SYMBOLIC SOFTWARE FOR NONLINEAR PDEs: INTEGRABILITY, SYMMETRIES AND EXACT SOLUTIONS

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II. **FOUR SYMBOLIC PROGRAMS**

Example 1 – Macsyma
Lie-point Symmetries

- System of m differential equations of order k

$$\Delta^i(x, u^{(k)}) = 0, \quad i = 1, 2, ..., m$$

with p independent and q dependent variables

$$\begin{aligned} x &= (x_1, x_2, ..., x_p) \in \mathbb{R}^p \\ u &= (u^1, u^2, ..., u^q) \in \mathbb{R}^q \end{aligned}$$

- The group transformations have the form

$$\tilde{x} = \Lambda_{group}(x, u), \quad \tilde{u} = \Omega_{group}(x, u)$$

where the functions Λ_{group} and Ω_{group} are to be determined

- Look for the Lie algebra $\mathcal{L}$ realized by the vector field

$$\alpha = \sum_{i=1}^{p} \eta^i(x, u)\frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x, u)\frac{\partial}{\partial u^l}$$

Example 2 – Macsyma

Painlevé Integrability Test

Integrability of a PDE requires that the only **movable singularities** in its solution are **poles**

Definition: A simple equation or system has the *Painlevé Property* if its solution in the complex plane has no worse singularities than movable poles

Aim: Verify if the PDE satisfies the **necessary criteria** to have the *Painlevé Property*

The solution f expressed as a Laurent series,

$$f = g^{\alpha} \sum_{k=0}^{\infty} u_k g^k$$

should only have movable poles

Example 3 - Mathematica
Conserved Densities

- **Purpose**

  Compute polynomial-type conservation laws of single PDEs and systems of PDEs

  Conservation law:

  $$\rho_t + J_x = 0$$

  both $\rho(u, u_x, u_{2x}, \ldots, u_{nx})$ and $J(u, u_x, u_{2x}, \ldots, u_{nx})$

  Consequently

  $$P = \int_{-\infty}^{+\infty} \rho dx = \text{constant}$$

  provided J vanishes at infinity

  Compare with constants of motions in classical mechanics

- **Example**

Consider the KdV equation

$$u_t + uu_x + u_{3x} = 0$$

Conserved densities:

$$\begin{aligned}
\rho_1 &= u \\
\rho_2 &= u^2 \\
\rho_3 &= u^3 - 3u_x^2 \\
&\vdots \\
\rho_6 &= u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \\
&+ \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2 \\
&\vdots
\end{aligned}$$

Integrable equations have ∞ conservation laws

- **Algorithm and Implementation**

Consider the scaling (weights) of the KdV

$$u \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$$

Compute building blocks of ρ_3

(i) Start with building block u^3

Divide by u and differentiate twice $(u^2)_{2x}$

Produces the list of terms

$[u_x^2, u u_{2x}] \longrightarrow [u_x^2]$

Second list: remove terms that are total derivative
with respect to x or total derivative
up to terms earlier in the list

Divide by u^2 and differentiate twice $(u)_{4x}$

Produces the list: $[u_{4x}] \longrightarrow [\,]$

$[\,]$ is the empty list

Gather the terms:

$$\rho_3 = u^3 + c[1]u_x^2$$

where the constant c_1 must be determined

(ii) Compute $\dfrac{\partial \rho_3}{\partial t} = 3u^2 u_t + 2c_1 u_x u_{xt}$

Replace u_t by $-(uu_x + u_{xxx})$ and u_{xt} by $-(uu_x + u_{xxx})_x$

(iii) Integrate the result with respect to x

Carry out all integrations by parts

$$\frac{\partial \rho_3}{\partial t} = -[\frac{3}{4}u^4 + (c_1 - 3)uu_x^2 + 3u^2 u_{xx} - c_1 u_{xx}^2 + 2c_1 u_x u_{xxx}]_x$$
$$-(c_1 + 3)u_x^3$$

The last non-integrable term must vanish

Thus, $c_1 = -3$

Result:

$$\rho_3 = u^3 - 3u_x^2$$

(iv) Expression [...] yields

$$J_3 = \frac{3}{4}u^4 - 6uu_x^2 + 3u^2 u_{xx} + 3u_{xx}^2 - 6u_x u_{xxx}$$

Computer building blocks of ρ_6

(i) Start with u^6

Divide by u and differentiate twice

$(u^5)_{2x}$ produces the list of terms

$[u^3 u_x^2, u^4 u_{2x}] \longrightarrow [u^3 u_x^2]$

Next, divide u^6 by u^2, and compute $(u^4)_{4x}$

Corresponding list:

