Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations

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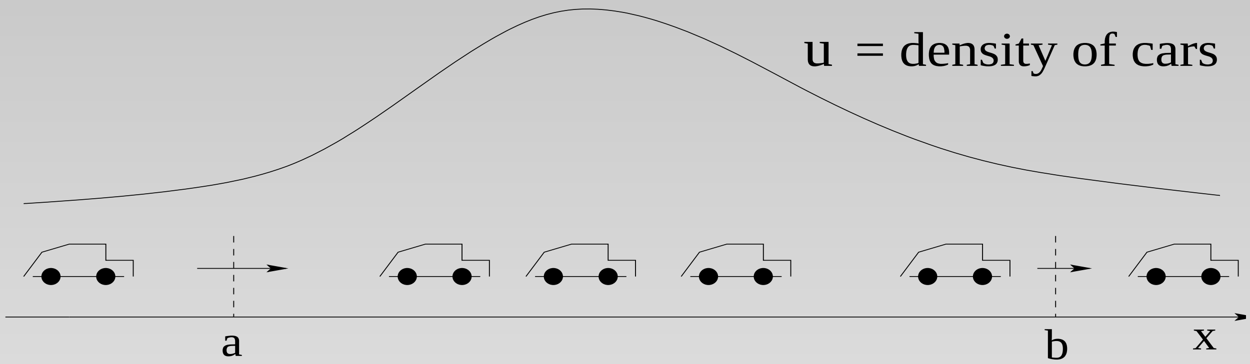
Outline

- Examples of conservation laws
- Famous example in historical perspective
- Example: Shallow water wave equations
- Symbolic computation of conservation laws
- Computer demonstration
- Tools:
  - The variational derivative (testing exactness)
  - The homotopy operator (inverting D_x and Div)
- Application to shallow water wave equations
- Additional examples
- Conclusions and future work

Examples of Conservation Laws

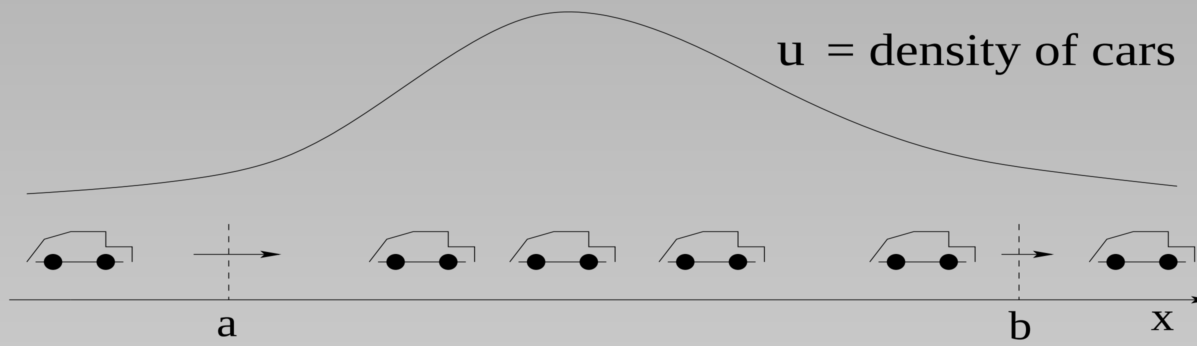
Example 1: Traffic Flow



Modeling the density of cars (Bressan, 2009)

$u(x,t)$ density of cars on a highway (e.g. number of cars per 100 meters)

$s(u)$ mean (equilibrium) speed of the cars (depends on the density)



Change in number of cars in segment $[a, b]$ equals the difference between cars entering at a and leaving at b during time interval $[t_1, t_2]$:

$$\int_a^b \big(u(x,t_2) - u(x,t_1)\big)\;dx \;=\; \int_{t_1}^{t_2} \big(J(a,t) - J(b,t)\big)\;dt$$

$$\int_a^b \left(\int_{t_1}^{t_2} u_t(x,t)\,dt\right)\;dx \;=\; -\int_{t_1}^{t_2} \left(\int_a^b J_x(x,t)\,dx\right)\;dt$$

where $J(x,t) = u(x,t)s(u(x,t))$ is the traffic flow (e.g. in cars per hour) at location x and time t

Then, $\int_a^b \int_{t_1}^{t_2} (u_t + J_x)\, dt\, dx = 0$ holds $\forall (a, b, t_1, t_2)$

Yields the conservation law:

$$\boxed{u_t + [s(u)\, u]_x = 0} \quad \text{or} \quad \boxed{\mathrm{D}_t \rho + \mathrm{D}_x J = 0}$$

$\rho = u$ is the conserved density

$J(u) = s(u)\, u$ is the associated flux

A simple Lighthill-Whitham-Richards model:

$$s(u) = s_{\max} \left(1 - \frac{u}{u_{\max}}\right), \qquad 0 \leq u \leq u_{\max}$$

$s_{\max}$ is posted highway speed, $u_{\max}$ is the jam density

Example 2: Fluid and Gas Dynamics

Euler equations for a compressible, non-viscous fluid:

$$\rho_t + \nabla.(\rho\,\mathbf{u}) = 0$$

$$(\rho\,\mathbf{u})_t + \nabla.\left(\mathbf{u}\otimes(\rho\mathbf{u})\right) + \nabla p = 0$$

$$E_t + \nabla.\left((E+p)\,\mathbf{u}\right) = 0$$

or, in components

$$\rho_t + \nabla.(\rho\,\mathbf{u}) = 0$$

$$(\rho\,u_i)_t + \nabla.(\rho\,u_i\,\mathbf{u} + p\,\mathbf{e}_i) = 0 \qquad (i = 1,2,3)$$

$$E_t + \nabla.\left((E+p)\,\mathbf{u}\right) = 0$$

Express conservation of mass, momentum, energy

$\otimes$ is the dyadic product

ρ is the mass density

$\mathbf{u} = u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2 + u_3\,\mathbf{e}_3$ is the velocity

p is the pressure $p(\rho, e)$

E is the energy density per unit volume
$E = \frac{1}{2}\rho|\mathbf{u}|^2 + \rho e$

e is internal energy density per unit of mass (related to temperature)

Conservation Laws for PDEs

- System of evolution equations of order M

$$\boxed{\mathbf{u}_t = \mathbf{F}(\mathbf{u}^{(M)}(\mathbf{x}))}$$

  with $\mathbf{u} = (u, v, w, \ldots)$ and $\mathbf{x} = (x, y, z)$

- Conservation law in (1+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \mathrm{D}_x J = 0} \text{ (on PDE)}$$

  conserved density ρ and flux J

- Conservation law in (3+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \nabla \cdot \mathbf{J} = \mathrm{D}_t \rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 + \mathrm{D}_z J_3 = 0} \text{ (on PDE)}$$

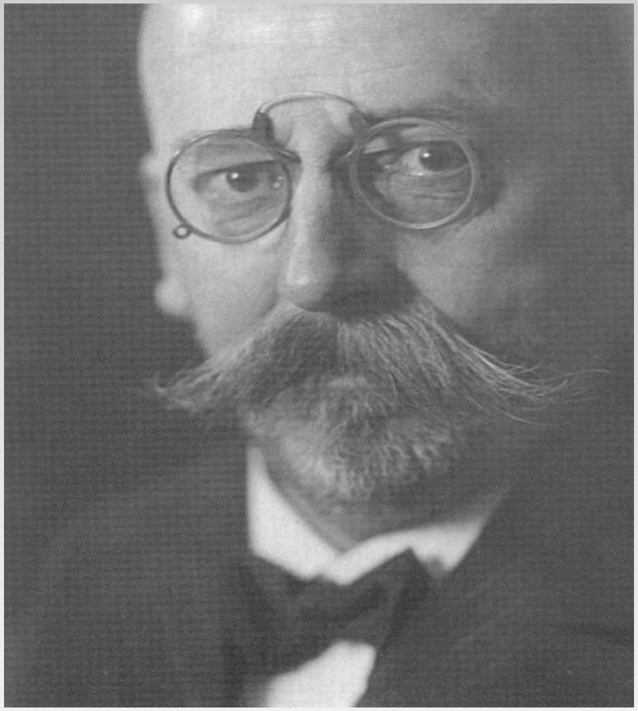
  conserved density ρ and flux $\mathbf{J} = (J_1, J_2, J_3)$

Famous Example in Historical Perspective

- Example: Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \text{or} \quad u_t + uu_x + u_{xxx} = 0$$

