

$$\{u^3, u_x^2, uu_{xx}, u_y^2, uu_{yy}, u_xu_y, uu_{xy}, u_{4x}, u_{3xy}, u_{2x2y}, u_{x3y}, u_{4y}\}$$

Remove divergences and divergence-equivalent terms.

Candidate density of rank 6:

$$\boxed{\rho = c_1u^3 + c_2u_x^2 + c_3u_y^2 + c_4u_xu_y}$$

- Step 3: Compute the undetermined coefficients

Compute

$$\mathrm{D}_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[u_t]$$

$$= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} \frac{\partial \rho}{\partial u_{kx\,\ell y}} \mathrm{D}_x^k \, \mathrm{D}_y^\ell \, u_t$$

$$= \Big( 3c_1 u^2 \mathrm{I} + 2c_2 u_x \mathrm{D}_x + 2c_3 u_y \mathrm{D}_y + c_4 (u_y \mathrm{D}_x + u_x \mathrm{D}_y) \Big) u_t$$

Substitute $u_t = -\Big( \alpha u u_x + \beta (u_{xx} + u_{yy})_x \Big)$.

$$
\begin{aligned}
E \;=\; -\mathsf{D}_t\rho \;=\; & 3c_1u^2(\alpha u u_x + \beta(u_{xx}+u_{xy})_x) \\
& + 2c_2u_x(\alpha u u_x + \beta(u_{xx}+u_{yy})_x)_x + 2c_3u_y(\alpha u u_x \\
& + \beta(u_{xx}+u_{yy})_x)_y + c_4(u_y(\alpha u u_x + \beta(u_{xx}+u_{yy})_x)_x \\
& + u_x(\alpha u u_x + \beta(u_{xx}+u_{yy})_x)_y)
\end{aligned}
$$

Apply the Euler operator (variational derivative)

$$
\begin{aligned}
\mathcal{L}_{u(x,y)}E &= \sum_{k=0}^{M_x}\sum_{\ell=0}^{M_y}(-\mathsf{D}_x)^k(-\mathsf{D}_y)^\ell \frac{\partial E}{\partial u_{kx\,\ell y}} \\
&= -2\Big((3c_1\beta + c_3\alpha)u_xu_{yy} + 2(3c_1\beta + c_3\alpha)u_yu_{xy} \\
&\qquad +2c_4\alpha u_xu_{xy} + c_4\alpha u_yu_{xx} + 3(3c_1\beta + c_2\alpha)u_xu_{xx}\Big) \\
&\equiv 0
\end{aligned}
$$

Solve a parameterized linear system for the $c_i$:

$$3c_1\beta + c_3\alpha = 0, \;\; c_4\alpha = 0, \;\; 3c_1\beta + c_2\alpha = 0$$

Solution:

$$c_1 = 1, \;\; c_2 = -\tfrac{3\beta}{\alpha}, \;\; c_3 = -\tfrac{3\beta}{\alpha}, \;\; c_4 = 0$$

Substitute the solution into the candidate density

$$\rho = c_1 u^3 + c_2 u_x^2 + c_3 u_y^2 + c_4 u_x u_y$$

Final density of rank 6:

$$\boxed{\rho = u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2)}$$

- Step 4: Compute the flux

Use the homotopy operator to invert $\mathrm{Div}$:

$$\mathbf{J} = \mathrm{Div}^{-1} E = \left( \mathcal{H}^{(x)}_{u(x,y)} E, \, \mathcal{H}^{(y)}_{u(x,y)} E \right)$$

where

$$\mathcal{H}^{(x)}_{u(x,y)} E = \int_0^1 (I^{(x)}_u E)[\lambda u] \, \frac{d\lambda}{\lambda}$$

with

$$\mathcal{I}^{(x)}_u E = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \Bigg( \sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix\,jy} \frac{\binom{i+j}{i} \binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} (-\mathrm{D}_x)^{k-i-1} \, (-\mathrm{D}_y)^{\ell-j} \Bigg) \frac{\partial E}{\partial u_{kx\,\ell y}}$$

Similar formulas for $\mathcal{H}^{(y)}_{u(x,y)} E$ and $\mathcal{I}^{(y)}_u E$.

