Symbolic Computation of Conservation Laws of Nonlinear PDEs in Multi-dimensions

Willy Hereman

Department of Mathematical and Computer Sciences
Colorado School of Mines
Golden, Colorado, U.S.A.
whereman@mines.edu
http://www.mines.edu/fs_home/whereman/

Department of Mathematics and Statistics
University of Otago
Dunedin, New Zealand

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Outline

- Conservation Laws of PDEs in multi-dimensions
- Example: Shallow water wave equations (Dellar)
- Algorithmic Methods for conservation laws
- Computer Demonstration
- Tools:
  - Euler operators (testing exactness)
  - Calculus-based formulas for homotopy operator
    - ⋆ symbolic integration by parts
    - ⋆ inversion of the total divergence operator
- Application to shallow water wave equations
- Conclusions and Future Work
- Software and Publications

Notation – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$
- Dependent variables $\mathbf{u} = (u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(N)})$
  In examples: $\mathbf{u} = (u, v, \theta, h, \ldots)$
- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kx\,ly} = \frac{\partial^{k+l} u}{\partial x^k y^l}$, etc.
- *Differential functions*
  Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$ for $u(x)$, $v(x)$
- Total derivative (with respect to x)

$$\mathrm{D}_x = \frac{\partial}{\partial x} + \sum_{k=0}^{M_x^{(1)}} u_{(k+1)x} \frac{\partial}{\partial u_{kx}} + \sum_{k=0}^{M_x^{(2)}} v_{(k+1)x} \frac{\partial}{\partial v_{kx}}$$

$M_x^{(1)}$ is the order of f in u (with respect to x), $etc.$

- Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$
  $M_x^{(1)} = 1$ and $M_x^{(2)} = 2$

- Total derivative with respect to x:

$$
\begin{aligned}
\mathrm{D}_x f &= \frac{\partial f}{\partial x} + \sum_{k=0}^{1} u_{(k+1)x} \frac{\partial f}{\partial u_{kx}} + \sum_{k=0}^{2} v_{(k+1)x} \frac{\partial f}{\partial v_{kx}} \\
&= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{2x} \frac{\partial f}{\partial u_x} \\
&\quad v_x \frac{\partial f}{\partial v} + v_{2x} \frac{\partial f}{\partial v_x} + v_{3x} \frac{\partial f}{\partial v_{2x}} \\
&= 2x u_x^3 v_x + u_x (v v_x) + u_{xx} (3x^2 u_x^2 v_x + v_{xx}) \\
&\quad + v_x (u v_x) + v_{xx} (uv + x^2 u_x^3) + v_{xxx} (u_x)
\end{aligned}
$$

Conservation Laws

- Conservation law in $(1+1)$ dimensions

$$\boxed{\mathrm{D}_t \rho + \mathrm{D}_x J = 0} \quad \text{(on PDE)}$$

conserved density ρ and flux J

- Example: Korteweg-de Vries (KdV) equation

$$u_t + u u_x + u_{3x} = 0$$

- Example of conservation law

$$\mathrm{D}_t \left(u^3 - 3{u_x}^2 \right) +$$

$$\mathrm{D}_x \left(\frac{3}{4} u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x} \right) = 0$$

- Key property: Dilation invariance
- Example: KdV equation and its density-flux pairs are invariant under the scaling symmetry

$$(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u),$$

λ is arbitrary parameter.

- Some density-flux pairs for the KdV equation:

$$\rho^{(1)} = u \qquad J^{(1)} = \frac{u^2}{2} + u_{2x}$$

$$\rho^{(2)} = u^2 \qquad J^{(2)} = \frac{2u^3}{3} + 2uu_{2x} - {u_x}^2$$

$$\rho^{(3)} = u^3 - 3{u_x}^2$$

$$J^{(3)} = \frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2u_{2x} + 3{u_{2x}}^2 - 6u_xu_{3x}$$

$$\vdots$$

$$\rho^{(6)} = u^6 - 60\,u^3{u_x}^2 - 30\,{u_x}^4 + 108\,u^2{u_{2x}}^2 + \frac{720}{7}\,{u_{2x}}^3 - \frac{648}{7}\,u{u_{3x}}^2 + \frac{216}{7}\,{u_{4x}}^2$$

$$\vdots$$

- Conservation law in $(3+1)$ dimensions

$$\boxed{\mathrm{D}_t\rho + \nabla\cdot\mathbf{J} = \mathrm{D}_t\rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 + \mathrm{D}_z J_3 = 0}\ \text{(on PDE)}$$

conserved density ρ and flux $\mathbf{J} = (J_1, J_2, J_3)$

- Example: Shallow water wave (SWW) equations

[P. Dellar, Phys. Fluids **15** (2003) 292-297]

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\,\mathbf{\Omega}\times\mathbf{u} + \nabla(\theta h) - \frac{1}{2}h\nabla\theta = \mathbf{0}$$

$$\theta_t + \mathbf{u}\cdot(\nabla\theta) = 0$$

$$h_t + \nabla\cdot(\mathbf{u}h) = 0$$

where $\mathbf{u}(x,y,t), \theta(x,y,t)$ and $h(x,y,t)$.

- In components:

$$u_t + uu_x + vu_y - 2\,\Omega v + \frac{1}{2} h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\,\Omega u + \frac{1}{2} h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- SWW equations are invariant under

$$(x, y, t, u, v, h, \theta, \Omega) \rightarrow$$

$$(\lambda^{-1}x, \lambda^{-1}y, \lambda^{-b}t, \lambda^{b-1}u, \lambda^{b-1}v, \lambda^{a}h, \lambda^{2b-a-2}\theta, \lambda^{b}\Omega)$$

where $W(h) = a$ and $W(\Omega) = b \quad (a, b \in \mathbb{Q})$.