$[u_x^4, u u_x^2 u_{2x}, u^2 u_{2x}^2, u^2 u_x u_{3x}, u^3 u_{4x}] \longrightarrow [u_x^4, u^2 u_{2x}^2]$

Proceed with $(\frac{u^6}{u^3})_{6x} = (u^3)_{6x}, (\frac{u^6}{u^4})_{8x} = (u^2)_{8x}$

and $(\frac{u^6}{u^5})_{10x} = (u)_{10x}$

Obtain the lists:

$[u_{2x}^3, u_x u_{2x} u_{3x}, u u_{3x}^2, u_x^2 u_{4x}, u u_{2x} u_{4x}, u u_x u_{5x}, u^2 u_{6x}] \longrightarrow$
$[u_{2x}^3, u u_{3x}^2]$

$[u_{4x}^2, u_{3x} u_{5x}, u_{2x} u_{6x}, u_x u_{7x}, u u_{8x}] \longrightarrow [u_{4x}^2]$

and $[u_{10x}] \longrightarrow [\,]$

Gather the terms:

$$\rho_6 = u^6 + c_1 u^3 u_x^2 + c_2 u_x^4 + c_3 u^2 u_{2x}^2 + c_4 u_{2x}^3 + c_5 u u_{3x}^2 + c_6 u_{4x}^2$$

where the constants c_i must be determined

(ii) Compute $\frac{\partial}{\partial t}\rho_6$

Replace $u_t, u_{xt}, \ldots, u_{nx,t}$ by $-(uu_x + u_{xxx}), \ldots$

(iii) Integrate the result with respect to x

Carry out all integrations by parts

Require that non-integrabe part vanishes

Set to zero all the coefficients of the independent combinations involving powers of u and its derivatives with respect to x

Solve the linear system for unknowns $c_1, c_2, \ldots, c_6$

Result:

$$\begin{aligned} \rho_6 \ &= \ u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \\ &+ \ \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2 \end{aligned}$$

(iv) Flux J_6 can be computed by substituting the constants into the integrable part of ρ_6

- Further Examples

  * Conservation laws of generalized Schamel equation

$$n^2u_t + (n+1)(n+2)u^{\frac{2}{n}}u_x + u_{xxx} = 0$$

n positive integer

$$\begin{aligned}
\rho_1 &= u \\
\rho_2 &= u^2 \\
\rho_3 &= \frac{1}{2}u_x^2 - \frac{n^2}{2}u^{2+\frac{2}{n}}
\end{aligned}$$

no further conservation laws

* Conserved densities of modified vector derivative nonlinear Schrödinger equation

$$\frac{\partial \mathbf{B}_\perp}{\partial t} + \frac{\partial}{\partial x}(B_\perp^2 \mathbf{B}_\perp) + \alpha \mathbf{B}_{\perp 0}\mathbf{B}_{\perp 0} \cdot \frac{\partial \mathbf{B}_\perp}{\partial x} + \mathbf{e}_x \times \frac{\partial^2 \mathbf{B}_\perp}{\partial x^2} = 0$$

Replace vector equation by

$$\begin{aligned}
u_t + \big(u(u^2+v^2) + \beta u - v_x\big)_x = 0 \\
v_t + \big(v(u^2+v^2) + u_x\big)_x = 0
\end{aligned}$$

u and v denote the components of $\mathbf{B}_\perp$ parallel and perpendicular to $\mathbf{B}_{\perp 0}$ and $\beta = \alpha B_{\perp 0}^2$

The first 5 conserved densities are:

$$\rho_1 = u^2 + v^2$$

$$\rho_2 = \frac{1}{2}(u^2 + v^2)^2 - uv_x + u_x v + \beta u^2$$

$$\rho_3 = \frac{1}{4}(u^2 + v^2)^3 + \frac{1}{2}(u_x^2 + v_x^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4)$$

$$\rho_4 = \frac{1}{4}(u^2 + v^2)^4 - \frac{2}{5}(u_x v_{xx} - u_{xx} v_x) + \frac{4}{5}(uu_x + vv_x)^2$$

$$+ \frac{6}{5}(u^2 + v^2)(u_x^2 + v_x^2) - (u^2 + v^2)^2(uv_x - u_x v)$$

$$+ \frac{\beta}{5}(2u_x^2 - 4u^3 v_x + 2u^6 + 3u^4 v^2 - v^6) + \frac{\beta^2}{5}u^4$$