Let $A = \alpha u u_x + \beta(u_{xxx} + u_{xyy})$ so that

$$E = 3u^2 A - \frac{6\beta}{\alpha} u_x A_x - \frac{6\beta}{\alpha} u_y A_y$$

Then,

$$\begin{aligned}
\mathbf{J} &= \left(\mathcal{H}^{(x)}_{u(x,y)} E,\ \mathcal{H}^{(y)}_{u(x,y)} E\right) \\
&= \Big(\tfrac{3}{4}\alpha u^4 + \beta u^2(3u_{xx} + 2u_{yy}) - \beta u(6u_x^2 + 2u_y^2) \\
&\quad + \frac{3\beta^2}{4\alpha} u(u_{2x2y} + u_{4y}) - \frac{\beta^2}{\alpha} u_x(\tfrac{7}{2} u_{xyy} + 6u_{xxx}) \\
&\quad - \frac{\beta^2}{\alpha} u_y(4u_{xxy} + \tfrac{3}{2} u_{yyy}) + \frac{\beta^2}{\alpha}(3u_{xx}^2 + \tfrac{5}{2} u_{xy}^2 + \tfrac{3}{4} u_{yy}^2) \\
&\quad + \frac{5\beta^2}{4\alpha} u_{xx} u_{yy}, \quad \beta u^2 u_{xy} - 4\beta u u_x u_y \\
&\quad - \frac{3\beta^2}{4\alpha} u(u_{x3y} + u_{3xy}) - \frac{\beta^2}{4\alpha} u_x(13u_{xxy} + 3u_{yyy}) \\
&\quad - \frac{5\beta^2}{4\alpha} u_y(u_{xxx} + 3u_{xyy}) + \frac{9\beta^2}{4\alpha} u_{xy}(u_{xx} + u_{yy})\Big)
\end{aligned}$$

However, $\mathrm{Div}^{-1}E$ is not unique.

Indeed, $\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{K}$, where $\mathbf{K} = (\mathsf{D}_y\theta, \, -\mathsf{D}_x\theta)$ is a curl term.

For example,

$$\theta = 2\beta u^2 u_y + \frac{\beta^2}{4\alpha}\Big(3u(u_{xxy} + u_{yyy}) + 10 u_x u_{xy} + 5u_y(3u_{yy} + u_{xx})\Big)$$

Shorter flux:

$$\begin{aligned}\tilde{\mathbf{J}} &= \mathbf{J} - \mathbf{K} \\ &= \Big(\tfrac{3\alpha}{4}u^4 + 3\beta u^2 u_{xx} - 6\beta u(u_x^2 + u_y^2) + \tfrac{3\beta^2}{\alpha}\Big(u_{xx}^2 - u_{yy}^2\Big) \\ &\qquad - \tfrac{6\beta^2}{\alpha}(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy})), \\ &\qquad 3\beta u^2 u_{xy} + \tfrac{6\beta^2}{\alpha}u_{xy}(u_{xx} + u_{yy})\Big)\end{aligned}$$

## Additional Examples

- Manakov-Santini system

$$u_{tx} + u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y = 0$$
$$v_{tx} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} = 0$$

- Conservation laws for Manakov-Santini system:

$$\mathsf{D}_t\Big(f u_x v_x\Big) + \mathsf{D}_x\Big(f(uu_x v_x - u_x v_x v_y - u_y v_y)$$
$$- f' y(u_t + uu_x - u_x v_y)\Big) + \mathsf{D}_y\Big(f(u_x v_y + u_y v_x + u_x v_x^2)$$
$$+ f'(u - yu_y - yu_x v_x)\Big) = 0$$

where $f = f(t)$ is arbitrary.

Conservation laws – continued:

$$\mathsf{D}_t\Big(f(2u+v_x^2-yu_xv_x)\Big)+\mathsf{D}_x\Big(f(u^2+uv_x^2+u_yv$$
$$-v_y^2-v_x^2v_y-y(uu_xv_x-u_xv_xv_y-u_yv_y))$$
$$-f'y(v_t+uv_x-v_xv_y)-(2fx-f')y^2(u_t+uu_x-u_xv_y)\Big)$$
$$-\mathsf{D}_y\Big(f(u_xv-2v_xv_y-v_x^3+y(u_xv_x^2+u_xv_y+u_yv_x))$$
$$-f'(v-y(2u+v_y+v_x^2))+(2fx-f'y^2)(u_xv_x+u_y)\Big)=0$$

where $f=f(t)$ is arbitrary.

There are three additional conservation laws.

