# Symbolic Computation of Conserved Densities of Nonlinear Evolution Equations and Differential-Difference Equations

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- **Purpose**

  Design and implement an algorithm to compute polynomial conservation laws for nonlinear systems of evolution equations and differential-difference equations

- **Motivation**

  – Conservation laws describe the conservation of fundamental physical quantities such as linear momentum and energy. Compare with constants of motion (first integrals) in mechanics

  – For nonlinear PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws assures complete integrability

  – Conservation laws provide a simple and efficient method to study both quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures

  – Conservation laws can be used to test numerical integrators

# PART I: Evolution Equations

## • Conservation Laws for PDEs

Consider a single nonlinear evolution equation

$$u_t = \mathrm{F}(u, u_x, u_{2x}, ..., u_{nx})$$

or a system of $N$ nonlinear evolution equations

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, ..., \mathbf{u}_{nx})$$

where $\mathbf{u} = [u_1, \ldots, u_N]^T$ and

$$u_t \stackrel{\text{def}}{=} \frac{\partial u}{\partial t}, \qquad u^{(n)} = u_{nx} \stackrel{\text{def}}{=} \frac{\partial^n u}{\partial x^n}$$

All components of $\mathbf{u}$ depend on $x$ and $t$

*Conservation law:*

$$D_t \rho + D_x J = 0$$

$\rho$ is the density, $J$ is the flux

Both are polynomial in $u, u_x, u_{2x}, u_{3x}, ...$

Consequently

$$P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}$$

if $J$ vanishes at infinity

- **The Euler Operator (calculus of variations)**

Useful tool to verify if an expression is a total derivative

**Theorem:**

If

$$f = f(x, y_1, \ldots, y_1^{(n)}, \ldots, y_N, \ldots, y_N^{(n)})$$

then

$$\mathcal{L}_{\mathbf{y}}(f) \equiv \mathbf{0}$$

if and only if

$$f = D_x g$$

where

$$g = g(x, y_1, \ldots, y_1^{(n-1)}, \ldots, y_N, \ldots, y_N^{(n-1)})$$

Notations:

$$\mathbf{y} = [y_1, \ldots, y_N]^T$$

$$\mathcal{L}_{\mathbf{y}}(f) = [\mathcal{L}_{y_1}(f), \ldots, \mathcal{L}_{y_N}(f)]^T$$

$$\mathbf{0} = [0, \ldots, 0]^T$$

($T$ for transpose)

and **Euler Operator**:

$$\mathcal{L}_{y_i} = \frac{\partial}{\partial y_i} - \frac{d}{dx}(\frac{\partial}{\partial {y_i}'}) + \frac{d^2}{dx^2}(\frac{\partial}{\partial {y_i}''}) + \cdots + (-1)^n \frac{d^n}{dx^n}(\frac{\partial}{\partial {y_i}^{(n)}})$$

- **Example: Korteweg-de Vries (KdV) equation**

$$u_t + uu_x + u_{3x} = 0$$

Conserved densities:

$$\rho_1 = u, \qquad (u)_t + (\frac{u^2}{2} + u_{2x})_x = 0$$

$$\rho_2 = u^2, \qquad (u^2)_t + (\frac{2u^3}{3} + 2uu_{2x} - {u_x}^2)_x = 0$$

$$\rho_3 = u^3 - 3{u_x}^2,$$

$$\left(u^3 - 3{u_x}^2\right)_t + \left(\frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2u_{2x} + 3{u_{2x}}^2 - 6u_xu_{3x}\right)_x = 0$$

$\vdots$

$$\rho_6 = u^6 - 60u^3{u_x}^2 - 30{u_x}^4 + 108u^2{u_{2x}}^2$$

$$+\frac{720}{7}{u_{2x}}^3 - \frac{648}{7}u{u_{3x}}^2 + \frac{216}{7}{u_{4x}}^2, \qquad \text{...... long ......}$$