Diederik Korteweg



Gustav de Vries

- First six (of ∞ many) densities-flux pairs:

$$\mathrm{D}_t\left(u\right)+\mathrm{D}_x\left(\frac{u^2}{2}+u_{xx}\right)=0$$

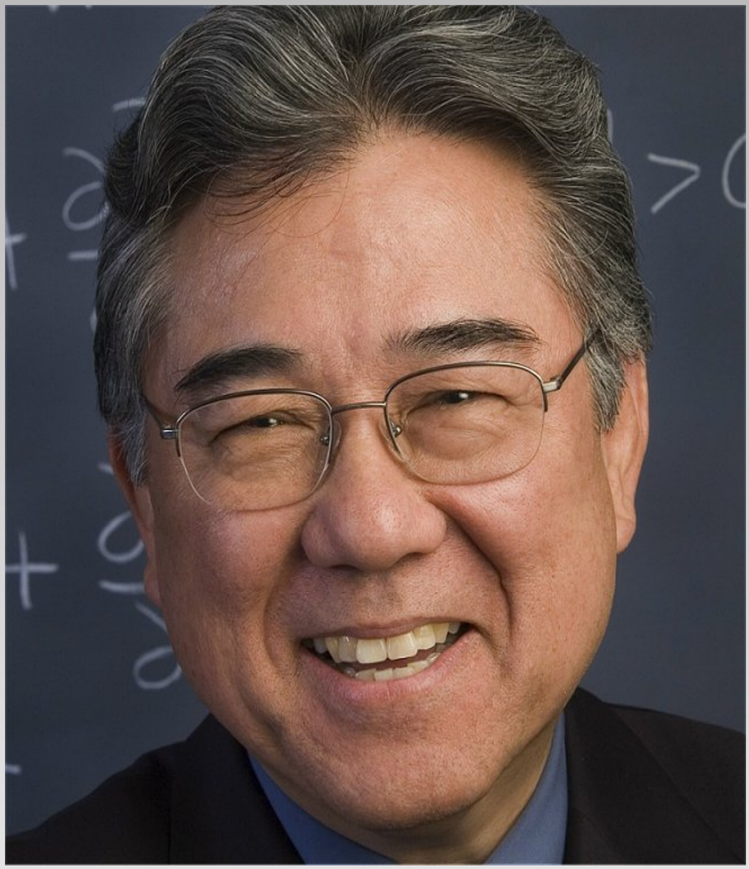
$$\mathrm{D}_t\left(u^2\right)+\mathrm{D}_x\left(\frac{2}{3}u^3-{u_x}^2+2uu_{xx}\right)=0$$

$$\mathrm{D}_t\left(u^3-3{u_x}^2\right)+$$
$$\mathrm{D}_x\left(\frac{3}{4}u^4-6u{u_x}^2+3u^2u_{xx}+3{u_{xx}}^2-6u_xu_{xxx}\right)=0$$

$$\mathrm{D}_t\left(u^5-30\,u^2{u_x}^2+36\,u{u_{xx}}^2-\frac{108}{7}u_{xxx}^2\right)+$$
$$\mathrm{D}_x\left(\frac{5}{6}u^6-40u^3u_x^2-\ldots-\frac{216}{7}u_{xxx}u_{5x}\right)=0$$

$$
\mathrm{D}_t \left(u^6 - 60\, u^3 {u_x}^2 - 30\, {u_x}^4 + 108\, u^2 {u_{xx}}^2 + \frac{720}{7}\, {u_{xx}}^3 - \frac{648}{7}\, u {u_{xxx}}^2 + \frac{216}{7}\, {u_{4x}}^2 \right) + \mathrm{D}_x \left(\frac{6}{7} u^7 - 75 u^4 u_x^2 - \ldots + \frac{432}{7} u_{4x} u_{6x} \right) = 0
$$

- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Sixth: algebraic mistake, 1966
- Seventh (sixth thru tenth): Robert Miura, 1966

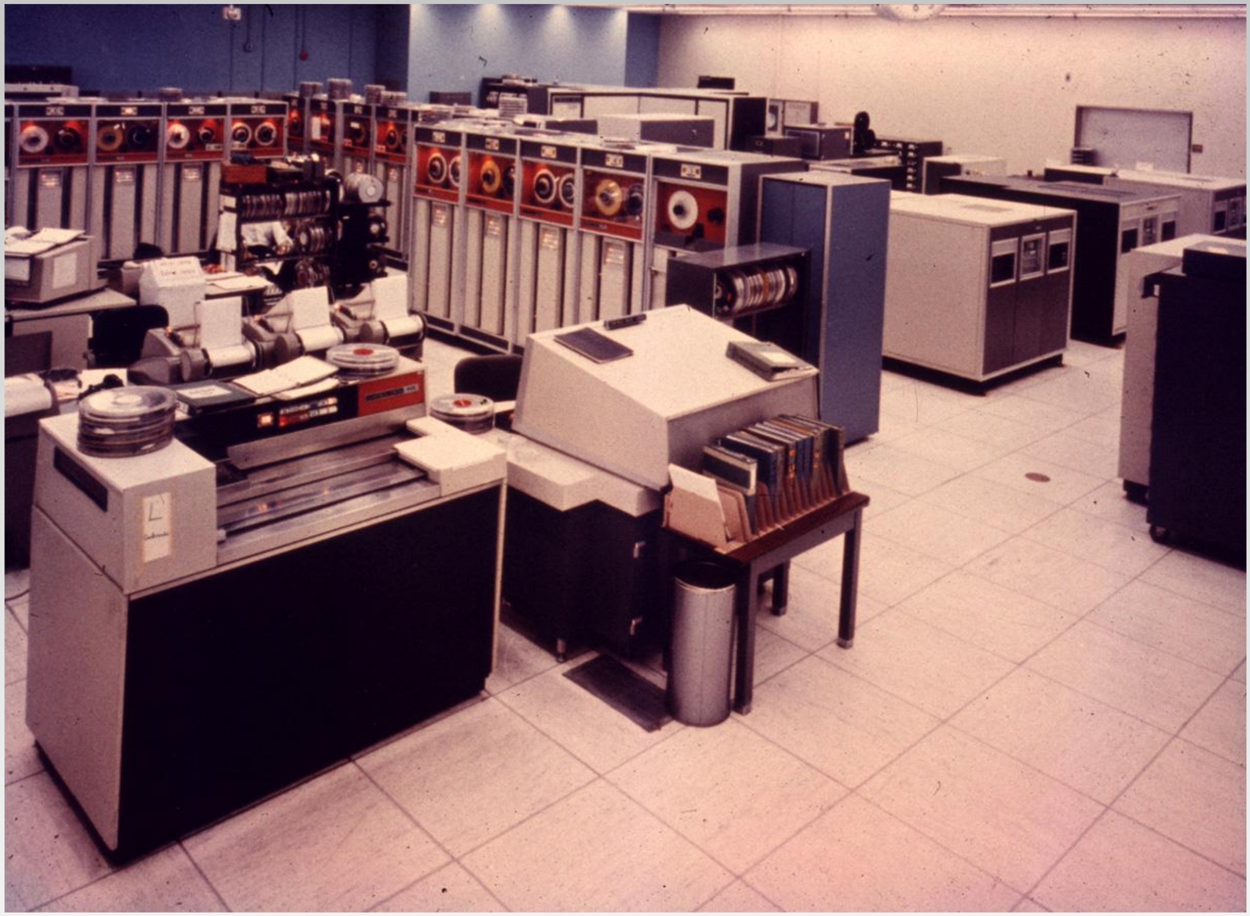


Robert Miura

- First five: IBM 7094 computer with FORMAC (1966) $\longrightarrow$ storage space problem!

IBM 7094 Computer

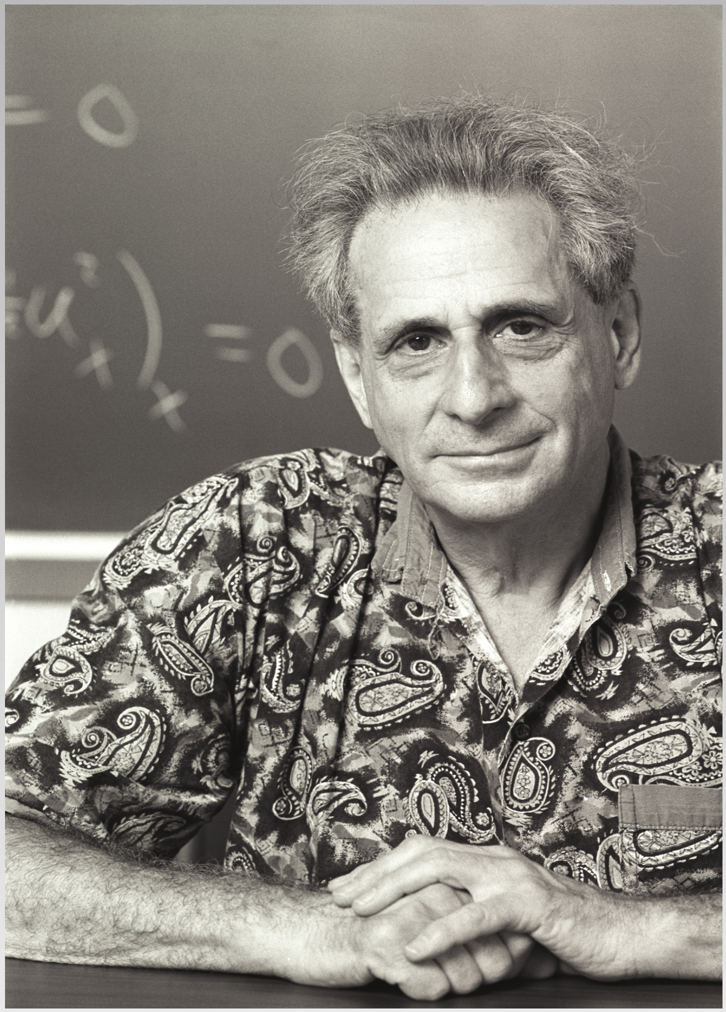
- First eleven densities: Control Data Computer CDC-6600 computer (2.2 seconds)
  $\longrightarrow$ large integers problem!