$$
\begin{aligned}
\rho_5 \;=\;& \frac{7}{16}(u^2+v^2)^5+\frac{1}{2}(u_{xx}^2+v_{xx}^2)\\
-\;& \frac{5}{2}(u^2+v^2)(u_xv_{xx}-u_{xx}v_x)+5(u^2+v^2)(uu_x+vv_x)^2\\
+\;& \frac{15}{4}(u^2+v^2)^2(u_x^2+v_x^2)^2-\frac{35}{16}(u^2+v^2)^3(uv_x-u_xv)\\
+\;& \frac{\beta}{8}(5u^8+10u^6v^2-10u^2v^6-5v^8+20u^2u_x^2\\
-\;& 12u^5v_x+60uv^4v_x-20v^2v_x^2)\\
+\;& \frac{\beta^2}{4}(u^6+v^6)
\end{aligned}
$$

Conserved Densities, Lax Pairs & Bäcklund Transformations

- Lax pairs by Ito (Reduce, 1985)
- Conserved densities by Ito & Kako (Reduce, 1985)
- Conserved densities in DELiA by Bocharov (Pascal, 1990)
- Lax pairs & Bäcklund transformations by Conte & Musette (AMP, C++, 1991-1993)
- Conserved densities by Gerdt (Reduce, 1993)
- Conserved densities by Roelofs and Sanders (Reduce, 1994)
- Conserved densities by Hereman, Verheest and Göktas (Mathematica, 1993-1995)

Example 4 – Macsyma/Mathematica
Solitons – Hirota's Method

- Hirota's Direct Method
  allows to construct soliton solutions of
  - nonlinear evolution equations
  - wave equations
  - coupled systems
- Test conditions for existence of soliton solutions
- Examples:
  - Korteweg-de Vries equation (KdV)

$$u_t + 6uu_x + u_{3x} = 0$$

  - Kadomtsev-Petviashvili equation (KP)

$$(u_t + 6uu_x + u_{3x})_x + 3u_{2y} = 0$$

  - Sawada-Kotera equation (SK)

$$u_t + 45u^2u_x + 15u_xu_{2x} + 15uu_{3x} + u_{5x} = 0$$

Hirota's Method

Korteweg-de Vries equation

$$u_t + 6uu_x + u_{3x} = 0$$

Substitute

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

Integrate with respect to x

$$f f_{xt} - f_x f_t + f f_{4x} - 4 f_x f_{3x} + 3 f_{2x}^2 = 0$$

Bilinear form

$$B(f \cdot f) \stackrel{\text{def}}{=} \left(D_x D_t + D_x^4\right)(f \cdot f) = 0$$

Introduce the bilinear operator

$$D_x^m\, D_t^n (f \cdot g) = (\partial x - \partial x')^m (\partial t - \partial t')^n f(x,t)\, g(x',t')|_{x'=x,t'=t}$$

Use the expansion

$$f = 1 + \sum_{n=1}^{\infty} \epsilon^n\, f_n$$

Substitute f into the bilinear equation

Collect powers in ϵ (book keeping parameter)

$$
\begin{aligned}
O(\epsilon^0) &: B(1\cdot 1) = 0 \\
O(\epsilon^1) &: B(1\cdot f_1 + f_1\cdot 1) = 0 \\
O(\epsilon^2) &: B(1\cdot f_2 + f_1\cdot f_1 + f_2\cdot 1) = 0 \\
O(\epsilon^3) &: B(1\cdot f_3 + f_1\cdot f_2 + f_2\cdot f_1 + f_3\cdot 1) = 0 \\
O(\epsilon^4) &: B(1\cdot f_4 + f_1\cdot f_3 + f_2\cdot f_2 + f_3\cdot f_1 + f_4\cdot 1) = 0 \\
O(\epsilon^n) &: B(\sum_{j=0}^{n} f_j\cdot f_{n-j}) = 0 \qquad \text{with } f_0 = 1
\end{aligned}
$$