$\vdots$

**Note:** KdV equation and conservation laws are invariant under dilation (scaling) symmetry

$$(x, t, u) \to (\lambda x, \lambda^3 t, \lambda^{-2} u)$$

$u$ and $t$ carry the weights of 2 and 3 derivatives with respect to $x$

$$u \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$$

- **Key Steps of the Algorithm**

  1. Determine weights (scaling properties) of variables & parameters

  2. Construct the form of the density (building blocks)

  3. Determine the unknown constant coefficients

- **Example: KdV equation**

$$u_t + uu_x + u_{3x} = 0$$

Compute the density of rank 6

(i) Compute the weights by solving a linear system

$$w(u) + w(\frac{\partial}{\partial t}) = 2w(u) + w(x) = w(u) + 3w(x)$$

With $w(x) = 1$, $w(\frac{\partial}{\partial t}) = 3, \;\; w(u) = 2.$

Thus, $\qquad (x, t, u) \to (\lambda x, \lambda^3 t, \lambda^{-2} u)$

(ii) Take all the variables, except $(\frac{\partial}{\partial t})$, with positive weight and list all possible powers of $u$, up to rank 6 : $\;\; [u, u^2, u^3]$

Introduce $x$ derivatives to 'complete' the rank

$u$ has weight 2, introduce $\frac{\partial^4}{\partial x^4}$

$u^2$ has weight 4, introduce $\frac{\partial^2}{\partial x^2}$

$u^3$ has weight 6, no derivatives needed

Apply the derivatives and remove terms that are total derivatives with respect to $x$ or total derivative up to terms kept earlier in the list

$$[u_{4x}] \;\rightarrow\; [\,]\;\text{ empty list}$$
$$[{u_x}^2, uu_{2x}] \;\rightarrow\; [{u_x}^2]\;\text{ since }\; uu_{2x} = (uu_x)_x - {u_x}^2$$
$$[u^3] \;\rightarrow\; [u^3]$$

Combine the building blocks: $\rho = c_1 u^3 + c_2 {u_x}^2$

(iii) Determine the coefficients $c_1$ and $c_2$

1. Compute $D_t\rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt}$
2. Replace $u_t$ by $-(uu_x + u_{3x})$ and $u_{xt}$ by $-(uu_x + u_{3x})_x$
3. Apply the Euler operator or integrate by parts

$$\begin{aligned} D_t\rho \;&= -[\frac{3}{4}c_1 u^4 - (3c_1 - c_2)uu_x^2 + 3c_1 u^2 u_{2x} - c_2 {u_{2x}}^2 + 2c_2 u_x u_{3x}]_x \\ &\quad - (3c_1 + c_2){u_x}^3 \end{aligned}$$

4. The non-integrable term must vanish. Thus, $c_1 = -\frac{1}{3}c_2$.
   Set $c_2 = -3$, hence, $c_1 = 1$

   Result:

$$\rho = u^3 - 3{u_x}^2$$

   Expression $[\ldots]$ yields

$$J = \frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x}$$

- **Example: Boussinesq equation**

$$u_{tt} - u_{2x} + 3uu_{2x} + 3{u_x}^2 + \alpha u_{4x} = 0$$

with nonzero parameter $\alpha$. Can be written as

$$\begin{aligned} u_t + v_x &= 0 \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0 \end{aligned}$$

The terms $u_x$ and $\alpha\ u_{3x}$ are not uniform in rank

Introduce auxiliary parameter $\beta$ with weight.
Replace the system by

$$\begin{aligned} u_t + v_x &= 0 \\ v_t + \beta u_x - 3uu_x - \alpha u_{3x} &= 0 \end{aligned}$$

The system is invariant under the scaling symmetry

$$(x, t, u, v, \beta) \rightarrow (\lambda x, \lambda^2 t, \lambda^{-2} u, \lambda^{-3} v, \lambda^{-2} \beta)$$

Hence

$$w(u) = 2,\ w(\beta) = 2,\ w(v) = 3 \ \text{ and } \ w(\frac{\partial}{\partial t}) = 2$$

or

$$u \sim \beta \sim \frac{\partial^2}{\partial x^2}, \qquad v \sim \frac{\partial^3}{\partial x^3}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^2}{\partial x^2}$$

Form $\rho$ of rank 6

$$\rho = c_1\ \beta^2 u + c_2\ \beta u^2 + c_3\ u^3 + c_4\ v^2 + c_5\ u_x v + c_6\ u_x^2$$

Compute the $c_i$. At the end set $\beta = 1$

$$\rho = u^2 - u^3 + v^2 + \alpha u_x^2$$

which is no longer uniform in rank!