- (2+1)-dimensional Camassa-Holm equation

$$\boxed{(\alpha u_t + \kappa u_x - u_{txx} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{yy} = 0}$$

Interchange $t$ with $y$

$$(\alpha u_y + \kappa u_x - u_{xxy} + 3\beta u u_x - 2u_x u_{xx} - u u_{xxx})_x + u_{tt} = 0$$

Set $v = u_t$ to get

$$u_t = v$$

$$v_t = -\alpha u_{xy} - \kappa u_{xx} + u_{3xy} - 3\beta u_x^2 - 3\beta u u_{xx} + 2u_{xx}^2 + 3u_x u_{xxx} + u u_{4x}$$

- Conservation laws for the Camassa-Holm equation

$$\mathsf{D}_t\Big(fu\Big) + \mathsf{D}_x\Big(\tfrac{1}{\alpha}f(\tfrac{3}{2}\beta u^2 + \kappa u - \tfrac{1}{2}u_x^2 - uu_{xx} - u_{tx})$$
$$- (\tfrac{1}{\alpha}fx - \tfrac{1}{2}f'y^2)(\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx}$$
$$- u_{txx})\Big) - \mathsf{D}_y\Big(u_y(\tfrac{1}{\alpha}fx - \tfrac{1}{2}f'y^2) + f'yu\Big) = 0$$

$$\mathsf{D}_t\Big(fyu\Big) + \mathsf{D}_x\Big(\tfrac{1}{\alpha}fy(\tfrac{3}{2}\beta u^2 + \kappa u - \tfrac{1}{2}u_x^2 - uu_{xx} - u_{tx})$$
$$- y(\tfrac{1}{\alpha}fx - \tfrac{1}{6}f'y^2)(\alpha u_t + \kappa u_x + 3\beta uu_x - 2u_xu_{xx} - uu_{xxx}$$
$$- u_{txx})\Big) - \mathsf{D}_y\Big(yu_y(\tfrac{1}{\alpha}fx - \tfrac{1}{6}f'y^2) - u(\tfrac{1}{\alpha}fx - \tfrac{1}{2}f'y^2)\Big) = 0$$

where $f = f(t)$ is an arbitrary function.

- Khoklov-Zabolotskaya equation

describes e.g. sound waves in nonlinear media

$$\boxed{(u_t - uu_x)_x - u_{yy} - u_{zz} = 0}$$

Conservation law:

$$\mathsf{D}_t\Big(fu\Big) - \mathsf{D}_x\Big(\tfrac{1}{2}fu^2 + (fx+g)(u_t - uu_x)\Big)$$
$$- \mathsf{D}_y\Big((f_y x + g_y)u - (fx+g)u_y\Big)$$
$$- \mathsf{D}_z\Big((f_z x + g_z)u - (fx+g)u_z\Big) = 0$$

under the constraints $\Delta f = 0$ and $\Delta g = f_t$

where $f = f(t,y,z)$ and $g = g(t,y,z)$.

- Shallow water wave model (atmosphere)

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\,\mathbf{\Omega}\times\mathbf{u} + \nabla(\theta h) - \tfrac{1}{2}h\nabla\theta = \mathbf{0}$$

$$\theta_t + \mathbf{u}\cdot(\nabla\theta) = 0$$

$$h_t + \nabla\cdot(\mathbf{u}h) = 0$$

where $\mathbf{u}(x,y,t), \theta(x,y,t)$ and $h(x,y,t)$.

- In components:

$$u_t + uu_x + vu_y - 2\,\Omega v + \tfrac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\,\Omega u + \tfrac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- First few conservation laws of SWW model:

$$\rho_{(1)} = h \qquad \mathbf{J}^{(1)} = h \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(2)} = h\,\theta \qquad \mathbf{J}^{(2)} = h\,\theta \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(3)} = h\,\theta^2 \qquad \mathbf{J}^{(3)} = h\,\theta^2 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(4)} = h\,(u^2 + v^2 + h\theta) \qquad \mathbf{J}^{(4)} = h \begin{pmatrix} u\,(u^2 + v^2 + 2h\theta) \\ v\,(v^2 + u^2 + 2h\theta) \end{pmatrix}$$

$$\rho_{(5)} = \theta\,(2\Omega + v_x - u_y)$$

$$\mathbf{J}^{(5)} = \tfrac{1}{2}\,\theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}$$