Control Data CDC-6600

Control Data CDC-6600 in Museum

- 2006 Leroy P. Steele Prize (AMS): Gardner, Greene, Kruskal, and Miura
- National Medal of Science, von Neumann Prize (SIAM), ...: Martin Kruskal



Martin D. Kruskal (1925-2006)

Reasons to Compute Conservation Laws

- Conservation of physical quantities (linear momentum, mass, energy, electric charge, ...)
- Verify the closure of a model
- Testing of complete integrability and application of Inverse Scattering Transform
- Testing of numerical integrators
- Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators, . . .)

- Key property: Dilation invariance
- Example: KdV equation and its density-flux pairs are invariant under the scaling symmetry

$$(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u) = (\tilde{x}, \tilde{t}, \tilde{u})$$

  λ is arbitrary parameter

- Examples of conservation laws

$$\mathrm{D}_t \left(u^2 \right) + \mathrm{D}_x \left(\frac{2}{3} u^3 - {u_x}^2 + 2 u u_{xx} \right) = 0$$

$$\mathrm{D}_t \left(u^3 - 3 {u_x}^2 \right) +$$

$$\mathrm{D}_x \left(\frac{3}{4} u^4 - 6 u {u_x}^2 + 3 u^2 u_{xx} + 3 {u_{xx}}^2 - 6 u_x u_{xxx} \right) = 0$$

Transcendental Equations in (1+1)-dimensions

- Example: sine-Gordon equation

$$U_{XT} = \sin U$$

or

$$u_{tt} - u_{xx} = \sin u$$

Written as a first order system:

$$\begin{aligned} u_t &= v \\ v_t &= u_{xx} + \alpha \sin u \end{aligned}$$

- Scaling invariance (trick!)

$$(x, t, u, v, \alpha) \to (\frac{x}{\lambda}, \frac{t}{\lambda}, \lambda^0 u, \lambda v, \lambda^2 \alpha)$$

λ is arbitrary parameter

- First few densities-flux pairs

$$\rho_{(1)} = 2\alpha\cos u + v^2 + {u_x}^2 \qquad J_{(1)} = -2u_x v$$

$$\rho_{(2)} = 2u_x v \qquad J_{(2)} = 2\alpha\cos u - v^2 - {u_x}^2$$

$$\rho_{(3)} = 12 v u_x \cos u + 2v^3 u_x + 2v{u_x}^3 - 16 v_x u_{xx}$$

$$\begin{aligned}\rho_{(4)} &= 2\cos^2 u - 2\sin^2 u + v^4 + 6v^2{u_x}^2 + {u_x}^4 \\ &+ \; 4v^2\cos u + 20{u_x}^2\cos u - 16{v_x}^2 - 16{u_{xx}}^2\end{aligned}$$

$J_{(3)}$ and $J_{(4)}$ are not shown (too long)

An Example in (2+1)-Dimensions

- Example: Shallow water wave (SWW) equations

  [P. Dellar, Phys. Fluids **15** (2003) 292-297]

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\,\mathbf{\Omega} \times \mathbf{u} + \nabla(\theta h) - \frac{1}{2}h\nabla\theta = \mathbf{0}$$

$$\theta_t + \mathbf{u}\cdot(\nabla\theta) = 0$$

$$h_t + \nabla\cdot(\mathbf{u}h) = 0$$

where $\mathbf{u}(x,y,t), \theta(x,y,t)$ and $h(x,y,t)$

- In components:

$$u_t + uu_x + vu_y - 2\,\Omega v + \frac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\,\Omega u + \frac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- SWW equations are invariant under

$$(x, y, t, u, v, h, \theta, \Omega) \longrightarrow$$

$$(\lambda^{-1}x, \lambda^{-1}y, \lambda^{-b}t, \lambda^{b-1}u, \lambda^{b-1}v, \lambda^{a}h, \lambda^{2b-a-2}\theta, \lambda^{b}\Omega)$$

where $W(h) = a$ and $W(\Omega) = b \quad (a, b \in \mathbb{Q})$

- First few densities-flux pairs of SWW system:

$$\rho_{(1)} = h \qquad \mathbf{J}^{(1)} = h \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(2)} = h\,\theta \qquad \mathbf{J}^{(2)} = h\,\theta \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(3)} = h\,\theta^2 \qquad \mathbf{J}^{(3)} = h\,\theta^2 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho_{(4)} = h\,(u^2 + v^2 + h\theta) \qquad \mathbf{J}^{(4)} = h \begin{pmatrix} u\,(u^2 + v^2 + 2h\theta) \\ v\,(v^2 + u^2 + 2h\theta) \end{pmatrix}$$

$$\rho_{(5)} = \theta\,(2\Omega + v_x - u_y)$$

$$\mathbf{J}^{(5)} = \frac{1}{2}\,\theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}$$

Generalizations:

$$\mathrm{D}_t\left(f(\theta)h\right) + \mathrm{D}_x\left(f(\theta)hu\right) + \mathrm{D}_y\left(f(\theta)hv\right) = 0$$

$$\begin{aligned}
&\mathrm{D}_t\left(g(\theta)(2\Omega + v_x - u_x)\right) \\
&+ \mathrm{D}_x\left(\frac{1}{2}g(\theta)(4\Omega u - 2uu_y + 2uv_x - h\theta_y)\right) \\
&+ \mathrm{D}_y\left(\frac{1}{2}g(\theta)(4\Omega v - 2u_yv + 2vv_x + h\theta_x)\right) = 0
\end{aligned}$$

for any functions $f(\theta)$ and $g(\theta)$

Notation – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$

- Dependent variables
  $\mathbf{u} = (u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(N)})$
  In examples: $\mathbf{u} = (u, v, \theta, h, \ldots)$

- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kx\,ly} = \frac{\partial^{k+l} u}{\partial x^k y^l}$, etc.
  Examples: $u_{xxxxx} = u_{5x} = \frac{\partial^5 u}{\partial x^5}$
  $u_{xx\,yyyy} = u_{2x\,4y} = \frac{\partial^6 u}{\partial x^2 y^4}$

- *Differential functions*
  Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

- *Total derivatives*: $\mathrm{D}_t,\ \mathrm{D}_x, \mathrm{D}_y, \ldots$

Example: Let $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

Then

$$\begin{aligned}
\mathrm{D}_x f &= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{xx} \frac{\partial f}{\partial u_x} \\
&\quad + v_x \frac{\partial f}{\partial v} + v_{xx} \frac{\partial f}{\partial v_x} + v_{xxx} \frac{\partial f}{\partial v_{xx}} \\
&= 2x u_x^3 v_x + u_x (v v_x) + u_{xx} (3x^2 u_x^2 v_x + v_{xx}) \\
&\quad + v_x (u v_x) + v_{xx} (uv + x^2 u_x^3) + v_{xxx} (u_x) \\
&= 2x u_x^3 v_x + v u_x v_x + 3x^2 u_x^2 v_x u_{xx} + u_{xx} v_{xx} \\
&\quad + u v_x^2 + u v v_{xx} + x^2 u_x^3 v_{xx} + u_x v_{xxx}
\end{aligned}$$

A Method to Compute Conservation Laws

- Density is linear combination of scaling invariant terms with undetermined coefficients
- Compute $\mathrm{D}_t \rho$ with total derivative operator
- Use variational derivative (Euler operator) to compute the undetermined coefficients
- Use homotopy operator to compute flux $\mathbf{J}$ (invert D_x or Div)
- Use linear algebra, calculus, and variational calculus (algorithmic)
- Work with linearly independent pieces in finite dimensional spaces

Algorithm for PDEs in (1+1)-dimensions

- **Example: Rank** 6 density for KdV equation

$$u_t + u u_x + u_{xxx} = 0$$

- **Step 1: Compute the dilation symmetry**

  Set $(x,t,u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^a}, \lambda^b u) = (\tilde{x}, \tilde{t}, \tilde{u})$