- **Application: A Class of Fifth-Order Evolution Equations**

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

where $\alpha, \beta, \gamma$ are nonzero parameters, and $u \sim \frac{\partial^2}{\partial x^2}$

Special cases:

| | | | |
|---|---|---|---|
| $\alpha = 30$ | $\beta = 20$ | $\gamma = 10$ | Lax |
| $\alpha = 5$ | $\beta = 5$ | $\gamma = 5$ | Sawada − Kotera |
| $\alpha = 20$ | $\beta = 25$ | $\gamma = 10$ | Kaup−Kupershmidt |
| $\alpha = 2$ | $\beta = 6$ | $\gamma = 3$ | Ito |

Under what conditions for the parameters $\alpha, \beta$ and $\gamma$ does this equation admit a density of fixed rank?

- **Rank 2:**
  No condition

$$\rho = u$$

- **Rank 4:**
  Condition: $\beta = 2\gamma$ (Lax and Ito cases)

$$\rho = u^2$$

– **Rank 6:**
Condition:

$$10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2$$

(Lax, SK, and KK cases)

$$\rho = u^3 + \frac{15}{(-2\beta + \gamma)} {u_x}^2$$

– **Rank 8:**

1. $\beta = 2\gamma$ (Lax and Ito cases)

$$\rho = u^4 - \frac{6\gamma}{\alpha} u{u_x}^2 + \frac{6}{\alpha} {u_{2x}}^2$$

2. $\alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45}$ (SK, KK and Ito cases)

$$\rho = u^4 - \frac{135}{2\beta + \gamma} u{u_x}^2 + \frac{675}{(2\beta + \gamma)^2} {u_{2x}}^2$$

– **Rank 10:**

Condition:

$$\beta = 2\gamma$$

and

$$10\alpha = 3\gamma^2$$

(Lax case)

$$\rho = u^5 - \frac{50}{\gamma} u^2{u_x}^2 + \frac{100}{\gamma^2} u{u_{2x}}^2 - \frac{500}{7\gamma^3} {u_{3x}}^2$$

What are the necessary conditions for the parameters $\alpha, \beta$ and $\gamma$ for this equation to admit infinitely many polynomial conservation laws?

– If $\alpha = \frac{3}{10}\gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case)

– If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case)

– If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \frac{5}{2}\gamma$ then there is a sequence (with gaps!) of conserved densities (KK case)

– If

$$\alpha = -\frac{2\beta^2 - 7\beta\gamma + 4\gamma^2}{45}$$

or

$$\beta = 2\gamma$$

then there is a conserved density of rank 8

Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case)

# PART II: Differential-difference Equations

## • Conservation Laws for DDEs

Consider a system of DDEs, continuous in time, discretized in space

$$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

$\mathbf{u}_n$ and $\mathbf{F}$ are vector dynamical variables

*Conservation law*:

$$\dot{\rho}_n = J_n - J_{n+1}$$

$\rho_n$ is the density, $J_n$ is the flux

Both are polynomials in $\mathbf{u}_n$ and its shifts

$$\frac{\mathrm{d}}{\mathrm{dt}}(\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1})$$

If $J_n$ is bounded for all $n$, with suitable boundary or periodicity conditions

$$\sum_n \rho_n = \text{constant}$$

## • Definitions

Define: $D$ *shift-down* operator, $U$ *shift-up* operator

$$Dm = m|_{n \to n-1} \qquad Um = m|_{n \to n+1}$$

For example,

$$Du_{n+2}v_n = u_{n+1}v_{n-1} \qquad Uu_{n-2}v_{n-1} = u_{n-1}v_n$$

Compositions of $D$ and $U$ define an *equivalence relation*
All shifted monomials are *equivalent*, e.g.

