Solving Nonlinear Wave Equations and Lattices with Mathematica

Willy Hereman

Department of Mathematical and Computer Sciences
Colorado School of Mines
Golden, Colorado, USA
http://www.mines.edu/fs_home/whereman/
whereman@mines.edu

Plenary Talk SANUM 2001
Department of Applied Mathematics
University of Stellenbosch
Matieland, South Africa

Wednesday, April 11 2001, 14:00

Collaborators: Ünal Göktaş (WRI), Doug Baldwin (CSM)
Ryan Martino, Joel Miller, Linda Hong (REU–NSF '99)
Steve Formenac, Andrew Menz (REU–NSF '00)

Research supported in part by NSF
Grants DMS-9732069, DMS-9912293 and CCR-9901929

OUTLINE

- Purpose & Motivation
- Typical Examples
- Algorithm for Tanh Solutions
- Algorithm for Sech Solutions
- Algorithm for Mixed Tanh-Sech Solutions
- Extension: Tanh Solutions for Differential-difference Equations (DDEs)
- Solving/Analyzing Systems of Algebraic Equations with Parameters
- Implementation Issues – Demo Mathematica Software Package
- Future Work

Purpose & Motivation

- **Develop** and implement various **methods** to find exact (closed form) solutions of nonlinear PDEs and DDEs: Lie symmetry methods, similarity methods, etc.
- Fully **automate** the tanh and sech methods to compute closed form solitary wave solutions of nonlinear (systems) of partial differential equations (PDEs) and differential-difference equations (DDEs or lattices).
- **Class** of nonlinear PDEs and DDEs solvable with the tanh/sech method includes famous evolution and wave equations.
  Typical examples: Korteweg-de Vries, Fisher and Boussinesq PDEs, Toda and Volterra lattices (DDEs).
- Solutions of tanh or sech type **model** solitary waves in fluid dynamics, plasmas, electrical circuits, optical fibers, bio-genetics, etc.
- **Benchmark** solutions for numerical PDE solvers.
- **Research aspect**: Design a high-quality application package for the computation of exact solitary wave solutions of large classes of nonlinear evolution and wave equations.
- **Educational aspect**: Software as course ware for courses in nonlinear PDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software. REU Projects.
- **Users**: scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.

Typical Examples of Single PDEs and Systems of PDEs

- The Korteweg-de Vries (KdV) equation:

$$u_t + \alpha u u_x + u_{3x} = 0.$$

Solitary wave solution:

$$u(x,t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2\left[\mathrm{c_1 x + c_2 t} + \Delta\right],$$

or, equivalently,

$$u(x,t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \operatorname{sech}^2\left[\mathrm{c_1 x + c_2 t} + \Delta\right].$$

- The modified Korteweg-de Vries (mKdV) equation:

$$u_t + \alpha u^2 u_x + u_{3x} = 0.$$

Solitary wave solution:

$$u(x,t) = \pm\sqrt{\frac{6}{\alpha}}\, c_1 \operatorname{sech}\left[\mathrm{c_1 x - c_1^3 t} + \Delta\right].$$

- Three-dimensional modified Korteweg-de Vries equation:

$$u_t + 6u^2 u_x + u_{xyz} = 0.$$

Solitary wave solution:

$$u(x,y,z,t) = \pm\sqrt{c_2 c_3}\, \operatorname{sech}\left[\mathrm{c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t} + \Delta\right].$$

- The combined KdV-mKdV equation:

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + \gamma u_{3x} = 0.$$

Real solitary wave solution:

$$u(x,t) = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{\gamma}{\beta}} c_1 \ \mathrm{sech}(\mathrm{c_1 x} + \frac{\mathrm{c_1}}{2\beta}(3\alpha^2 - 2\beta\gamma \mathrm{c_1^2})\mathrm{t} + \Delta).$$

Complex solutions:

$$u(x,t) = -\frac{\alpha}{2\beta} \pm i\sqrt{\frac{\gamma}{\beta}} c_1 \ \tanh(\mathrm{c_1 x} + \frac{\mathrm{c_1}}{2\beta}(3\alpha^2 + 4\beta\gamma \mathrm{c_1^2})\mathrm{t} + \Delta),$$

$$u(x,t) = -\frac{\alpha}{2\beta} + \frac{1}{2}\sqrt{\frac{\gamma}{\beta}} c_1 \left(\mathrm{sech}\xi \pm \mathrm{i}\tanh\xi \right),$$

and

$$u(x,t) = -\frac{\alpha}{2\beta} - \frac{1}{2}\sqrt{\frac{\gamma}{\beta}} c_1 \left(\mathrm{sech}\xi \mp \mathrm{i}\tanh\xi \right)$$

with $\xi = c_1 x + \frac{c_1}{2\beta}(3\alpha^2 + \beta\gamma c_1^2)t + \Delta$.

- The Fisher equation:

$$u_t - u_{xx} - u\,(1-u) = 0.$$

Solitary wave solution:

$$u(x,t) = \frac{1}{4} \pm \frac{1}{2}\tanh\xi + \frac{1}{4}\tanh^2\xi,$$

with

$$\xi = \pm\frac{1}{2\sqrt{6}}\, x \pm \frac{5}{12}\, t + \Delta.$$

- The generalized Kuramoto-Sivashinski equation:

$$u_t + uu_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0.$$

Solitary wave solutions

(ignoring symmetry $u \to -u, x \to -x, \sigma \to -\sigma$) :

For $\sigma = 4$:

$$u(x,t) = 9 - 2c_2 - 15\,\tanh\xi\ (1 + \ \tanh\xi - \ \tanh^2\xi)$$

with $\xi = \frac{x}{2} + c_2 t + \Delta$.