  Apply change of variables (chain rule)

$$\lambda^{-(a+b)} \tilde{u}_{\tilde{t}} + \lambda^{-(2b+1)} \tilde{u} \tilde{u}_{\tilde{x}} + \lambda^{-(b+3)} \tilde{u}_{3\tilde{x}} = 0$$

  Solve $a + b = 2b + 1 = b + 3.$

  Solution: $a = 3$ and $b = 2$

$$(x,t,u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u)$$

- **Step 2: Determine the form of the density**

List powers of u, up to rank $6:\quad [u, u^2, u^3]$

Differentiate with respect to x to increase the rank

u has weight 2 $\longrightarrow$ apply D_x^4

u^2 has weight 4 $\longrightarrow$ apply D_x^2

u^3 has weight 6 $\longrightarrow$ no derivatives needed

Apply the D_x derivatives

Remove total and highest derivative terms:

$$\mathrm{D}_x^4 u \;\rightarrow\; \{u_{4x}\} \;\rightarrow\; \text{empty list}$$

$$\mathrm{D}_x^2 u^2 \;\rightarrow\; \{{u_x}^2, u u_{xx}\} \;\rightarrow\; \{{u_x}^2\}$$

$$\text{since } u u_{xx} = (u u_x)_x - {u_x}^2$$

$$\mathrm{D}_x^0 u^3 \;\rightarrow\; \{u^3\} \;\rightarrow\; \{u^3\}$$

Linearly combine the “building blocks”

Candidate density:

$$\boxed{\rho = c_1 u^3 + c_2 {u_x}^2}$$

- **Step 3**: **Compute the coefficients** c_i

  Compute

$$\begin{aligned} \mathrm{D}_t\rho &= \frac{\partial \rho}{\partial t} + \rho'(u)[u_t] \\ &= \frac{\partial \rho}{\partial t} + \sum_{k=0}^{M} \frac{\partial \rho}{\partial u_{kx}} \mathrm{D}_x^k u_t \\ &= (3c_1 u^2 \mathrm{I} + 2c_2 u_x \mathrm{D}_x) u_t \end{aligned}$$

Substitute u_t by $-(uu_x + u_{xxx})$

$$\begin{aligned} E &= -\mathrm{D}_t\rho = (3c_1 u^2 \mathrm{I} + 2c_2 u_x \mathrm{D}_x)(uu_x + u_{xxx}) \\ &= 3c_1 u^3 u_x + 2c_2 u_x^3 + 2c_2 u u_x u_{xx} \\ &\quad +3\,c_1 u^2 u_{xxx} + 2c_2 u_x u_{4x} \end{aligned}$$

Apply the Euler operator (variational derivative)

$$\mathcal{L}_u(x) = \frac{\delta}{\delta u} = \sum_{k=0}^{m} (-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}}$$

Here, E has order $m = 4$, thus

$$\mathcal{L}_u(x)E = \frac{\partial E}{\partial u} - \mathrm{D}_x \frac{\partial E}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial E}{\partial u_{xx}} - \mathrm{D}_x^3 \frac{\partial E}{\partial u_{3x}} + \mathrm{D}_x^4 \frac{\partial E}{\partial u_{4x}}$$
$$= -6(3c_1 + c_2) u_x u_{xx}$$

This term must vanish!

So, $c_1 = -\frac{1}{3} c_2$. Set $c_2 = -3$, then $c_1 = 1$

Hence, the final form density is

$$\boxed{\rho = u^3 - 3{u_x}^2}$$

- **Step 4: Compute the flux J**

Method 1: Integrate by parts (simple cases)

Now,

$$E = 3u^3 u_x + 3u^2 u_{xxx} - 6u_x^3 - 6u u_x u_{xx} - 6u_x u_{xxxx}$$

Integration of $\mathrm{D}_x J = E$ yields the flux

$$\boxed{J = \frac{3}{4}u^4 - 6u {u_x}^2 + 3u^2 u_{xx} + 3{u_{xx}}^2 - 6u_x u_{xxx}}$$

Method 2: Build the form of J (cumbersome)

Note: Rank $J =$ Rank ρ + Rank $\mathrm{D}_t - 1$

Build up form of J. Compute

$$\mathrm{D}_x J = \frac{\partial J}{\partial x} + \sum_{k=0}^{m} \frac{\partial J}{\partial u_{kx}} u_{(k+1)x}$$

m is the order of J.

Match $\mathrm{D}_x J = E$

Method 3: Use the homotopy operator

$$J = \mathrm{D}_x^{-1} E = \int E \, dx = \mathcal{H}_{\mathbf{u}(x)} E = \int_0^1 (I_u E)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

with integrand

$$I_u E = \sum_{k=1}^{M} \left(\sum_{i=0}^{k-1} u_{ix} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}}$$

Here $M = 4$, thus

$$
\begin{aligned}
I_u E \;=\; & (u\mathrm{I})(\frac{\partial E}{\partial u_x}) + (u_x\mathrm{I} - u\mathrm{D}_x)(\frac{\partial E}{\partial u_{xx}}) \\
& +(u_{xx}\mathrm{I} - u_x\mathrm{D}_x + u\mathrm{D}_x^2)(\frac{\partial E}{\partial u_{xxx}}) \\
& +(u_{xxx}\mathrm{I} - u_{xx}\mathrm{D}_x + u_x\mathrm{D}_x^2 - u\mathrm{D}_x^3)(\frac{\partial E}{\partial u_{xxxx}}) \\
= \; & (u\mathrm{I})(3u^3 + 18u_x^2 - 6uu_{xx} - 6u_{xxxx}) \\
& +(u_x\mathrm{I} - u\mathrm{D}_x)(-6uu_x) \\
& +(u_{xx}\mathrm{I} - u_x\mathrm{D}_x + u\mathrm{D}_x^2)(3u^2) \\
& +(u_{xxx}\mathrm{I} - u_{xx}\mathrm{D}_x + u_x\mathrm{D}_x^2 - u\mathrm{D}_x^3)(-6u_x) \\
= \; & 3u^4 - 18uu_x^2 + 9u^2u_{xx} + 6u_{xx}^2 - 12u_xu_{xxx}
\end{aligned}
$$

Note: correct terms but incorrect coefficients!

Finally,

$$J = \mathcal{H}_{u(x)} E = \int_0^1 (I_u E)[\lambda u] \, \frac{d\lambda}{\lambda}$$
$$= \int_0^1 \Big(3\lambda^3 u^4 - 18\lambda^2 u u_x^2 + 9\lambda^2 u^2 u_{xx} + 6\lambda u_{xx}^2$$
$$-12\lambda u_x u_{xxx} \Big) \, d\lambda$$
$$= \frac{3}{4} u^4 - 6 u u_x^2 + 3 u^2 u_{xx} + 3 u_{xx}^2 - 6 u_x u_{xxx}$$

Final form of the flux:

$$\boxed{J = \frac{3}{4} u^4 - 6 u {u_x}^2 + 3 u^2 u_{xx} + 3 {u_{xx}}^2 - 6 u_x u_{xxx}}$$

Computer Demonstration

Review of Vector Calculus

- Definition: $\mathbf{F}$ is *conservative* if $\mathbf{F} = \boldsymbol{\nabla} f$
- Definition: $\mathbf{F}$ is *irrotational* or *curl free* if $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$
- Theorem: $\mathbf{F} = \boldsymbol{\nabla} f$ iff $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$

The curl annihilates gradients!

Review of Vector Calculus

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  The curl annihilates gradients!

- Definition: $\mathbf{F}$ is *incompressible* or *divergence free* if $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$
- Theorem: $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{G}$ iff $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$

  The divergence annihilates curls!

  Question: How can one test that $f = \boldsymbol{\nabla} \cdot \mathbf{F}$?

Review of Vector Calculus

- Definition: $\mathbf{F}$ is *conservative* if $\mathbf{F} = \boldsymbol{\nabla} f$
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- Theorem: $\mathbf{F} = \boldsymbol{\nabla} f$ iff $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$

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- Definition: $\mathbf{F}$ is *incompressible* or *divergence free* if $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$
- Theorem: $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{G}$ iff $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$

  The divergence annihilates curls!

Question: How can one test that $f = \boldsymbol{\nabla} \cdot \mathbf{F}$?

No theorem from vector calculus!

Tools from the Calculus of Variations

- Definition:

  A differential function f is *exact* iff $f = \mathrm{D}_x F$

- Theorem (exactness test):

  $f = \mathrm{D}_x F$ iff $\mathcal{L}_{u^{(j)}(x)} f \equiv 0, \quad j = 1, 2, \ldots, N$

- Definition:

  A differential function f is a *divergence* iff

  $f = \mathrm{Div}\, \mathbf{F}$

- Theorem (exactness test):

  $f = \mathrm{Div}\, \mathbf{F}$ iff $\mathcal{L}_{u^{(j)}(\mathbf{x})} f \equiv 0, \quad j = 1, 2, \ldots, N$

The Euler operator annihilates divergences!