- First few densities-flux pairs of SWW system:

$$\rho^{(1)} = h \qquad \mathbf{J}^{(1)} = \begin{pmatrix} uh \\ vh \end{pmatrix}$$

$$\rho^{(2)} = h\theta \qquad \mathbf{J}^{(2)} = \begin{pmatrix} uh\theta \\ vh\theta \end{pmatrix}$$

$$\rho^{(3)} = h\theta^2 \qquad \mathbf{J}^{(3)} = \begin{pmatrix} uh\theta^2 \\ vh\theta^2 \end{pmatrix}$$

$$\rho^{(4)} = (u^2 + v^2)h + h^2\theta \qquad \mathbf{J}^{(4)} = \begin{pmatrix} u^3h + uv^2h + 2uh^2\theta \\ v^3h + u^2vh + 2vh^2\theta \end{pmatrix}$$

$$\rho^{(5)} = v_x\theta - u_y\theta + 2\Omega\theta$$

$$\mathbf{J}^{(5)} = \frac{1}{2} \begin{pmatrix} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{pmatrix}$$

Algorithmic Methods for Conservation Laws

- Use Noether's Theorem (Lagrangian formulation).
- Direct methods (Anderson, Bluman, Anco, Wolf, etc.) based on solving ODEs (or PDEs).
- Strategy (linear algebra and variational calculus).
  - Density is linear combination of scaling invariant terms with undetermined coefficients.
  - Use variational derivative (Euler operator) to compute the undetermined coefficients.
  - Use the homotopy operator to compute the flux (invert D_x or Div)
  - Work with linearly independent pieces in finite dimensional spaces.

Computer Demonstration

Review of Vector Calculus

- Definition: $\mathbf{F}$ is *conservative* if $\mathbf{F} = \boldsymbol{\nabla} f$
- Definition: $\mathbf{F}$ is *irrotational* or *curl free* if $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$
- Theorem (gradient test): $\mathbf{F} = \boldsymbol{\nabla} f$ iff $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$

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Question: How can one test that $f = \boldsymbol{\nabla} \cdot \mathbf{F}$?

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The divergence annihilates curls!

Question: How can one test that $f = \boldsymbol{\nabla} \cdot \mathbf{F}$?

No theorem from vector calculus!

Tools from the Calculus of Variations

- Definition: a differential function f is *exact* iff $f = \mathrm{D}_x F$
- Theorem (exactness test): $f = \mathrm{D}_x F$ iff $\mathcal{L}^{(0)}_{u^{(j)}(x)} f \equiv 0, \quad j = 1, 2, \ldots, N$
- Definition: a differential function f is a *divergence* if $f = \mathrm{Div}\, \mathbf{F}$
- Theorem (divergence test): $f = \mathrm{Div}\, \mathbf{F}$ iff $\mathcal{L}^{(0)}_{u^{(j)}(\mathbf{x})} f \equiv 0, \quad j = 1, 2, \ldots, N$

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The Euler operator annihilates divergences!

Formula for Euler operator (variational derivative)

in 1D:

$$\mathcal{L}^{(0)}_{u^{(j)}(x)} = \sum_{k=0}^{M_x^{(j)}} (-\mathrm{D}_x)^k \frac{\partial}{\partial u^{(j)}_{kx}}$$

$$= \frac{\partial}{\partial u^{(j)}} - \mathrm{D}_x \frac{\partial}{\partial u^{(j)}_{x}} + \mathrm{D}_x^2 \frac{\partial}{\partial u^{(j)}_{2x}} - \mathrm{D}_x^3 \frac{\partial}{\partial u^{(j)}_{3x}} + \cdots$$

where $j = 1, 2, \ldots, N.$

Formula for Euler operator in 2D:

$$\mathcal{L}^{(0,0)}_{u^{(j)}(x,y)} = \sum_{k_x=0}^{M_x^{(j)}} \sum_{k_y=0}^{M_y^{(j)}} (-\mathrm{D}_x)^{k_x} (-\mathrm{D}_y)^{k_y} \frac{\partial}{\partial u^{(j)}_{k_x x\, k_y y}}$$

$$= \frac{\partial}{\partial u^{(j)}} - \mathrm{D}_x \frac{\partial}{\partial u^{(j)}_x} - \mathrm{D}_y \frac{\partial}{\partial u^{(j)}_y}$$

$$+ \mathrm{D}_x^2 \frac{\partial}{\partial u^{(j)}_{2x}} + \mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u^{(j)}_{xy}} + \mathrm{D}_y^2 \frac{\partial}{\partial u^{(j)}_{2y}} - \mathrm{D}_x^3 \frac{\partial}{\partial u^{(j)}_{3x}} \cdots$$

where $j = 1, 2, \ldots, N$.

Formula for Euler operator in 3D:

$$\mathcal{L}^{(0,0,0)}_{u^{(j)}(x,y,z)}$$

$$= \sum_{k_x=0}^{M_x^{(j)}} \sum_{k_y=0}^{M_y^{(j)}} \sum_{k_z=0}^{M_z^{(j)}} (-\mathrm{D}_x)^{k_x} (-\mathrm{D}_y)^{k_y} (-\mathrm{D}_z)^{k_z} \frac{\partial}{\partial u^{(j)}_{k_x x\, k_y y\, k_z z}}$$

$$= \frac{\partial}{\partial u^{(j)}} - \mathrm{D}_x \frac{\partial}{\partial u^{(j)}_x} - \mathrm{D}_y \frac{\partial}{\partial u^{(j)}_y} - \mathrm{D}_z \frac{\partial}{\partial u^{(j)}_z}$$

$$+ \mathrm{D}_x^2 \frac{\partial}{\partial u^{(j)}_{2x}} + \mathrm{D}_y^2 \frac{\partial}{\partial u^{(j)}_{2y}} + \mathrm{D}_z^2 \frac{\partial}{\partial u^{(j)}_{2z}}$$

$$+ \mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u^{(j)}_{xy}} + \mathrm{D}_x \mathrm{D}_z \frac{\partial}{\partial u^{(j)}_{xz}} + \mathrm{D}_y \mathrm{D}_z \frac{\partial}{\partial u^{(j)}_{yz}} - \cdots$$

where $j = 1, 2, \ldots, N.$

Application: Testing Exactness

Consider, for example,

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x}$$

for $u(x)$ and $v(x)$

- f is exact
- After integration by parts (by hand):

$$F = \int f\, dx = 4\, v_x^2 + u_x^2\, \cos u - 3\, v^2\, \cos u$$

- Exactness test with Euler operator:

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6vv_x \cos u + 2u_x u_{2x} \cos u + 8v_x v_{2x}$$

$$\mathcal{L}_{u(x)}^{(0)} f = \frac{\partial f}{\partial u} - \mathrm{D}_x \frac{\partial f}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial f}{\partial u_{2x}}$$

$$= 3u_x\, v^2\, \cos u - u_x^3\, \cos u + 6v\, v_x\, \sin u - 2u_x\, u_{2x}\, \sin u$$

$$-\mathrm{D}_x[3v^2\, \sin u - 3u_x^2\, \sin u + 2u_{2x}\, \cos u] + \mathrm{D}_x^2[2u_x\, \cos u]$$

$$= 3u_x\, v^2\, \cos u - u_x^3\, \cos u + 6v\, v_x\, \sin u - 2u_x\, u_{2x}\, \sin u$$

$$-[3u_x v^2\, \cos u + 6v\, v_x\, \sin u - 3u_x^3\, \cos u - 6u\, u_{2x}\, \sin u$$

$$-2u_x\, u_{2x}\, \sin u + 2u_{3x}\, \cos u]$$

$$+[-2u_{3x} \cos u - 6u_x\, u_{2x} \sin u + 2u_{3x} \cos u] = 0$$

Similarly, $\mathcal{L}_{v(x)}^{(0)} f \equiv 0$

Inverting D_x and Div

Problem Statement

- **In 1D:**

  Example: For $u(x)$ and $v(x)$

  $$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x}$$

- Find $F = \int f \, dx$ so, $f = D_x F$

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- Result (by hand):

$$F = 4\, v_x^2 + u_x^2 \, \cos u - 3\, v^2 \, \cos u$$

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Mathematica cannot compute this integral!

- **In 2D:**

  Example: For $u(x, y)$ and $v(x, y)$

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ so, $f = \mathrm{Div}\,\mathbf{F}$

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- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ so, $f = \mathrm{Div}\, \mathbf{F}$
- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

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Can this be reduced to single integral in one variable?

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  Example: For $u(x,y)$ and $v(x,y)$

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- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Mathematica cannot do this!

Can this be done without integration by parts?

Can this be reduced to single integral in one variable?

Yes! With the Homotopy operator

Tools from Differential Geometry

Higher Euler Operators

- In 1D (variable x):

$$\mathcal{L}^{(i)}_{u^{(j)}(x)} = \sum_{k=i}^{M_x^{(j)}} \binom{k}{i} (-\mathrm{D}_x)^{k-i} \frac{\partial}{\partial u^{(j)}_{kx}}$$

Examples for component $u(x)$:

$$\mathcal{L}^{(1)}_{u(x)} = \frac{\partial}{\partial u_x} - 2\mathrm{D}_x \frac{\partial}{\partial u_{2x}} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x^3 \frac{\partial}{\partial u_{4x}} + \cdots$$

$$\mathcal{L}^{(2)}_{u(x)} = \frac{\partial}{\partial u_{2x}} - 3\mathrm{D}_x \frac{\partial}{\partial u_{3x}} + 6\mathrm{D}_x^2 \frac{\partial}{\partial u_{4x}} - 10\mathrm{D}_x^3 \frac{\partial}{\partial u_{5x}} + \cdots$$

$$\mathcal{L}^{(3)}_{u(x)} = \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x \frac{\partial}{\partial u_{4x}} + 10\mathrm{D}_x^2 \frac{\partial}{\partial u_{5x}} - 20\mathrm{D}_x^3 \frac{\partial}{\partial u_{6x}} + \cdots$$

- In 2D (variables x and y):

$$\mathcal{L}_{u^{(j)}(x,y)}^{(i_x,i_y)} = \sum_{k_x=i_x}^{M_x^{(j)}} \sum_{k_y=i_y}^{M_y^{(j)}} \binom{k_x}{i_x} \binom{k_y}{i_y} (-\mathrm{D}_x)^{k_x-i_x} (-\mathrm{D}_y)^{k_y-i_y} \frac{\partial}{\partial u_{k_x x\, k_y y}^{(j)}}$$

Examples for component $u(x,y)$:

$$\mathcal{L}_{u(x,y)}^{(1,0)} = \frac{\partial}{\partial u_x} - 2\mathrm{D}_x \frac{\partial}{\partial u_{2x}} - \mathrm{D}_y \frac{\partial}{\partial u_{xy}} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u_{3x}} + \cdots$$

$$\mathcal{L}_{u(x,y)}^{(0,1)} = \frac{\partial}{\partial u_y} - 2\mathrm{D}_y \frac{\partial}{\partial u_{2y}} - \mathrm{D}_x \frac{\partial}{\partial u_{yx}} + 3\mathrm{D}_y^2 \frac{\partial}{\partial u_{3y}} + \cdots$$

$$\mathcal{L}_{u(x,y)}^{(1,1)} = \frac{\partial}{\partial u_{xy}} - 2\mathrm{D}_x \frac{\partial}{\partial u_{2xy}} - 2\mathrm{D}_y \frac{\partial}{\partial u_{x2y}} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u_{3xy}} + \cdots$$

$$\mathcal{L}_{u(x,y)}^{(2,1)} = \frac{\partial}{\partial u_{2xy}} - 3\mathrm{D}_x \frac{\partial}{\partial u_{3xy}} - 2\mathrm{D}_y \frac{\partial}{\partial u_{2x2y}} + 6\mathrm{D}_x^2 \frac{\partial}{\partial u_{4xy}} + \cdots$$