- More general conservation laws for SWW model:

$$\mathrm{D}_t\Big(f(\theta)h\Big) + \mathrm{D}_x\Big(f(\theta)hu\Big) + \mathrm{D}_y\Big(f(\theta)hv\Big) = 0$$

$$\begin{aligned}
&\mathrm{D}_t\Big(g(\theta)(2\Omega + v_x - u_x)\Big) \\
&+ \mathrm{D}_x\Big(\tfrac{1}{2}g(\theta)(4\Omega u - 2uu_y + 2uv_x - h\theta_y)\Big) \\
&+ \mathrm{D}_y\Big(\tfrac{1}{2}g(\theta)(4\Omega v - 2u_y v + 2vv_x + h\theta_x)\Big) = 0
\end{aligned}$$

for any functions $f(\theta)$ and $g(\theta)$.

- Kadomtsev-Petviashvili (KP) Equation

$$\boxed{(u_t + \alpha u u_x + u_{xxx})_x + \sigma^2 u_{yy} = 0}$$

parameter $\alpha \in \mathbb{R}$ and $\sigma^2 = \pm 1$.

Equation be written as a conservation law

$$\mathrm{D}_t(u_x) + \mathrm{D}_x(\alpha u u_x + u_{xxx}) + \mathrm{D}_y(\sigma^2 u_y) = 0.$$

Exchange $y$ and $t$ and set $u_t = v$

$$\begin{aligned} u_t &= v \\ v_t &= -\frac{1}{\sigma^2}(u_{xy} + \alpha u_x^2 + \alpha u u_{xx} + u_{xxxx}) \end{aligned}$$

- Examples of conservation laws for KP equation (explicitly dependent on $t, x$, and $y$)

$$\mathrm{D}_t\Big(xu_x\Big) + \mathrm{D}_x\Big(3u^2 - u_{xx} - 6xuu_x + xu_{xxx}\Big) + \mathrm{D}_y\Big(\alpha x u_y\Big) = 0$$

$$\mathrm{D}_t\Big(yu_x\Big) + \mathrm{D}_x\Big(y(\alpha uu_x + u_{xxx})\Big) + \mathrm{D}_y\Big(\sigma^2(yu_y - u)\Big) = 0$$

$$\mathrm{D}_t\Big(\sqrt{t}u\Big) + \mathrm{D}_x\Big(\tfrac{\alpha}{2}\sqrt{t}u^2 + \sqrt{t}u_{xx} + \frac{\sigma^2 y^2}{4\sqrt{t}}u_t + \frac{\sigma^2 y^2}{4\sqrt{t}}u_{xxx}$$

$$+ \frac{\alpha\sigma^2 y^2}{4\sqrt{t}}uu_x - x\sqrt{t}u_t - \alpha x\sqrt{t}uu_x - x\sqrt{t}u_{xxx}\Big)$$

$$+\mathrm{D}_y\Big(x\sqrt{t}u_y + \frac{y^2 u_y}{4\sqrt{t}} - \frac{yu}{2\sqrt{t}}\Big) = 0$$

- More general conservation laws for KP equation:

$$\mathrm{D}_t\Big(fu\Big) + \mathrm{D}_x\Big(f(\tfrac{\alpha}{2}u^2 + u_{xx})$$
$$+(\tfrac{\sigma^2}{2}f'y^2 - fx)(u_t + \alpha u u_x + u_{3x})\Big)$$
$$+\mathrm{D}_y\Big((\tfrac{1}{2}f'y^2 - \sigma^2 fx)u_y - f'yu\Big) = 0$$

$$\mathrm{D}_t\Big(fyu\Big) + \mathrm{D}_x\Big(fy(\tfrac{\alpha}{2}u^2 + u_{xx})$$
$$+y(\tfrac{\sigma^2}{6}f'y^2 - fx)(u_t + \alpha u u_x + u_{3x})\Big)$$
$$+\mathrm{D}_y\Big(y(\tfrac{1}{6}f'y^2 - \sigma^2 fx)u_y + (\sigma^2 fx - \tfrac{1}{2}f'y^2)u\Big) = 0$$

where $f(t)$ is arbitrary function.