Formula for Euler operator in 1D

for dependent variable $u(x)$

$$\mathcal{L}_{u(x)} = \sum_{k=0}^{M} (-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}}$$

$$= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{xxx}} + \cdots$$

Formula for Euler operator in 2D

for dependent variable $u(x,y)$

$$\mathcal{L}_{u(x,y)} = \sum_{k=0}^{M_x} \sum_{\ell=0}^{M_y} (-\mathrm{D}_x)^k (-\mathrm{D}_y)^\ell \frac{\partial}{\partial u_{kx\,\ell y}}$$

$$= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} - \mathrm{D}_y \frac{\partial}{\partial u_y}$$

$$+ \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} + \mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u_{xy}} + \mathrm{D}_y^2 \frac{\partial}{\partial u_{yy}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{xxx}} \cdots$$

Application: Testing Exactness

Example:

$$f = 8 v_x v_{xx} - u_x^3 \sin u + 2 u_x u_{xx} \cos u - 6 v v_x \cos u + 3 u_x v^2 \sin u$$

where $u(x)$ and $v(x)$

- f is exact
- After integration by parts (by hand):

$$F = \int f \, dx = 4 \, v_x^2 + u_x^2 \, \cos u - 3 \, v^2 \, \cos u$$

- Exactness test with Euler operator:

$$f = 8v_x v_{xx} - u_x^3 \sin u + 2u_x u_{xx} \cos u - 6v v_x \cos u + 3u_x v^2 \sin u$$

$$\mathcal{L}_{u(x)} f = \frac{\partial f}{\partial u} - \mathrm{D}_x \frac{\partial f}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial f}{\partial u_{xx}} \equiv 0$$

$$\mathcal{L}_{v(x)} f = \frac{\partial f}{\partial v} - \mathrm{D}_x \frac{\partial f}{\partial v_x} + \mathrm{D}_x^2 \frac{\partial f}{\partial v_{xx}} \equiv 0$$

Inverting D_x and Div

Problem Statement in 1D

- Example:

$$f = 8 v_x v_{xx} - u_x^3 \sin u + 2 u_x u_{xx} \cos u - 6 v v_x \cos u + 3 u_x v^2 \sin u$$

- Find $F = \int f \, dx$. So, $f = D_x F$
- Result (by hand):

$$F = 4\, v_x^2 + u_x^2 \, \cos u - 3\, v^2 \, \cos u$$

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- Find $F = \int f \, dx$. So, $f = D_x F$
- Result (by hand):

$$F = 4\, v_x^2 + u_x^2 \, \cos u - 3\, v^2 \, \cos u$$

Mathematica cannot compute this integral!

Problem Statement in 2D

- Example: $f = u_x v_y - u_{xx} v_y - u_y v_x + u_{xy} v_x$ where $u(x, y)$ and $v(x, y)$
- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ so, $f = \mathrm{Div}\,\mathbf{F}$
- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Problem Statement in 2D

- Example: $f = u_x v_y - u_{xx} v_y - u_y v_x + u_{xy} v_x$ where $u(x, y)$ and $v(x, y)$
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Mathematica cannot do this!

Can this be done without integration by parts?

Problem Statement in 2D

- Example: $f = u_x v_y - u_{xx} v_y - u_y v_x + u_{xy} v_x$ where $u(x, y)$ and $v(x, y)$
- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ so, $f = \mathrm{Div}\,\mathbf{F}$
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Mathematica cannot do this!

Can this be done without integration by parts?

Can this be reduced to single integral in one variable?

Problem Statement in 2D

- Example: $f = u_x v_y - u_{xx} v_y - u_y v_x + u_{xy} v_x$ where $u(x, y)$ and $v(x, y)$
- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ so, $f = \mathrm{Div}\,\mathbf{F}$
- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Mathematica cannot do this!

Can this be done without integration by parts?

Can this be reduced to single integral in one variable?

Yes! With the Homotopy operator

Using the Homotopy Operator

- Theorem (integration with homotopy operator):

- In 1D: If f is exact then

$$F = \mathrm{D}_x^{-1} f = \int f \, dx = \mathcal{H}_{\mathbf{u}(x)} f$$

- In 2D: If f is a divergence then

$$\mathbf{F} = \mathrm{Div}^{-1} f = (\mathcal{H}_{\mathbf{u}(x,y)}^{(x)} f, \mathcal{H}_{\mathbf{u}(x,y)}^{(y)} f)$$

- Homotopy Operator in 1D (variable x):

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^{N} (I_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

with integrand

$$I_{u^{(j)}} f = \sum_{k=1}^{M_x^{(j)}} \left(\sum_{i=0}^{k-1} u_{ix}^{(j)} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}^{(j)}}$$

N is the number of dependent variables and $(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u} \rightarrow \lambda \mathbf{u}$, $\mathbf{u}_x \rightarrow \lambda \mathbf{u}_x$, *etc.*

More general: $\mathbf{u} \rightarrow \lambda(\mathbf{u} - \mathbf{u}_0) + \mathbf{u}_0$

$\mathbf{u}_x \rightarrow \lambda(\mathbf{u}_x - \mathbf{u}_{x0}) + \mathbf{u}_{x0}$ etc.

Application of Homotopy Operator in 1D

Example:

$$f = 8v_x v_{xx} - u_x^3 \sin u + 2u_x u_{xx} \cos u - 6 v v_x \cos u + 3u_x v^2 \sin u$$

- Compute

$$\begin{aligned} I_u f &= u \frac{\partial f}{\partial u_x} + \left(u_x \mathrm{I} - u \mathrm{D}_x\right) \frac{\partial f}{\partial u_{xx}} \\ &= -u u_x^2 \sin u + 3 u v^2 \sin u + 2 u_x^2 \cos u \end{aligned}$$

- Similarly,

$$
\begin{aligned}
I_v f &= v\frac{\partial f}{\partial v_x} + (v_x \mathrm{I} - v\mathrm{D}_x)\,\frac{\partial f}{\partial v_{xx}} \\
&= -6v^2\cos u + 8v_x^2
\end{aligned}
$$

- Finally,

$$
\begin{aligned}
F = \mathcal{H}_{\mathbf{u}(x)} f &= \int_0^1 (I_u f + I_v f)\,[\lambda \mathbf{u}]\,\frac{d\lambda}{\lambda} \\
&= \int_0^1 \Big(3\lambda^2 u v^2 \sin(\lambda u) - \lambda^2 u u_x^2 \sin(\lambda u) \\
&\quad + 2\lambda u_x^2 \cos(\lambda u) - 6\lambda v^2 \cos(\lambda u) + 8\lambda v_x^2\Big)\,d\lambda \\
&= 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u
\end{aligned}
$$

- Homotopy Operator in 2D (variables x and y):

$$\boxed{\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(x)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}}$$

$$\boxed{\mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(y)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}}$$

where for dependent variable $u(x,y)$

$$\mathcal{I}^{(x)}_{u} f = \sum_{k=1}^{M_x} \sum_{\ell=0}^{M_y} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{\ell} u_{ix\,jy} \frac{\binom{i+j}{i} \binom{k+\ell-i-j-1}{k-i-1}}{\binom{k+\ell}{k}} \right.$$

$$\left. (-\mathrm{D}_x)^{k-i-1} \, (-\mathrm{D}_y)^{\ell-j} \right) \frac{\partial f}{\partial u_{kx\,\ell y}}$$

Application of Homotopy Operator in 2D

- Example: $f = u_x v_y - u_{xx} v_y - u_y v_x + u_{xy} v_x$
- By hand: $\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$
- Compute

$$
\begin{aligned}
I_u^{(x)} f &= u \frac{\partial f}{\partial u_x} + (u_x \mathrm{I} - u \mathrm{D}_x) \frac{\partial f}{\partial u_{xx}} \\
&\quad + \left(\frac{1}{2} u_y \mathrm{I} - \frac{1}{2} u \mathrm{D}_y \right) \frac{\partial f}{\partial u_{xy}} \\
&= u v_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy}
\end{aligned}
$$

- Similarly,

$$I_v^{(x)} f = v \frac{\partial f}{\partial v_x} = -u_y v + u_{xy} v$$

- Hence,

$$\begin{aligned} F_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \left(I_u^{(x)} f + I_v^{(x)} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \lambda \left(u v_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy} - u_y v + u_{xy} v \right) d\lambda \\ &= \frac{1}{2} u v_y + \frac{1}{4} u_y v_x - \frac{1}{2} u_x v_y + \frac{1}{4} u v_{xy} - \frac{1}{2} u_y v + \frac{1}{2} u_{xy} v \end{aligned}$$