- In 3D (variables x, y, and z):

$$\mathcal{L}^{(i_x,i_y,i_z)}_{u^{(j)}(x,y,z)} = \sum_{k_x=i_x}^{M_x^{(j)}} \sum_{k_y=i_y}^{M_y^{(j)}} \sum_{k_z=i_z}^{M_z^{(j)}} \binom{k_x}{i_x}\binom{k_y}{i_y}\binom{k_z}{i_z} (-\mathrm{D}_x)^{k_x-i_x}(-\mathrm{D}_y)^{k_y-i_y}(-\mathrm{D}_z)^{k_z-i_z} \frac{\partial}{\partial u^{(j)}_{k_x x\, k_y y\, k_z z}}$$

- Theorem:

[Kruskal et al, J. Math. Phys. **11** (1970) 952-960]

$f = D_x^r F$ iff $\mathcal{L}^{(i)}_{u^{(j)}(x)} f \equiv 0$ for $i = 0, 1, \ldots, r-1$, $j = 1, \ldots, N$

Using the Homotopy Operator

- Theorem (integration with homotopy operator):

- In 1D: If f is exact then

$$F = \mathrm{D}_x^{-1} f = \int f \, dx = \mathcal{H}_{\mathbf{u}(x)} f$$

- In 2D: If f is a divergence then

$$\mathbf{F} = \mathrm{Div}^{-1} f = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f)$$

- In 3D: If f is a divergence then

$$\mathbf{F} = \mathrm{Div}^{-1} f = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y,z)} f, \mathcal{H}^{(y)}_{\mathbf{u}(x,y,z)} f, \mathcal{H}^{(z)}_{\mathbf{u}(x,y,z)} f)$$

- Homotopy Operator in 1D (variable x):

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^{N} (I_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

with integrand

$$I_{u^{(j)}} f = \sum_{i=0}^{M_x^{(j)}-1} \mathrm{D}_x^i \left(u^{(j)} \, \mathcal{L}_{u^{(j)}(x)}^{(i+1)} f \right)$$

N is the number of dependent variables and $(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u}(x) \rightarrow \lambda \mathbf{u}(x)$, $\mathbf{u}_x(x) \rightarrow \lambda \mathbf{u}_x(x)$, *etc.*

Example: $\mathbf{u}(x) = (u^{(1)}(x), u^{(2)}(x)) = (u(x), v(x))$:

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 (I_u f + I_v f)\,[\lambda \mathbf{u}]\,\frac{d\lambda}{\lambda}$$

where

$$I_u f = \sum_{i=0}^{M_x^{(1)}-1} \mathrm{D}_x^i \left(u\, \mathcal{L}_{u(x)}^{(i+1)} f \right)$$

and

$$I_v f = \sum_{i=0}^{M_x^{(2)}-1} \mathrm{D}_x^i \left(v\, \mathcal{L}_{v(x)}^{(i+1)} f \right)$$

Simplified Formula in 1D (variable x)

$$\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^{N} (I_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

where

$$I_{u^{(j)}} f = \sum_{i=0}^{M_x^{(j)}-1} \mathrm{D}_x^i \left(u^{(j)} \, \mathcal{L}_{u^{(j)}(x)}^{(i+1)} f \right)$$

$$= \sum_{i=0}^{M_x^{(j)}-1} u_{ix}^{(j)} \sum_{k=i+1}^{M_x^{(j)}} (-\mathcal{D}_x)^{k-(i+1)} \, \frac{\partial f}{\partial u_{kx}^{(j)}}$$

Application of Homotopy Operator in 1D

Example:

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x}$$

- Compute

$$\begin{aligned} I_u f &= u \frac{\partial f}{\partial u_x} + u_x \frac{\partial f}{\partial u_{2x}} - u \mathrm{D}_x \frac{\partial f}{\partial u_{2x}} \\ &= 3uv^2 \sin u - u u_x^2 \sin u + 2u_x^2 \cos u \end{aligned}$$

- Similarly,

$$\begin{aligned} I_v f &= v\frac{\partial f}{\partial v_x} + v_x\frac{\partial f}{\partial v_{2x}} - v\mathrm{D}_x\frac{\partial f}{\partial v_{2x}} \\ &= -6v^2\cos u + 8v_x^2 \end{aligned}$$

- Finally,

$$F = \mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 (I_u f + I_v f)\,[\lambda\mathbf{u}]\,\frac{d\lambda}{\lambda}$$

$$= \int_0^1 \Big(3\lambda^2 u v^2 \sin(\lambda u) - \lambda^2 u u_x^2 \sin(\lambda u) + 2\lambda u_x^2 \cos(\lambda u)$$

$$-6\lambda v^2\cos(\lambda u) + 8\lambda v_x^2\Big)\,d\lambda$$

$$= 4v_x^2 + u_x^2\cos u - 3v^2\cos u$$

Why does this work?