- Potential KP equation

  Replace $u$ by $u_x$ and integrate with respect to $x$.

$$\boxed{u_{xt} + \alpha u_x u_{xx} + u_{xxxx} + \sigma^2 u_{yy} = 0}$$

- Examples of conservation laws
  (not explicitly dependent on $x, y, t$):

$$\mathrm{D}_t\Big(u_x\Big) + \mathrm{D}_x\left(\tfrac{1}{2}\alpha u_x^2 + u_{xxx}\right) + \mathrm{D}_y\left(\sigma^2 u_y\right) = 0$$

$$\mathrm{D}_t\left(u_x^2\right) + \mathrm{D}_x\left(\tfrac{2}{3}\alpha u_x^3 - u_{xx}^2 + 2u_x u_{xxx} - \sigma^2 u_{yy}\right)$$

$$+\mathrm{D}_y\left(2\sigma^2 u_x u_y\right) = 0$$

Conservation laws for pKP equation – continued:

$$\mathrm{D}_t\Big(u_x u_y\Big) + \mathrm{D}_x\Big(\alpha u_x^2 u_y + u_t u_y + 2u_{xxx}u_y - 2u_{xx}u_{xy}\Big)$$
$$+\mathrm{D}_y\Big(\sigma^2 u_y^2 - \tfrac{1}{3}u_x^3 - u_t u_x + u_{xx}^2\Big) = 0$$

$$\mathrm{D}_t\Big(2\alpha u u_x u_{xx} + 3uu_{4x} - 3\sigma^2 u_y^2\Big) + \mathrm{D}_x\Big(2\alpha u_t u_x^2 + 3u_t^2$$
$$-2\alpha u u_x u_{tx} - 3u_{tx}u_{xx} + 3u_t u_{xxx} + 3u_x u_{t\,xx} - 3uu_{t\,xxx}\Big)$$
$$+\mathrm{D}_y\Big(6\sigma^2 u_t u_y\Big) = 0$$

Various generalizations exist.

- Generalized Zakharov-Kuznetsov equation

$$\boxed{u_t + \alpha u^n u_x + \beta(u_{xx} + u_{yy})_x = 0}$$

where $n$ is rational, $n \neq 0$.

Conservation laws:

$$\mathsf{D}_t\Big(u\Big) + \mathsf{D}_x\Big(\tfrac{\alpha}{n+1}u^{n+1} + \beta u_{xx}\Big) + \mathsf{D}_y\Big(\beta u_{xy}\Big) = 0$$

$$\mathsf{D}_t\Big(u^2\Big) + \mathsf{D}_x\Big(\tfrac{2\alpha}{n+2}u^{n+2} - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\Big)$$
$$- \mathsf{D}_y\Big(2\beta u_x u_y\Big) = 0$$

- Third conservation law for gZK equation:

$$\mathsf{D}_t\Big(u^{n+2} - \tfrac{(n+1)(n+2)\beta}{2\alpha}(u_x^2 + u_y^2)\Big)$$

$$+ \mathsf{D}_x\Big(\tfrac{(n+2)\alpha}{2(n+1)}u^{2(n+1)} + (n+2)\beta u^{n+1}u_{xx}$$

$$- (n+1)(n+2)\beta u^n(u_x^2 + u_y^2) + \tfrac{(n+1)(n+2)\beta^2}{2\alpha}(u_{xx}^2 - u_{yy}^2)$$

$$- \tfrac{(n+1)(n+2)\beta^2}{\alpha}(u_x(u_{xxx} + u_{xyy}) + u_y(u_{xxy} + u_{yyy}))\Big)$$

$$+ \mathsf{D}_y\Big((n+2)\beta u^{n+1}u_{xy} + \tfrac{(n+1)(n+2)\beta^2}{\alpha}u_{xy}(u_{xx} + u_{yy})\Big) = 0.$$

## Conclusions and Future Work

- The power of Euler and homotopy operators:
  - Testing exactness
  - Integration by parts: $\mathrm{D}_x^{-1}$ and $\mathrm{Div}^{-1}$
- Integration of non-exact expressions

  Example: $f = u_x v + u v_x + u^2 u_{xx}$

  $\int f dx = uv + \int u^2 u_{xx}\, dx$
- Use other homotopy formulas (prevent curl terms)

- Broader class of PDEs (beyond evolution type)

  Example: short pulse equation (nonlinear optics)

$$u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2 u_{xx}$$

  with non-polynomial conservation law

$$\mathrm{D}_t \left( \sqrt{1 + 6u_x^2} \right) - \mathrm{D}_x \left( 3u^2 \sqrt{1 + 6u_x^2} \right) = 0$$

- Continue the implementation in *Mathematica*
- Software: http://inside.mines.edu/∼whereman

# Thank You