- Analogously,

$$F_2 = \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f = \int_0^1 \left(I_u^{(y)} f + I_v^{(y)} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$= \int_0^1 \left(\lambda \left(-u v_x - \frac{1}{2} u v_{xx} + \frac{1}{2} u_x v_x \right) + \lambda \left(u_x v - u_{xx} v \right) \right) d\lambda$$

$$= -\frac{1}{2} u v_x - \frac{1}{4} u v_{xx} + \frac{1}{4} u_x v_x + \frac{1}{2} u_x v - \frac{1}{2} u_{xx} v$$

- So,

$$\mathbf{F} = \frac{1}{4} \begin{pmatrix} 2uv_y + u_y v_x - 2u_x v_y + u v_{xy} - 2u_y v + 2u_{xy} v \\ -2uv_x - u v_{xx} + u_x v_x + 2u_x v - 2u_{xx} v \end{pmatrix}$$

Let $\mathbf{K} = \tilde{\mathbf{F}} - \mathbf{F}$ then

$$\mathbf{K} = \frac{1}{4}\begin{pmatrix} 2uv_y - u_y v_x - 2u_x v_y - uv_{xy} + 2u_y v - 2u_{xy} v \\ -2uv_x + uv_{xx} + 3u_x v_x - 2u_x v + 2u_{xx} v \end{pmatrix}$$

then $\mathrm{Div}\,\mathbf{K} = 0$

- Also, $\mathbf{K} = (\mathrm{D}_y \phi, -\mathrm{D}_x \phi)$ with $\phi = \frac{1}{4}(2uv - uv_x - 2u_x v)$

(*curl* in 2D)

Needed: Strategy to avoid curl terms dynamically!

Why does this work?

Sketch of Derivation and Proof

(in 1D with variable x, and for one component u)

Definition: Degree operator $\mathcal{M}$

$$\mathcal{M} f = \sum_{i=0}^{M} u_{ix} \frac{\partial f}{\partial u_{ix}} = u \frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} + u_{2x} \frac{\partial f}{\partial u_{2x}} + \cdots + u_{Mx} \frac{\partial f}{\partial u_{Mx}}$$

f is of order M in x

Example: $f = u^p u_x^q u_{3x}^r$ (p, q, r non-negative integers)

$$g = \mathcal{M} f = \sum_{i=0}^{3} u_{ix} \frac{\partial f}{\partial u_{ix}} = (p + q + r)\, u^p u_x^q u_{3x}^r$$

Application of $\mathcal{M}$ computes the total *degree*

Theorem (inverse operator) $\mathcal{M}^{-1} g(u) = \int_0^1 g[\lambda u] \, \frac{d\lambda}{\lambda}$

Proof:

$$\frac{d}{d\lambda} g[\lambda u] = \sum_{i=0}^{M} \frac{\partial g[\lambda u]}{\partial \lambda u_{ix}} \frac{d\lambda u_{ix}}{d\lambda} = \frac{1}{\lambda} \sum_{i=0}^{M} u_{ix} \frac{\partial g[\lambda u]}{\partial u_{ix}} = \frac{1}{\lambda} \mathcal{M} g[\lambda u]$$

Integrate both sides with respect to λ

$$\begin{aligned} \int_0^1 \frac{d}{d\lambda} g[\lambda u] \, d\lambda &= g[\lambda u]\Big|_{\lambda=0}^{\lambda=1} = g(u) - g(0) \\ &= \int_0^1 \mathcal{M} g[\lambda u] \, \frac{d\lambda}{\lambda} = \mathcal{M} \int_0^1 g[\lambda u] \, \frac{d\lambda}{\lambda} \end{aligned}$$

Assuming $g(0) = 0$,

$$\mathcal{M}^{-1} g(u) = \int_0^1 g[\lambda u] \, \frac{d\lambda}{\lambda}$$

Example:

If $g(u) = (p+q+r)\, u^p u_x^q u_{3x}^r$, then

$g[\lambda u] = (p+q+r)\lambda^{p+q+r}\, u^p u_x^q u_{3x}^r$

Hence,

$$
\begin{aligned}
\mathcal{M}^{-1} g &= \int_0^1 (p+q+r)\, \lambda^{p+q+r-1}\, u^p u_x^q u_{3x}^r \, d\lambda \\
&= u^p u_x^q u_{3x}^r \;\; \lambda^{p+q+r} \Big|_{\lambda=0}^{\lambda=1} = u^p u_x^q u_{3x}^r
\end{aligned}
$$

Theorem: If f is an exact differential function, then

$$F = \mathcal{D}_x^{-1} f = \int f \, dx = \mathcal{H}_{u(x)} f$$

Proof: Multiply

$$\mathcal{L}_{u(x)}^{(0)} f = \sum_{k=0}^{M} (-\mathcal{D}_x)^k \frac{\partial f}{\partial u_{kx}}$$

by u to restore the degree.

Split off $u \frac{\partial f}{\partial u}$. Integrate by parts.

Split off $u_x \frac{\partial f}{\partial u_x}$. Repeat the process.

Lastly, split off $u_{Mx} \frac{\partial f}{\partial u_{Mx}}$.

Explicitly,

$$
\begin{aligned}
u\mathcal{L}^{(0)}_{u(x)} f &= u \sum_{k=0}^{M} (-\mathcal{D}_x)^k \frac{\partial f}{\partial u_{kx}} \\
&= u\frac{\partial f}{\partial u} - \mathcal{D}_x \left(u \sum_{k=1}^{M} (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \right) + u_x \sum_{k=1}^{M} (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \\
&= u\frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} - \mathcal{D}_x \left(u \sum_{k=1}^{M} (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \right. \\
&\qquad \left. + u_x \sum_{k=2}^{M} (-\mathcal{D}_x)^{k-2} \frac{\partial f}{\partial u_{kx}} \right) + u_{2x} \sum_{k=2}^{M} (-\mathcal{D}_x)^{k-2} \frac{\partial f}{\partial u_{kx}} \\
&= \ldots
\end{aligned}
$$

$$
\begin{aligned}
&= u\frac{\partial f}{\partial u} + u_x\frac{\partial f}{\partial u_x} + \ldots + u_{Mx}\frac{\partial f}{\partial u_{Mx}} \\
&\quad -\mathcal{D}_x\left(u\sum_{k=1}^{M}(-\mathcal{D}_x)^{k-1}\frac{\partial f}{\partial u_{kx}} + u_x\sum_{k=2}^{M}(-\mathcal{D}_x)^{k-2}\frac{\partial f}{\partial u_{kx}}\right. \\
&\quad \left. + \ldots + u_{(M-1)x}\sum_{k=M}^{M}(-\mathcal{D}_x)^{k-M}\frac{\partial f}{\partial u_{kx}}\right) \\
&= \sum_{i=0}^{M} u_{ix}\frac{\partial f}{\partial u_{ix}} - \mathcal{D}_x\left(\sum_{i=0}^{M-1} u_{ix}\sum_{k=i+1}^{M}(-\mathcal{D}_x)^{k-(i+1)}\frac{\partial f}{\partial u_{kx}}\right) \\
&= \mathcal{M}f - \mathcal{D}_x\left(\sum_{i=0}^{M-1} u_{ix}\sum_{k=i+1}^{M}(-\mathcal{D}_x)^{k-(i+1)}\frac{\partial f}{\partial u_{kx}}\right) \\
&= 0
\end{aligned}
$$

$$\mathcal{M} f = \mathcal{D}_x \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right)$$

Apply $\mathcal{M}^{-1}$ and use $\mathcal{M}^{-1}\mathcal{D}_x = \mathcal{D}_x \mathcal{M}^{-1}$.

$$f = \mathcal{D}_x \left(\mathcal{M}^{-1} \sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right)$$

Apply $\mathcal{D}_x^{-1}$ and use the formula for $\mathcal{M}^{-1}$.

$$\begin{aligned} F &= \mathcal{D}_x^{-1} f = \int_0^1 \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right) [\lambda u] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \left(\sum_{k=1}^{M} \left(\sum_{i=0}^{k-1} u_{ix} (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}} \right) [\lambda u] \frac{d\lambda}{\lambda} \\ &= \mathcal{H}_{u(x)} f \end{aligned}$$

Computation of Conservation Laws for SWW

Quick Recapitulation

- Conservation law in (2+1) dimensions

$$\boxed{\mathrm{D}_t \rho + \nabla \cdot \mathbf{J} = \mathrm{D}_t \rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 = 0} \quad \text{(on PDE)}$$

conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$

- Example: Shallow water wave (SWW) equations

$$u_t + u u_x + v u_y - 2\,\Omega v + \frac{1}{2} h \theta_x + \theta h_x = 0$$

$$v_t + u v_x + v v_y + 2\,\Omega u + \frac{1}{2} h \theta_y + \theta h_y = 0$$

$$\theta_t + u \theta_x + v \theta_y = 0$$

$$h_t + h u_x + u h_x + h v_y + v h_y = 0$$

- Typical density-flux pair:

$$\rho_{(5)} = \theta\,(v_x - u_y + 2\Omega)$$

$$\mathbf{J}^{(5)} = \frac{1}{2}\,\theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}$$

Algorithm for PDEs in (2+1)-dimensions

- **Step 1: Construct the form of the density**

  The SWW equations are invariant under the scaling symmetries

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda\theta, \lambda h, \lambda^2\Omega)$$

and

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda^2\theta, \lambda^0 h, \lambda^2\Omega)$$

Construct a candidate density, for example,

$$\rho = c_1\Omega\theta + c_2 u_y\theta + c_3 v_y\theta + c_4 u_x\theta + c_5 v_x\theta$$

which is scaling invariant under *both* symmetries.