Sketch of Derivation and Proof

(in 1D with variable x, and for one component u)

Definition: Degree operator $\mathcal{M}$

$$\mathcal{M}f = \sum_{i=0}^{M} u_{ix} \frac{\partial f}{\partial u_{ix}} = u\frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} + u_{2x}\frac{\partial f}{\partial u_{2x}} + \cdots + u_{Mx}\frac{\partial f}{\partial u_{Mx}}$$

f is of order M in x

Example: $f = u^p u_x^q u_{3x}^r$ (p, q, r non-negative integers)

$$g = \mathcal{M}f = \sum_{i=0}^{3} u_{ix}\frac{\partial f}{\partial u_{ix}} = (p+q+r)\, u^p u_x^q u_{3x}^r$$

Application of $\mathcal{M}$ computes the total *degree*

Theorem (inverse operator) $\mathcal{M}^{-1} g(u) = \int_0^1 g[\lambda u] \, \frac{d\lambda}{\lambda}$

Proof:

$$\frac{d}{d\lambda} g[\lambda u] = \sum_{i=0}^{M} \frac{\partial g[\lambda u]}{\partial \lambda u_{ix}} \frac{d\lambda u_{ix}}{d\lambda} = \frac{1}{\lambda} \sum_{i=0}^{M} u_{ix} \frac{\partial g[\lambda u]}{\partial u_{ix}} = \frac{1}{\lambda} \mathcal{M} g[\lambda u]$$

Integrate both sides with respect to λ

$$\begin{aligned} \int_0^1 \frac{d}{d\lambda} g[\lambda u] \, d\lambda &= g[\lambda u] \Big|_{\lambda=0}^{\lambda=1} = g(u) - g(0) \\ &= \int_0^1 \mathcal{M} g[\lambda u] \, \frac{d\lambda}{\lambda} = \mathcal{M} \int_0^1 g[\lambda u] \, \frac{d\lambda}{\lambda} \end{aligned}$$

Assuming $g(0) = 0$,

$$\mathcal{M}^{-1} g(u) = \int_0^1 g[\lambda u] \, \frac{d\lambda}{\lambda}$$

Example:

If $g(u) = (p+q+r)\, u^p u_x^q u_{3x}^r$, then

$g[\lambda u] = (p+q+r)\lambda^{p+q+r}\, u^p u_x^q u_{3x}^r$

Hence,

$$\begin{aligned}
\mathcal{M}^{-1} g &= \int_0^1 (p+q+r)\, \lambda^{p+q+r-1}\, u^p u_x^q u_{3x}^r \, d\lambda \\
&= u^p u_x^q u_{3x}^r \;\; \lambda^{p+q+r} \Big|_{\lambda=0}^{\lambda=1} = u^p u_x^q u_{3x}^r
\end{aligned}$$

Theorem: If f is an exact differential function, then

$$F = \mathcal{D}_x^{-1} f = \int f\, dx = \mathcal{H}_{u(x)} f$$

Proof: Multiply

$$\mathcal{L}_{u(x)}^{(0)} f = \sum_{k=0}^{M} (-\mathcal{D}_x)^k \frac{\partial f}{\partial u_{kx}}$$

by u to restore the degree.

Split off $u \frac{\partial f}{\partial u}$. Integrate by parts.

Split off $u_x \frac{\partial f}{\partial u_x}$. Repeat the process.

Lastly, split off $u_{Mx} \frac{\partial f}{\partial u_{Mx}}$.

Explicitly,

$$
\begin{aligned}
u\mathcal{L}^{(0)}_{u(x)} f &= u \sum_{k=0}^{M} (-\mathcal{D}_x)^k \frac{\partial f}{\partial u_{kx}} \\
&= u\frac{\partial f}{\partial u} - \mathcal{D}_x \left(u \sum_{k=1}^{M} (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \right) + u_x \sum_{k=1}^{M} (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \\
&= u\frac{\partial f}{\partial u} + u_x \frac{\partial f}{\partial u_x} - \mathcal{D}_x \left(u \sum_{k=1}^{M} (-\mathcal{D}_x)^{k-1} \frac{\partial f}{\partial u_{kx}} \right. \\
&\quad \left. + u_x \sum_{k=2}^{M} (-\mathcal{D}_x)^{k-2} \frac{\partial f}{\partial u_{kx}} \right) + u_{2x} \sum_{k=2}^{M} (-\mathcal{D}_x)^{k-2} \frac{\partial f}{\partial u_{kx}} \\
&= \ldots
\end{aligned}
$$

$$
\begin{aligned}
&= u\frac{\partial f}{\partial u} + u_x\frac{\partial f}{\partial u_x} + \ldots + u_{Mx}\frac{\partial f}{\partial u_{Mx}} \\
&\quad -\mathcal{D}_x\left(u\sum_{k=1}^{M}(-\mathcal{D}_x)^{k-1}\frac{\partial f}{\partial u_{kx}} + u_x\sum_{k=2}^{M}(-\mathcal{D}_x)^{k-2}\frac{\partial f}{\partial u_{kx}} \right. \\
&\quad \left. + \ldots + u_{(M-1)x}\sum_{k=M}^{M}(-\mathcal{D}_x)^{k-M}\frac{\partial f}{\partial u_{kx}} \right) \\
&= \sum_{i=0}^{M} u_{ix}\frac{\partial f}{\partial u_{ix}} - \mathcal{D}_x\left(\sum_{i=0}^{M-1} u_{ix}\sum_{k=i+1}^{M}(-\mathcal{D}_x)^{k-(i+1)}\frac{\partial f}{\partial u_{kx}}\right) \\
&= \mathcal{M}f - \mathcal{D}_x\left(\sum_{i=0}^{M-1} u_{ix}\sum_{k=i+1}^{M}(-\mathcal{D}_x)^{k-(i+1)}\frac{\partial f}{\partial u_{kx}}\right) \\
&= 0
\end{aligned}
$$

So,

$$\mathcal{M}f = \mathcal{D}_x \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right)$$

Apply $\mathcal{M}^{-1}$ and use $\mathcal{M}^{-1}\mathcal{D}_x = \mathcal{D}_x\mathcal{M}^{-1}$.