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}$$

Use *equivalence criterion*:

If two monomials, $m_1$ and $m_2$, are equivalent, $m_1 \equiv m_2$, then

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial $M_n$

For example, $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}]$

with $M_n = u_{n-2}u_n$

*Main representative* of an equivalence class; the monomial with label $n$ on $u$ (or $v$)

For example, $u_nu_{n+2}$ is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3}$, etc.

Use lexicographical ordering to resolve conflicts

For example, $u_nv_{n+2}$ (not $u_{n-2}v_n$) is the main representative of the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4}$, etc.

- **Algorithm: Toda Lattice**

$$m\ddot{y}_n = a[e^{(y_{n-1}-y_n)} - e^{(y_n-y_{n+1})}]$$

Take $m = a = 1$ (scale on $t$), and set $u_n = \dot{y}_n, \;\; v_n = e^{(y_n-y_{n+1})}$

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1})$$

Simplest conservation law (by hand):

$$\dot{u}_n = \dot{\rho}_n = v_{n-1} - v_n = J_n - J_{n+1} \quad \text{with} \quad J_n = v_{n-1}$$

*First* pair:

$$\rho_n^{(1)} = u_n, \qquad\qquad J_n^{(1)} = v_{n-1}$$

*Second* pair:

$$\rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n, \qquad J_n^{(2)} = u_n v_{n-1}$$

*Key observation*: The DDE and the two conservation laws, $\dot{\rho}_n = J_n - J_{n+1}$, with

$$\rho_n^{(1)} = u_n, \qquad\qquad J_n^{(1)} = v_{n-1}$$

$$\rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n, \qquad J_n^{(2)} = u_n v_{n-1}$$

are invariant under the scaling symmetry

$$(t, u_n, v_n) \rightarrow (\lambda t, \lambda^{-1} u_n, \lambda^{-2} v_n)$$

Dimensional analysis:
$u_n$ corresponds to one derivative with respect to $t$

For short, $u_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$, and similarly, $v_n \sim \frac{\mathrm{d}^2}{\mathrm{dt}^2}$

Our algorithm exploits this symmetry to find conserved densities:

1. Determining the weights
2. Constructing the form of density
3. Determining the unknown coefficients

- **Step 1: Determine the weights**

The *weight*, $w$, of a variable is equal to the number of derivatives with respect to $t$ the variable carries.

Weights are positive, rational, and independent of $n$.

Requiring uniformity in rank for each equation

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1})$$

allows one to compute the weights of the dependent variables.

Solve the linear system

$$\begin{aligned} w(u_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) &= w(v_n) \\ w(v_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) &= w(v_n) + w(u_n) \end{aligned}$$

Set $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$, then $w(u_n) = 1$, and $w(v_n) = 2$

which is consistent with the scaling symmetry

$$(t, u_n, v_n) \rightarrow (\lambda t, \lambda^{-1} u_n, \lambda^{-2} v_n)$$

- **Step 2: Construct the form of the density**

The *rank* of a monomial is the total weight of the monomial.

For example, compute the form of the density of rank 3

List all monomials in $u_n$ and $v_n$ of rank 3 or less:

$$\mathcal{G} = \{{u_n}^3, {u_n}^2, u_n v_n, u_n, v_n\}$$

Next, for each monomial in $\mathcal{G}$, introduce enough $t$-derivatives, so that each term exactly has weight 3. Use the DDE to remove $\dot{u}_n$ and $\dot{v}_n$

$$\frac{\mathrm{d}^0}{\mathrm{dt}^0}({u_n}^3) = {u_n}^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,$$

$$\frac{\mathrm{d}}{\mathrm{dt}}({u_n}^2) = 2u_n v_{n-1} - 2u_n v_n, \qquad \frac{\mathrm{d}}{\mathrm{dt}}(v_n) = u_n v_n - u_{n+1} v_n,$$

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n$$

Gather the resulting terms in a set

$$\mathcal{H} = \{{u_n}^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n\}$$

Identify members of the same equivalence classes and replace them by the main representatives.