- **Step 2: Determine the constants** c_i

Compute $E = -\mathrm{D}_t \rho$ and remove time derivatives

$$
\begin{aligned}
E &= -(\frac{\partial \rho}{\partial u_x} u_{tx} + \frac{\partial \rho}{\partial u_y} u_{ty} + \frac{\partial \rho}{\partial v_x} v_{tx} + \frac{\partial \rho}{\partial v_y} v_{ty} + \frac{\partial \rho}{\partial \theta} \theta_t) \\
&= c_4 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_x \\
&\quad + c_2 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_y \\
&\quad + c_5 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_x \\
&\quad + c_3 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_y \\
&\quad + (c_1 \Omega + c_2 u_y + c_3 v_y + c_4 u_x + c_5 v_x)(u \theta_x + v \theta_y)
\end{aligned}
$$

Require that

$$\mathcal{L}_{u(x,y)} E = \mathcal{L}_{v(x,y)} E = \mathcal{L}_{\theta(x,y)} E = \mathcal{L}_{h(x,y)} E \equiv 0.$$

- Solution: $c_1 = 2,\; c_2 = -1,\; c_3 = c_4 = 0,\; c_5 = 1$ gives

$$\boxed{\rho = \theta\,(2\Omega - u_y + v_x)}$$

- **Step 3: Compute the flux $\mathbf{J}$**

$$\begin{aligned} E \;=\; & \theta(u_x v_x + u v_{xx} + v_x v_y + v v_{xy} + 2\Omega u_x \\ & + \tfrac{1}{2}\theta_x h_y - u_x u_y - u u_{xy} - u_y v_y - u_{yy} v \\ & + 2\Omega v_y - \tfrac{1}{2}\theta_y h_x) \\ & + 2\Omega u \theta_x + 2\Omega v \theta_y - u u_y \theta_x \\ & - u_y v \theta_y + u v_x \theta_x + v v_x \theta_y \end{aligned}$$

Apply the 2D homotopy operator:

$$\mathbf{J} = (J_1, J_2) = \mathrm{Div}^{-1} E = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E)$$

Compute

$$I_u^{(x)} E = u \frac{\partial E}{\partial u_x} + \left(\frac{1}{2} u_y \mathrm{I} - \frac{1}{2} u \mathrm{D}_y \right) \frac{\partial E}{\partial u_{xy}}$$

$$= u v_x \theta + 2\Omega u \theta + \frac{1}{2} u^2 \theta_y - u u_y \theta$$

Similarly, compute

$$I_v^{(x)} E = v v_y \theta + \frac{1}{2} v^2 \theta_y + u v_x \theta$$

$$I_\theta^{(x)} E = \frac{1}{2} \theta^2 h_y + 2\Omega u \theta - u u_y \theta + u v_x \theta$$

$$I_h^{(x)} E = -\frac{1}{2} \theta \theta_y h$$

Next,

$$
\begin{aligned}
J_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E \\
&= \int_0^1 \left(I_u^{(x)} E + I_v^{(x)} E + I_\theta^{(x)} E + I_h^{(x)} E \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
&= \int_0^1 \Bigg(4\lambda\Omega u\theta + \lambda^2 \left(3uv_x\theta + \frac{1}{2}u^2\theta_y - 2uu_y\theta + vv_y\theta \right. \\
&\qquad \left. + \frac{1}{2}v^2\theta_y + \frac{1}{2}\theta^2 h_y - \frac{1}{2}\theta\theta_y h \right) \Bigg) d\lambda \\
&= \; 2\Omega u\theta - \frac{2}{3}uu_y\theta + uv_x\theta + \frac{1}{3}vv_y\theta + \frac{1}{6}u^2\theta_y \\
&\quad + \frac{1}{6}v^2\theta_y - \frac{1}{6}h\theta\theta_y + \frac{1}{6}h_y\theta^2
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
J_2 &= \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E \\
&= 2\Omega v\theta + \frac{2}{3} v v_x \theta - v u_y \theta - \frac{1}{3} u u_x \theta - \frac{1}{6} u^2 \theta_x - \frac{1}{6} v^2 \theta_x \\
&\quad + \frac{1}{6} h\theta\theta_x - \frac{1}{6} h_x \theta^2
\end{aligned}
$$

Hence,

$$
\mathbf{J} = \frac{1}{6}\begin{pmatrix} 12\Omega u\theta - 4uu_y\theta + 6uv_x\theta + 2vv_y\theta + (u^2+v^2)\theta_y - h\theta\theta_y + h_y\theta^2 \\ 12\Omega v\theta + 4vv_x\theta - 6vu_y\theta - 2uu_x\theta - (u^2+v^2)\theta_x + h\theta\theta_x - h_x\theta^2 \end{pmatrix}
$$

There are curl terms in $\mathbf{J}$

Indeed, subtract $\mathbf{K}$ where $\mathrm{Div}\,\mathbf{K} = 0$

Here

$$\mathbf{K} = \frac{1}{6}\begin{pmatrix} -(2uu_y\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y + 2h\theta\theta_y + h_y\theta^2) \\ 2vv_x\theta + 2uu_x\theta + u^2\theta_x + v^2\theta_x + 2h\theta\theta_x + h_x\theta^2 \end{pmatrix}$$

Note that $\mathbf{K} = (\mathrm{D}_y\phi, -\mathrm{D}_x\phi)$ with $\phi = -(h\theta^2 + u^2\theta + v^2\theta)$

($curl$ in 2D)

After removing the curl term $\mathbf{K}$,

$$\tilde{\mathbf{J}}^{(5)} = \frac{1}{2}\,\theta \begin{pmatrix} 4\Omega u - 2uu_y + 2uv_x - h\theta_y \\ 4\Omega v + 2vv_x - 2vu_y + h\theta_x \end{pmatrix}$$

Needed: Strategy to avoid curl terms dynamically!

Additional Examples

- Example: Kadomtsev-Petviashvili (KP) Equation

$$\boxed{(u_t + \alpha u u_x + u_{xxx})_x + \sigma^2 u_{yy} = 0}$$

parameter $\alpha \in \mathbb{R}$ and $\sigma^2 = \pm 1$.

The equation be written as a conservation law

$$\mathrm{D}_t(u_x) + \mathrm{D}_x(\alpha u u_x + u_{xxx}) + \mathrm{D}_y(\sigma^2 u_y) = 0.$$

Exchange y and t and set $u_t = v$

$$\begin{aligned} u_t &= v \\ v_t &= -\frac{1}{\sigma^2}(u_{xy} + \alpha u_x^2 + \alpha u u_{xx} + u_{xxxx}) \end{aligned}$$

- Examples of conservation laws explicitly dependent on t, x, and y

$$\mathrm{D}_t\left(xu_x\right)+\mathrm{D}_x\left(3u^2-u_{xx}-6xuu_x+xu_{xxx}\right)+\mathrm{D}_y\left(\alpha xu_y\right)=0$$

$$\mathrm{D}_t(yu_x)+\mathrm{D}_x\left(y(\alpha uu_x+u_{xxx})\right)+\mathrm{D}_y\left(\sigma^2(yu_y-u)\right)=0$$

$$\mathrm{D}_t\left(\sqrt{t}u\right)+\mathrm{D}_x\left(\frac{1}{2}\alpha\sqrt{t}u^2+\sqrt{t}u_{xx}+\frac{\sigma^2y^2}{4\sqrt{t}}u_t+\frac{\sigma^2y^2}{4\sqrt{t}}u_{xxx}\right.$$

$$\left.+\frac{\alpha\sigma^2y^2}{4\sqrt{t}}uu_x-x\sqrt{t}u_t-\alpha x\sqrt{t}uu_x-x\sqrt{t}u_{xxx}\right)$$

$$+\mathrm{D}_y\left(-\frac{yu}{2\sqrt{t}}+\frac{y^2u_y}{4\sqrt{t}}+x\sqrt{t}u_y\right)=0$$

- More general conservation laws for KP equation:

$$\mathrm{D}_t\Big(f(t)u\Big) + \mathrm{D}_x\Big(f(t)(\tfrac{1}{2}\alpha u^2 + u_{xx})$$
$$+(\tfrac{1}{2}\sigma^2 f'(t)y^2 - f(t)x)(u_t + \alpha u u_x + u_{3x})\Big)$$
$$+\mathrm{D}_y\Big(u_y(\tfrac{1}{2}f'(t)y^2 - \sigma^2 f(t)x) - f'(t)yu\Big) = 0$$

$$\mathrm{D}_t\Big(f(t)yu\Big) + \mathrm{D}_x\Big(f(t)y(\tfrac{1}{2}\alpha u^2 + u_{xx})$$
$$+(\tfrac{1}{6}\sigma^2 f'(t)y^3 - f(t)xy)(u_t + \alpha u u_x + u_{3x})\Big)$$
$$+\mathrm{D}_y\Big(u_y(\tfrac{1}{6}f'(t)y^3 - \sigma^2 f(t)xy) - u(\tfrac{1}{2}f'(t)y^2 - \sigma^2 f(t)x)\Big) = 0$$

where $f(t)$ is arbitrary function.