Start with

$$f_1 = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp\left(k_i\, x - \omega_i\, t + \delta_i\right)$$

k_i, ω_i and δ_i are constants

Dispersion law

$$\omega_i = k_i^3 \qquad (i = 1, 2, ..., N)$$

If the original PDE admits a N-soliton solution

then the expansion will truncate at level $n = N$

Consider the case N=3

Terms generated by $B(f_1, f_1)$ determine

$$
\begin{aligned}
f_2 &= a_{12}\,\exp(\theta_1+\theta_2) + a_{13}\,\exp(\theta_1+\theta_3) + a_{23}\,\exp(\theta_2+\theta_3) \\
&= a_{12}\,\exp\left[(k_1+k_2)\,x - (\omega_1+\omega_2)\,t + (\delta_1+\delta_2)\right] \\
&+ a_{13}\,\exp\left[(k_1+k_3)\,x - (\omega_1+\omega_3)\,t + (\delta_1+\delta_3)\right] \\
&+ a_{23}\,\exp\left[(k_2+k_3)\,x - (\omega_2+\omega_3)\,t + (\delta_2+\delta_3)\right]
\end{aligned}
$$

Calculate the constants a_{12}, a_{13} and a_{23}

$$
a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \qquad i, j = 1, 2, 3
$$

Terms from $B(f_1 \cdot f_2 + f_2 \cdot f_1)$ determine

$$
\begin{aligned}
f_3 &= b_{123}\exp(\theta_1+\theta_2+\theta_3) \\
&= b_{123}\exp\left[(k_1+k_2+k_3)x-(\omega_1+\omega_2+\omega_3)t+(\delta_1+\delta_2+\delta_3)\right]
\end{aligned}
$$

with

$$
b_{123} = a_{12}\,a_{13}\,a_{23} = \frac{(k_1-k_2)^2\,(k_1-k_3)^2\,(k_2-k_3)^2}{(k_1+k_2)^2\,(k_1+k_3)^2\,(k_2+k_3)^2}
$$

Subsequently, $f_i = 0$ for $i > 3$

Set $\epsilon = 1$

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 \\ &+ a_{12}\exp(\theta_1+\theta_2) + a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) \\ &+ b_{123}\exp(\theta_1+\theta_2+\theta_3) \end{aligned}$$

Return to the original $u(x,t)$

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}$$

Single soliton solution

$$f = 1 + e^{\theta}\ , \qquad \theta = kx - \omega t + \delta$$

k, ω and δ are constants and $\omega = k^3$

Substituting f into

$$\begin{aligned} u(x,t) &= 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} \\ &= 2(\frac{f_{xx}f - f_x^2}{f^2}) \end{aligned}$$

Take $k = 2K$

$$u = 2K^2 \text{sech}^2 K(x - 4K^2 t + \delta)$$

Two-soliton solution

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}$$

$$\theta_i = k_i x - \omega_i t + \delta_i$$

with $\omega_i = k_i^3$, $(i = 1, 2)$ and $a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$

Select

$$\begin{aligned}
e^{\delta_i} &= \frac{c_i^2}{k_i} e^{k_i x - \omega_i t + \Delta_i} \\
\tilde{f} &= \frac{1}{4} f e^{-\frac{1}{2}(\tilde{\theta}_1 + \tilde{\theta}_2)} \\
\tilde{\theta}_i &= k_i x - \omega_i t + \Delta_i \\
c_i^2 &= \left(\frac{k_2 + k_1}{k_2 - k_1}\right) k_i
\end{aligned}$$

Return to $u(x, t)$

$$\begin{aligned}
u(x,t) &= \tilde{u}(x,t) = 2\frac{\partial^2 \ln \tilde{f}(x,t)}{\partial x^2} \\
&= \left(\frac{k_2^2 - k_1^2}{2}\right)\left(\frac{k_2^2 \text{cosech}^2 \frac{\tilde{\theta}_2}{2} + k_1^2 \text{sech}^2 \frac{\tilde{\theta}_1}{2}}{(k_2 \coth \frac{\tilde{\theta}_2}{2} - k_1 \tanh \frac{\tilde{\theta}_1}{2})^2}\right)
\end{aligned}$$

III. **PLANS FOR THE FUTURE**

Extension of Symbolic Software Packages (Macsyma/Mathematica)

- Lie symmetries of differential-difference equations
- Solver for systems of linear, homogeneous PDEs (Hereman)
- Painlevé test for systems of PDEs (Elmer, Göktaş & Coffey)
- Solitons via Hirota's method for bilinear equations (Zhuang)
- Simplification of Hirota's method (Hereman & Nuseir)
- Conservation laws of PDEs with variable coefficients (Göktaş)
- Lax pairs, special solutions, ...

New Software

- Wavelets (prototype/educational tool)
- Other methods for Differential Equations