For example, since $u_n v_{n-1} \equiv u_{n+1} v_n$ both are replaced by $u_n v_{n-1}$.
$\mathcal{H}$ is replaced by

$$\mathcal{I} = \{{u_n}^3, u_n v_{n-1}, u_n v_n\}$$

containing the building blocks of the density.

Form a linear combination of the monomials in $\mathcal{I}$

$$\rho_n = c_1\, {u_n}^3 + c_2\, u_n v_{n-1} + c_3\, u_n v_n$$

with constant coefficients $c_i$

- **Step 3: Determine the unknown coefficients**

Require that the conservation law, $\dot{\rho}_n = J_n - J_{n+1}$, holds

Compute $\dot{\rho}_n$ and use the equations to remove $\dot{u}_n$ and $\dot{v}_n$.

Group the terms

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2){u_n}^2 v_{n-1} + (c_3 - 3c_1){u_n}^2 v_n + (c_3 - c_2) v_{n-1} v_n \\ &\quad + c_2 u_{n-1} u_n v_{n-1} + c_2 {v_{n-1}}^2 - c_3 u_n u_{n+1} v_n - c_3 {v_n}^2\end{aligned}$$

Use the equivalence criterion to modify $\dot{\rho}_n$

Replace $u_{n-1} u_n v_{n-1}$ by $u_n u_{n+1} v_n + [u_{n-1} u_n v_{n-1} - u_n u_{n+1} v_n]$.

The goal is to introduce the main representatives. Therefore,

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2){u_n}^2 v_{n-1} + (c_3 - 3c_1){u_n}^2 v_n \\ &\quad + (c_3 - c_2) v_n v_{n+1} + [(c_3 - c_2) v_{n-1} v_n - (c_3 - c_2) v_n v_{n+1}] \\ &\quad + c_2 u_n u_{n+1} v_n + [c_2 u_{n-1} u_n v_{n-1} - c_2 u_n u_{n+1} v_n] \\ &\quad + c_2 {v_n}^2 + [c_2 {v_{n-1}}^2 - c_2 {v_n}^2] - c_3 u_n u_{n+1} v_n - c_3 {v_n}^2\end{aligned}$$

Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom. Rearrange the latter terms so that they match the pattern $[J_n - J_{n+1}]$. Hence,

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2){u_n}^2 v_{n-1} + (c_3 - 3c_1){u_n}^2 v_n \\ &\quad + (c_3 - c_2) v_n v_{n+1} + (c_2 - c_3) u_n u_{n+1} v_n + (c_2 - c_3){v_n}^2 \\ &\quad + [\{(c_3 - c_2) v_{n-1} v_n + c_2 u_{n-1} u_n v_{n-1} + c_2 {v_{n-1}}^2\} \\ &\quad - \{(c_3 - c_2) v_n v_{n+1} + c_2 u_n u_{n+1} v_n + c_2 {v_n}^2\}]\end{aligned}$$

The terms inside the square brackets determine:

$$J_n = (c_3 - c_2) v_{n-1} v_n + c_2 u_{n-1} u_n v_{n-1} + c_2 {v_{n-1}}^2$$

The terms outside the square brackets must vanish, thus

$$\mathcal{S} = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}$$

The solution is $3c_1 = c_2 = c_3$. Choose $c_1 = \frac{1}{3}$, thus $c_2 = c_3 = 1$

$$\rho_n = \tfrac{1}{3}\, {u_n}^3 + u_n (v_{n-1} + v_n), \qquad J_n = u_{n-1} u_n v_{n-1} + {v_{n-1}}^2$$