- Example: Potential KP Equation

  Replace u by u_x and integrate with respect to x.

$$\boxed{u_{xt} + \alpha u_x u_{xx} + u_{xxxx} + \sigma^2 u_{yy} = 0}$$

- Examples of conservation laws (not explicitly dependent on x, y, t)

$$\mathrm{D}_t\left(u_x\right) + \mathrm{D}_x\left(\frac{1}{2}\alpha u_x^2 + u_{xxx}\right) + \mathrm{D}_y\left(\sigma^2 u_y\right) = 0$$

$$\mathrm{D}_t\left(u_x^2\right) + \mathrm{D}_x\left(\frac{2}{3}\alpha u_x^3 - u_{xx}^2 + 2u_x u_{xxx} - \sigma^2 u_{yy}\right)$$
$$+\mathrm{D}_y\left(2\sigma^2 u_x u_y\right) = 0$$

$$\mathrm{D}_t\left(u_x u_y\right) + \mathrm{D}_x\left(\alpha u_x^2 u_y + u_t u_y + 2u_{xxx}u_y - 2u_{xx}u_{xy}\right)$$
$$+\mathrm{D}_y\left(\sigma^2 u_y^2 - \frac{1}{3}u_x^3 - u_t u_x + u_{xx}^2\right) = 0$$

$$\mathrm{D}_t\left(2\alpha u u_x u_{xx} + 3u u_{4x} - 3\sigma^2 u_y^2\right) + \mathrm{D}_x\left(2\alpha u_t u_x^2 + 3u_t^2\right.$$
$$\left.-2\alpha u u_x u_{tx} - 3u_{tx}u_{xx} + 3u_t u_{xxx} + 3u_x u_{t\,xx} - 3u u_{t\,xxx}\right)$$
$$+\mathrm{D}_y\left(6\sigma^2 u_t u_y\right) = 0$$

Various generalizations exist

- Example: Khoklov-Zabolotskaya Equation (describes e.g. sound waves in nonlinear media)

$$\boxed{(u_t - uu_x)_x - u_{yy} = 0}$$

- Examples of conservation laws (with $f(t)$):

$$\mathrm{D}_t(u_x) + \mathrm{D}_x(-uu_x) + \mathrm{D}_y(-u_y) = 0$$

$$\mathrm{D}_t\left(fu\right) + \mathrm{D}_x\left(-(fx + \tfrac{1}{2}f'y^2)(u_t - uu_x) - \frac{1}{2}fu^2\right)$$

$$+\mathrm{D}_y\left((fx + \tfrac{1}{2}f'y^2)u_y - f'yu\right) = 0$$

$$\mathrm{D}_t\left(fyu\right) + \mathrm{D}_x\left(-(fxy + \frac{1}{6}f'y^3)(u_t - uu_x) - \frac{1}{2}fyu^2\right)$$

$$+\mathrm{D}_y\left((fxy + \frac{1}{6}f'y^3)u_y - (fx + \frac{1}{2}f'y^2)u\right) = 0$$

- Example: Zakharov-Kuznetsov Equation

  (describes e.g. ion acoustic solitons in magnetic plasma)

$$\boxed{u_t + \alpha u u_x + \beta \nabla^2 u_x = 0}$$

in 2-D, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

- Examples of conservation laws:

$$\mathrm{D}_t\left(u\right) + \mathrm{D}_x\left(\frac{\alpha}{2}u^2 + \beta u_{xx}\right) + \mathrm{D}_y\left(\beta u_{xy}\right) = 0$$

$$\mathrm{D}_t\left(u^2\right) + \mathrm{D}_x\left(\frac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{xx} + u_{yy})\right)$$

$$+\mathrm{D}_y\left(-2\beta u_x u_y\right) = 0$$

$$\mathrm{D}_t\left(u^3-\frac{3\beta}{\alpha}(u_x^2+u_y^2)\right)+\mathrm{D}_x\left(\frac{3\alpha}{4}u^4+3\beta u^2u_{xx}\right.$$

$$-6\beta u(u_x^2+u_y^2)+\frac{3\beta^2}{\alpha}(u_{xx}^2-u_{yy}^2)$$

$$\left.-\frac{6\beta^2}{\alpha}(u_x(u_{xxx}+u_{xyy})+u_y(u_{xxy}+u_{yyy}))\right)$$

$$+\mathrm{D}_y\left(3\beta u^2u_{xy}+\frac{6\beta^2}{\alpha}u_{xy}(u_{xx}+u_{yy})\right)=0$$

$$
\begin{aligned}
&\mathrm{D}_t\left(tu^2-\frac{2}{\alpha}xu\right)+\mathrm{D}_x\left(t(\frac{2\alpha}{3}u^3-\beta(u_x^2-u_y^2)\right.\\
&\left.+2\beta u(u_{xx}+u_{yy}))-\frac{2}{\alpha}x(\frac{\alpha}{2}u^2+\beta u_{xx})+\frac{2\beta}{\alpha}u_x\right)\\
&-\mathrm{D}_y\left(2\beta(tu_xu_y+\frac{1}{\alpha}xu_{xy})\right)=0
\end{aligned}
$$

- Example: Navier's Equation

  (describes e.g. wave motion in elastic solids)

$$\boxed{\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \Delta \mathbf{u}}$$

  where $\mathbf{u} = (u, v, w)$, λ and μ are the Lamé constants

  In components,

$$\rho u_{2t} = (\lambda + \mu)\left(u_{xx} + v_{xy} + w_{xz}\right) + \mu\left(u_{xx} + u_{yy} + u_{zz}\right)$$
$$\rho v_{2t} = (\lambda + \mu)\left(u_{xy} + v_{yy} + w_{yz}\right) + \mu\left(v_{xx} + v_{yy} + v_{zz}\right)$$
$$\rho w_{2t} = (\lambda + \mu)\left(u_{xz} + v_{yz} + w_{zz}\right) + \mu\left(w_{xx} + w_{yy} + w_{zz}\right)$$

- Examples of densities (fluxes are long):

$$
\begin{aligned}
\rho_{(1)} &= \rho u_t \\
\rho_{(2)} &= \rho v_t \\
\rho_{(3)} &= \rho w_t \\
\rho_{(4)} &= u_x(v_{tz} - w_{ty}) - v_x(u_{tz} - w_{tx}) + w_x(u_{ty} - v_{tx}) \\
\rho_{(5)} &= u_y(v_{tz} - w_{ty}) + v_x w_{ty} - v_y u_{tz} - w_x v_{ty} + w_y u_{ty} \\
\rho_{(6)} &= (u_y - v_x)w_{tz} - (u_z - w_x)v_{tz} + (v_z - w_y)u_{tz} \\
\rho_{(7)} &= \rho(v_t u_{tz} - w_t(u_{ty} - v_{tx})) + \mu\,\big(u_y w_{zz} + v_x(u_{xz} - w_{yy} - w_{zz}) \\
&\quad + v_y u_{yz} + v_z u_{zz} + w_x(v_{xx} - u_{xy}) - w_y u_{yy}\big)
\end{aligned}
$$

and many more....

Conclusions and Future Work

- The power of Euler and homotopy operators:
  - Testing exactness
  - Integration by parts: D_x^{-1} and Div^{-1}
- Integration of non-exact expressions

  Example: $f = u_x v + u v_x + u^2 u_{xx}$

  $\int f dx = uv + \int u^2 u_{xx} \, dx$
- Use other homotopy formulas (prevent curl terms)

- Treat broader class of PDEs (beyond evolution type)

  Example: short pulse equation (nonlinear optics)

$$u_{xt} = u + (u^3)_{xx} = u + 6uu_x^2 + 3u^2 u_{xx}$$

  with non-polynomial conservation law

$$\mathrm{D}_t \left(\sqrt{1 + 6u_x^2} \right) - \mathrm{D}_x \left(3u^2 \sqrt{1 + 6u_x^2} \right) = 0$$

- Continue the implementation in *Mathematica*

Thank You

Software packages in *Mathematica*

Codes are available via the Internet:

URL: http://inside.mines.edu/~whereman/

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