$$f = \mathcal{D}_x \left(\mathcal{M}^{-1} \sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right)$$

Apply $\mathcal{D}_x^{-1}$ and use the formula for $\mathcal{M}^{-1}$

$$\begin{aligned} F &= \mathcal{D}_x^{-1} f = \int_0^1 \left(\sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right) [\lambda u] \frac{d\lambda}{\lambda} \\ &= \mathcal{H}_{u(x)} f \end{aligned}$$

using

$$
\begin{aligned}
I_u f &= \sum_{i=0}^{M-1} \mathrm{D}_x^i \left(u\, \mathcal{L}_{u(x)}^{(i+1)} f \right) \\
&= \sum_{i=0}^{M-1} \mathcal{D}_x^i \left(u \sum_{k=i+1}^{M} \binom{k}{i+1} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}} \right) \\
&= \sum_{i=0}^{M-1} u_{ix} \sum_{k=i+1}^{M} (-\mathcal{D}_x)^{k-(i+1)} \frac{\partial f}{\partial u_{kx}}
\end{aligned}
$$

The latter formula provides a fast algorithm to compute the integrand $I_u f$

- Homotopy Operator in 2D (variables x and y):

$$\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(x)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

with

$$I^{(x)}_{u^{(j)}} f = \sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \left(\frac{1+i_x}{1+i_x+i_y} \right) \mathrm{D}_x^{i_x} \mathrm{D}_y^{i_y} \left(u^{(j)} \mathcal{L}^{(1+i_x, i_y)}_{u^{(j)}(x,y)} f \right)$$

Analogous formulas for $\mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f$ and $I^{(y)}_{u^{(j)}} f$

Simplified Formula in 2D (variables x and y)

$$\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(x)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

$$\mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(y)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

where

$$\mathcal{I}^{(x)}_{u^{(j)}} f = \sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \binom{i_x+i_y}{i_x} u_{i_x x \, i_y y} \sum_{k_x=i_x+1}^{M_x^{(j)}} \sum_{k_y=i_y}^{M_y^{(j)}} \frac{\binom{k_x+k_y-i_x-i_y-1}{k_x-i_x-1}}{\binom{k_x+k_y}{k_x}} \left(-\mathcal{D}_x\right)^{k_x-i_x-1} \left(-\mathcal{D}_y\right)^{k_y-i_y} \frac{\partial f}{\partial u^{(j)}_{k_x x \, k_y y}}$$

and

$$\mathcal{I}^{(y)}_{u^{(j)}} f \;=\; \sum_{i_x=0}^{M_x^{(j)}} \sum_{i_y=0}^{M_y^{(j)}-1} \binom{i_x+i_y}{i_y} u_{i_x x\, i_y y} \sum_{k_x=i_x}^{M_x^{(j)}} \sum_{k_y=i_y+1}^{M_y^{(j)}} \frac{\binom{k_x+k_y-i_x-i_y-1}{k_y-i_y-1}}{\binom{k_x+k_y}{k_y}} \left(-\mathcal{D}_x\right)^{k_x-i_x} \left(-\mathcal{D}_y\right)^{k_y-i_y-1} \frac{\partial f}{\partial u^{(j)}_{k_x x\, k_y y}}$$

Application of Homotopy Operator in 2D

Example: $f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$

Recall (by hand): $\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$

- Compute

$$
\begin{aligned}
I_u^{(x)} f &= u \frac{\partial f}{\partial u_x} + u_x \frac{\partial f}{\partial u_{2x}} - u \mathrm{D}_x \frac{\partial f}{\partial u_{2x}} \\
&\quad + \frac{1}{2} u_y \frac{\partial f}{\partial u_{xy}} - \frac{1}{2} u \mathrm{D}_y \frac{\partial f}{\partial u_{xy}} \\
&= u v_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy}
\end{aligned}
$$

- Similarly,

$$I_v^{(x)} f = v \frac{\partial f}{\partial v_x} = -u_y v + u_{xy} v$$

- Hence,

$$\begin{aligned} F_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \left(I_u^{(x)} f + I_v^{(x)} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \lambda \left(u v_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy} - u_y v + u_{xy} v \right) d\lambda \\ &= \frac{1}{2} u v_y + \frac{1}{4} u_y v_x - \frac{1}{2} u_x v_y + \frac{1}{4} u v_{xy} - \frac{1}{2} u_y v + \frac{1}{2} u_{xy} v \end{aligned}$$

- Analogously,

$$F_2 = \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f = \int_0^1 \left(I_u^{(y)} f + I_v^{(y)} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$= \int_0^1 \left(\lambda \left(-u v_x - \frac{1}{2} u v_{2x} + \frac{1}{2} u_x v_x \right) + \lambda \left(u_x v - u_{2x} v \right) \right) d\lambda$$

$$= -\frac{1}{2} u v_x - \frac{1}{4} u v_{2x} + \frac{1}{4} u_x v_x + \frac{1}{2} u_x v - \frac{1}{2} u_{2x} v$$

- So,

$$\mathbf{F} = \begin{pmatrix} \frac{1}{2} u v_y + \frac{1}{4} u_y v_x - \frac{1}{2} u_x v_y + \frac{1}{4} u v_{xy} - \frac{1}{2} u_y v + \frac{1}{2} u_{xy} v \\ -\frac{1}{2} u v_x - \frac{1}{4} u v_{2x} + \frac{1}{4} u_x v_x + \frac{1}{2} u_x v - \frac{1}{2} u_{2x} v \end{pmatrix}$$

Let $\mathbf{K} = \tilde{\mathbf{F}} - \mathbf{F}$ then

$$\mathbf{K} = \begin{pmatrix} \frac{1}{2}uv_y - \frac{1}{4}u_y v_x - \frac{1}{2}u_x v_y - \frac{1}{4}uv_{xy} + \frac{1}{2}u_y v - \frac{1}{2}u_{xy} v \\ -\frac{1}{2}uv_x + \frac{1}{4}uv_{2x} + \frac{3}{4}u_x v_x - \frac{1}{2}u_x v + \frac{1}{2}u_{2x} v \end{pmatrix}$$

then $\mathrm{Div}\,\mathbf{K} = 0$

- Also, $\mathbf{K} = (\mathrm{D}_y \theta, -\mathrm{D}_x \theta)$ with $\theta = \frac{1}{2}uv - \frac{1}{4}uv_x - \frac{1}{2}u_x v$

($curl$ in 2D)

Needed: Fast algorithm to remove curl terms!