Analogously, conserved densities of rank $\leq 5$:

$$\rho_n^{(1)} = u_n, \qquad \rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n$$

$$\rho_n^{(3)} = \tfrac{1}{3}{u_n}^3 + u_n(v_{n-1} + v_n)$$

$$\rho_n^{(4)} = \tfrac{1}{4}{u_n}^4 + {u_n}^2(v_{n-1} + v_n) + u_n u_{n+1} v_n + \tfrac{1}{2}{v_n}^2 + v_n v_{n+1}$$

$$\begin{aligned}\rho_n^{(5)} = {} & \tfrac{1}{5}{u_n}^5 + {u_n}^3(v_{n-1} + v_n) + u_n u_{n+1} v_n (u_n + u_{n+1}) \\ & + u_n v_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_n v_n (v_{n-1} + v_n + v_{n+1})\end{aligned}$$

- **Application: A parameterized Toda lattice**

$$\dot{u}_n = \alpha \, v_{n-1} - v_n, \quad \dot{v}_n = v_n \, (\beta \, u_n - u_{n+1})$$

$\alpha$ and $\beta$ are *nonzero* parameters. The system is integrable if $\alpha = \beta = 1$

Compute the *compatibility conditions* for $\alpha$ and $\beta$, so that there is a conserved densities of, say, rank 3.

In this case, we have $\mathcal{S}$:

$\{3\alpha c_1 - c_2 = 0, \beta c_3 - 3c_1 = 0, \alpha c_3 - c_2 = 0, \beta c_2 - c_3 = 0, \alpha c_2 - c_3 = 0\}$

A non-trivial solution $3c_1 = c_2 = c_3$ will exist *iff* $\alpha = \beta = 1$

Analogously, the parameterized Toda lattice has density

$\rho_n^{(1)} = u_n$ of rank 1 if $\alpha = 1$

and density

$\rho_n^{(2)} = \frac{\beta}{2} {u_n}^2 + v_n$ of rank 2 if $\alpha \, \beta = 1$

Only when $\alpha = \beta = 1$ will the parameterized system have conserved densities of rank $\geq 3$

- **Example: Nonlinear Schrödinger (NLS) equation**

Ablowitz and Ladik discretization of the NLS equation:

$$i\,\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n^* u_n (u_{n+1} + u_{n-1})$$

where $u_n^*$ is the complex conjugate of $u_n$.

Treat $u_n$ and $v_n = u_n^*$ as independent variables, add the complex conjugate equation, and absorb $i$ in the scale on $t$

$$\begin{aligned} \dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}) \\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}) \end{aligned}$$

Since $v_n = u_n^*$, $\; w(v_n) = w(u_n)$.

No uniformity in rank! Circumvent this problem by introducing an auxiliary parameter $\alpha$ with weight,

$$\begin{aligned} \dot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}) \\ \dot{v}_n &= -\alpha(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

Uniformity in rank requires that

$$\begin{aligned} w(u_n) + 1 &= w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n) \\ w(v_n) + 1 &= w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n) \end{aligned}$$

which yields

$$w(u_n) = w(v_n) = \frac{1}{2}, \quad w(\alpha) = 1$$

Uniformity in rank is essential for the first two steps of the algorithm. After Step 2, you can already set $\alpha = 1$.

The computations now proceed as in the previous examples

Conserved densities:

$$
\begin{aligned}
\rho_n^{(1)} &= c_1 u_n v_{n-1} + c_2 u_n v_{n+1} \\[2ex]
\rho_n^{(2)} &= c_1 \big( \tfrac{1}{2} {u_n}^2 {v_{n-1}}^2 + u_n u_{n+1} v_{n-1} v_n + u_n v_{n-2} \big) \\
&+ c_2 \big( \tfrac{1}{2} {u_n}^2 {v_{n+1}}^2 + u_n u_{n+1} v_{n+1} v_{n+2} + u_n v_{n+2} \big) \\[2ex]
\rho_n^{(3)} &= c_1 \big[ \tfrac{1}{3} {u_n}^3 {v_{n-1}}^3 \\
&\quad + u_n u_{n+1} v_{n-1} v_n \big( u_n v_{n-1} + u_{n+1} v_n + u_{n+2} v_{n+1} \big) \\
&\quad + u_n v_{n-1} \big( u_n v_{n-2} + u_{n+1} v_{n-1} \big) \\
&\quad + u_n v_n \big( u_{n+1} v_{n-2} + u_{n+2} v_{n-1} \big) + u_n v_{n-3} \big] \\
&+ c_2 \big[ \tfrac{1}{3} {u_n}^3 {v_{n+1}}^3 \\
&\quad + u_n u_{n+1} v_{n+1} v_{n+2} \big( u_n v_{n+1} + u_{n+1} v_{n+2} + u_{n+2} v_{n+3} \big) \\
&\quad + u_n v_{n+2} \big( u_n v_{n+1} + u_{n+1} v_{n+2} \big) \\
&\quad + u_n v_{n+3} \big( u_{n+1} v_{n+1} + u_{n+2} v_{n+2} \big) + u_n v_{n+3} \big]
\end{aligned}
$$

- **Scope and Limitations of Algorithm & Software**

  – Systems of PDEs or DDEs must be polynomial in dependent variables

  – Only one space variable (continuous $x$ for PDEs, discrete $n$ for DDEs) is allowed

  – No terms should *explicitly* depend on $x$ and $t$ for PDEs, or $n$ for DDEs

  – Program only computes polynomial conserved densities;
  only polynomials in the dependent variables and their derivatives;
  no explicit dependencies on $x$ and $t$ for PDEs (or $n$ for DDEs)

  – No limit on the number of PDEs or DDEs.
  In practice: time and memory constraints

  – Input systems may have (nonzero) parameters.
  Program computes the compatibility conditions for parameters such that densities (of a given rank) exist

  – Systems can also have parameters with (unknown) weight.
  Allows one to test PDEs or DDEs of non-uniform rank

  – For systems where one or more of the weights are free,
  the program prompts the user to enter values for the free weights

  – Negative weights are not allowed

  – Fractional weights and ranks are permitted

  – Form of $\rho$ can be given in the data file (testing purposes)

- **Conserved Densities Software**

  - Conserved densities programs **CONSD** and **SYMCD** by Ito and Kako (Reduce, 1985, 1994 & 1996).
  - Conserved densities in **DELiA** by Bocharov (Pascal, 1990)
  - Conserved densities and formal symmetries **FS** by Gerdt and Zharkov (Reduce, 1993)
  - Formal symmetry approach by Mikhailov and Yamilov (MuMath, 1990)
  - Recursion operators and symmetries by Roelofs, Sanders and Wang (Reduce 1994, Maple 1995, Form 1995-present)
  - Conserved densities **condens.m** by Hereman and Göktaş (Mathematica, 1996)
  - Conservation laws, based on **CRACK** by Wolf (Reduce, 1995)
  - Conservation laws by Hickman (Maple, 1996)
  - Conserved densities by Ahner *et al.* (Mathematica, 1995). Project halted.
  - Conserved densities **diffdens.m** by Göktaş and Hereman (Mathematica, 1997)

- **Conclusions and Further Research**
  - Two *Mathematica* programs are available:
    *condens.m* for evolution equations (PDEs)
    *diffdens.m* for differential-difference equations (DDEs)
  - Usefulness
    * Testing models for integrability
    * Study of classes of nonlinear PDEs or DDEs
  - Comparison with other programs
    * Parameter analysis is possible
    * Not restricted to uniform rank equations
    * Not restricted to evolution equations provided that one can write the equation(s) as a system of evolution equations
  - Future work
    * Generalization towards broader classes of equations (e.g. $u_{xt}$)
    * Generalization towards more space variables (e.g. KP equation)
    * Conservation laws with time and space dependent coefficients
    * Conservation laws with $n$ dependent coefficients

* Exploit other symmetries in the hope to find conserved densities of non-polynomial form

* Constants of motion for dynamical systems
  (e.g. Lorenz and Hénon-Heiles systems)

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– Papers submitted to: J. Symb. Comp., Phys. Lett. A and Physica D

– Software: available via FTP, ftp site *mines.edu* in subdirectories

  pub/papers/math_cs_dept/software/condens
  pub/papers/math_cs_dept/software/diffdens

  or via the Internet

  URL: http://www.mines.edu/fs_home/whereman/