- Homotopy Operator in 3D (variables x, y, and z):

$$\mathcal{H}^{(x)}_{\mathbf{u}(x,y,z)} f = \int_0^1 \sum_{j=1}^{N} (I^{(x)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

with

$$I^{(x)}_{u^{(j)}} f =$$

$$\sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \sum_{i_z=0}^{M_z^{(j)}} \left(\frac{1+i_x}{1+i_x+i_y+i_z} \right) \mathrm{D}_x^{i_x} \mathrm{D}_y^{i_y} \mathrm{D}_z^{i_z} \left(u^{(j)} \, \mathcal{L}^{(1+i_x, i_y, i_x)}_{u^{(j)}(x,y,z)} f \right)$$

Analogous formulas for $\mathcal{H}^{(y)}_{\mathbf{u}(x,y,z)} f$, $\mathcal{H}^{(z)}_{\mathbf{u}(x,y,z)} f$, $I^{(y)}_{u^{(j)}} f$, and $I^{(z)}_{u^{(j)}} f$

Simplified Formula in 3D (variables $x, y,$ and z)

$$\mathcal{H}^{(x)}_{\mathbf{u}(x,y,z)} f = \int_0^1 \sum_{j=1}^{N} (I^{(x)}_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$\mathcal{H}^{(y)}_{\mathbf{u}(x,y,z)} f = \int_0^1 \sum_{j=1}^{N} (I^{(y)}_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

$$\mathcal{H}^{(z)}_{\mathbf{u}(x,y,z)} f = \int_0^1 \sum_{j=1}^{N} (I^{(z)}_{u^{(j)}} f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$

where

$$
\begin{aligned}
& I^{(x)}_{u^{(j)}(x,y,z)} f \\
&= \sum_{i_x=0}^{M_x^{(j)}-1} \sum_{i_y=0}^{M_y^{(j)}} \sum_{i_z=0}^{M_z^{(j)}} \binom{i_x+i_y+i_z}{i_x} \binom{i_y+i_z}{i_y} u^{(j)}_{i_x x\, i_y y\, i_z z} \\
&\quad \sum_{k_x=i_x+1}^{M_x^{(j)}} \sum_{k_y=i_y}^{M_y^{(j)}} \sum_{k_z=i_z}^{M_z^{(j)}} \frac{\binom{k_x+k_y+k_z-i_x-i_y-i_z-1}{k_x-i_x-1} \binom{k_y+k_z-i_y-i_z}{k_y-i_y}}{\binom{k_x+k_y+k_z}{k_x} \binom{k_y+k_z}{k_y}} \\
&\quad (-\mathcal{D}_x)^{k_x-i_x-1} \left(-\mathcal{D}_y\right)^{k_y-i_y} (-\mathcal{D}_z)^{k_z-i_z} \frac{\partial f}{\partial u^{(j)}_{k_x x\, k_y y\, k_z z}}
\end{aligned}
$$

Integrands $I^{(y)}_{u^{(j)}(x,y,z)} f$ and $I^{(z)}_{u^{(j)}(x,y,z)} f$ are similar.

Computation of Conservation Laws for SWW

Quick Recapitulation

- Conservation law in $(2+1)$ dimensions

$$\boxed{\mathrm{D}_t \rho + \nabla \cdot \mathbf{J} = \mathrm{D}_t \rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 = 0} \quad \text{(on PDE)}$$

conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$

- Example: Shallow water wave (SWW) equations

$$u_t + u u_x + v u_y - 2\,\Omega v + \frac{1}{2} h \theta_x + \theta h_x = 0$$

$$v_t + u v_x + v v_y + 2\,\Omega u + \frac{1}{2} h \theta_y + \theta h_y = 0$$

$$\theta_t + u \theta_x + v \theta_y = 0$$

$$h_t + h u_x + u h_x + h v_y + v h_y = 0$$

- Typical density-flux pair:

$$\rho^{(5)} = v_x\theta - u_y\theta + 2\Omega\theta$$

$$\mathbf{J}^{(5)} = \frac{1}{2}\begin{pmatrix} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{pmatrix}$$

Algorithm

- **Step 1: Construct the form of the density**

  The SWW equations are invariant under the scaling symmetries

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda\theta, \lambda h, \lambda^{2}\Omega)$$

and

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda^{2}\theta, \lambda^{0}h, \lambda^{2}\Omega)$$

  Construct a candidate density, for example,

$$\rho = c_1\Omega\theta + c_2u_y\theta + c_3v_y\theta + c_4u_x\theta + c_5v_x\theta$$

  which is scaling invariant under *both* symmetries.

- **Step 2: Determine the constants c_i**

Compute $E = -\mathrm{D}_t \rho$ and remove time derivatives

$$\begin{aligned} E &= -(\frac{\partial \rho}{\partial u_x} u_{tx} + \frac{\partial \rho}{\partial u_y} u_{ty} + \frac{\partial \rho}{\partial v_x} v_{tx} + \frac{\partial \rho}{\partial v_y} v_{ty} + \frac{\partial \rho}{\partial \theta} \theta_t) \\ &= c_4 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_x \\ &\quad + c_2 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_y \\ &\quad + c_5 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_x \\ &\quad + c_3 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_y \\ &\quad + (c_1 \Omega + c_2 u_y + c_3 v_y + c_4 u_x + c_5 v_x)(u \theta_x + v \theta_y) \end{aligned}$$

Require that

$$\mathcal{L}^{(0,0)}_{u(x,y)} E = \mathcal{L}^{(0,0)}_{v(x,y)} E = \mathcal{L}^{(0,0)}_{\theta(x,y)} E = \mathcal{L}^{(0,0)}_{h(x,y)} E \equiv 0$$

- Solution: $c_1 = 2,\ c_2 = -1,\ c_3 = c_4 = 0,\ c_5 = 1$ gives

$$\boxed{\rho = 2\Omega\theta - u_y\theta + v_x\theta}$$

- **Step 3: Compute the flux $\mathbf{J}$**

$$\begin{aligned} E \; = \; & \theta(u_x v_x + u v_{2x} + v_x v_y + v v_{xy} + 2\Omega u_x \\ & + \tfrac{1}{2}\theta_x h_y - u_x u_y - u u_{xy} - u_y v_y - u_{2y} v \\ & + 2\Omega v_y - \tfrac{1}{2}\theta_y h_x) \\ & + 2\Omega u \theta_x + 2\Omega v \theta_y - u u_y \theta_x \\ & - u_y v \theta_y + u v_x \theta_x + v v_x \theta_y \end{aligned}$$

Apply the 2D homotopy operator:

$$\mathbf{J} = (J_1, J_2) = \mathrm{Div}^{-1} E = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E)$$

Compute

$$I_u^{(x)} E = u \frac{\partial E}{\partial u_x} + u_x \frac{\partial E}{\partial u_{2x}} - u \mathrm{D}_x \frac{\partial E}{\partial u_{2x}} + \frac{1}{2} u_y \frac{\partial E}{\partial u_{xy}} - \frac{1}{2} u \mathrm{D}_y \frac{\partial E}{\partial u_{xy}}$$
$$= u v_x \theta + 2\Omega u \theta + \frac{1}{2} u^2 \theta_y - u u_y \theta$$

Similarly, compute

$$I_v^{(x)} E = v v_y \theta + \frac{1}{2} v^2 \theta_y + u v_x \theta$$
$$I_\theta^{(x)} E = \frac{1}{2} \theta^2 h_y + 2\Omega u \theta - u u_y \theta + u v_x \theta$$
$$I_h^{(x)} E = -\frac{1}{2} \theta \theta_y h$$

Next,

$$
\begin{aligned}
J_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E \\
&= \int_0^1 \left(I^{(x)}_u E + I^{(x)}_v E + I^{(x)}_\theta E + I^{(x)}_h E \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
&= \int_0^1 \Bigg(4\lambda \Omega u \theta + \lambda^2 \bigg(3 u v_x \theta + \frac{1}{2} u^2 \theta_y - 2 u u_y \theta + v v_y \theta \\
&\qquad + \frac{1}{2} v^2 \theta_y + \frac{1}{2} \theta^2 h_y - \frac{1}{2} \theta \theta_y h \bigg) \Bigg) d\lambda \\
&= 2\Omega u \theta - \frac{2}{3} u u_y \theta + u v_x \theta + \frac{1}{3} v v_y \theta + \frac{1}{6} u^2 \theta_y \\
&\quad + \frac{1}{6} v^2 \theta_y - \frac{1}{6} h \theta \theta_y + \frac{1}{6} h_y \theta^2
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
J_2 &= \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E \\
&= 2\Omega v\theta + \frac{2}{3} v v_x \theta - v u_y \theta - \frac{1}{3} u u_x \theta - \frac{1}{6} u^2 \theta_x - \frac{1}{6} v^2 \theta_x \\
&\qquad + \frac{1}{6} h\theta\theta_x - \frac{1}{6} h_x \theta^2
\end{aligned}
$$

Hence,

$$
\mathbf{J} = \frac{1}{6} \begin{pmatrix} 12\Omega u\theta - 4uu_y\theta + 6uv_x\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y - h\theta\theta_y + h_y\theta^2 \\ 12\Omega v\theta + 4vv_x\theta - 6vu_y\theta - 2uu_x\theta - u^2\theta_x - v^2\theta_x + h\theta\theta_x - h_x\theta^2 \end{pmatrix}
$$

After removing the curl term

$$\tilde{\mathbf{J}}^{(5)} = \frac{1}{2} \left(\begin{array}{c} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{array} \right)$$

Needed: Fast algorithm to remove curl terms!

Conclusions and Future Work

- Scope and limitations of the homotopy operator.

  Integration by parts, D_x^{-1}, and Div^{-1}.

- Integration of non-exact expressions.

  Example: $f = u_x v + u v_x + u^2 u_{2x}$

  $\int f dx = uv + \int u^2 u_{2x}\, dx.$

- Integration of parametrized differential functions.

  Example: $f = a u_x v + b u v_x$

  $\int f dx = uv$ if $a = b$.

- Broader class of PDEs (other than those of evolution type).

- Full implementation in *Mathematica*.

Software packages in *Mathematica*

Codes are available via the Internet:

URL: http://www.mines.edu/fs_home/whereman/

and via anonymous FTP from mines.edu in directory:

pub/papers/math_cs_dept/software/

Publications

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249-285.

2. W. Hereman, Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions, Int. J. Quan. Chem. **106**(1), 278-299 (2006).

3. W. Hereman, B. Deconinck, and L. D. Poole, Continuous and discrete homotopy operators: A theoretical approach made concrete, Math. Comput. Simul. **74**(4-5), 352-360 (2007).