For $\sigma = \frac{12}{\sqrt{47}}$:

$$u(x,t) = \frac{45 \mp 4418c_2}{47\sqrt{47}} \pm \frac{45}{47\sqrt{47}}\,\tanh\xi - \frac{45}{47\sqrt{47}}\,\tanh^2\xi \pm \frac{15}{47\sqrt{47}}\,\tanh^3\xi$$

with $\xi = \pm\frac{1}{2\sqrt{47}}\ x + c_2 t + \Delta$.

For $\sigma = 16/\sqrt{73}$:

$$u(x,t) = \frac{2\,(30 \mp 5329c_2)}{73\sqrt{73}} \pm \frac{75}{73\sqrt{73}}\,\tanh\xi - \frac{60}{73\sqrt{73}}\,\tanh^2\xi \pm \frac{15}{73\sqrt{73}}\,\tanh^3\xi$$

with $\xi = \pm\frac{1}{2\sqrt{73}}\ x + c_2 t + \Delta$.

For $\sigma = 0$:

$$u(x,t) = -2\sqrt{\frac{19}{11}}c_2 - \frac{135}{19}\sqrt{\frac{11}{19}}\,\tanh\xi + \frac{165}{19}\sqrt{\frac{11}{19}}\,\tanh^3\xi$$

with $\xi = \frac{1}{2}\sqrt{\frac{11}{19}}\ x + c_2 t + \Delta$.

- The Boussinesq (wave) equation:

$$u_{tt} - u_{2x} + 3uu_{2x} + 3{u_x}^2 + \alpha u_{4x} = 0,$$

or written as a first-order system (v auxiliary variable):

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0. \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \Delta\right], \\ v(x,t) &= b_0 + 4\alpha c_1 c_2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \Delta\right]. \end{aligned}$$

- The Broer-Kaup system:

$$\begin{aligned} u_{ty} + 2(uu_x)_y + 2v_{xx} - u_{xxy} &= 0, \\ v_t + 2(uv)_x + v_{xx} &= 0. \end{aligned}$$

Solitary wave solution:

$$\begin{aligned} u(x,t) &= -\frac{c_3}{2c_1} + c_1 \tanh\left[\mathrm{c_1 x + c_2 y + c_3 t} + \Delta\right], \\ v(x,t) &= c_1 c_2 - c_1 c_2 \tanh^2\left[\mathrm{c_1 x + c_2 y + c_3 t} + \Delta\right]. \end{aligned}$$

- System of three nonlinearly coupled equations (Gao & Tian, 2001):

$$
\begin{aligned}
u_t - u_x - 2v &= 0, \\
v_t + 2uw &= 0, \\
w_t + 2uv &= 0.
\end{aligned}
$$

Solutions:

$$
\begin{aligned}
u(x,t) &= \pm c_2 \tanh\xi, \\
v(x,t) &= \mp\frac{1}{2}c_2(c_1 - c_2)\,\mathrm{sech}^2\xi, \\
w(x,t) &= -\frac{1}{2}c_2(c_1 - c_2)\,\mathrm{sech}^2\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
u(x,t) &= \pm i c_2\,\mathrm{sech}\xi, \\
v(x,t) &= \pm\frac{1}{2}ic_2(c_1 - c_2)\,\tanh\xi\,\mathrm{sech}\xi, \\
w(x,t) &= \frac{1}{4}c_2(c_1 - c_2)\left(1 - 2\,\mathrm{sech}^2\xi\right),
\end{aligned}
$$

and also

$$
\begin{aligned}
u(x,t) &= \pm\frac{1}{2}ic_2\left(\mathrm{sech}\xi + \mathrm{i}\tanh\xi\right), \\
v(x,t) &= \pm\frac{1}{4}c_2(c_1 - c_2)\,\mathrm{sech}\xi\left(\mathrm{sech}\xi + \mathrm{i}\tanh\xi\right), \\
w(x,t) &= -\frac{1}{4}c_2(c_1 - c_2)\,\mathrm{sech}\xi\left(\mathrm{sech}\xi + \mathrm{i}\tanh\xi\right)
\end{aligned}
$$

with $\xi = c_1 x + c_2 t + \Delta$.

- Nonlinear sine-Gordon equation (light cone coordinates):

$$\Phi_{xt} = \sin\Phi.$$

Set $u = \Phi_x, \quad v = \cos(\Phi) - 1$,

$$\begin{aligned} u_{xt} - u - u\,v &= 0, \\ u_t^2 + 2v + v^2 &= 0. \end{aligned}$$

Solitary wave solution (kink):

$$\begin{aligned} u &= \pm\frac{1}{\sqrt{-c}}\,\mathrm{sech}[\frac{1}{\sqrt{-\mathrm{c}}}(\mathrm{x} - \mathrm{ct}) + \Delta], \\ v &= 1 - 2\,\mathrm{sech}^2[\frac{1}{\sqrt{-\mathrm{c}}}(\mathrm{x} - \mathrm{ct}) + \Delta]. \end{aligned}$$

Solution:

$$\Phi(x,t) = \int u(x,t)dx = \pm 4\ \arctan\left[\exp\left(\frac{1}{\sqrt{-c}}(x - ct) + \Delta\right)\right].$$

- ODEs from quantum field theory:

$$\begin{aligned} u_{xx} &= -u + u^3 + auv^2, \\ v_{xx} &= bv + cv^3 + av(u^2 - 1). \end{aligned}$$

Solitary wave solutions:

$$\begin{aligned} u &= \pm\tanh[\sqrt{\frac{\mathrm{a}^2 - \mathrm{c}}{2(\mathrm{a} - \mathrm{c})}}\mathrm{x} + \Delta], \\ v &= \pm\sqrt{\frac{1-a}{a-c}}\,\mathrm{sech}[\sqrt{\frac{\mathrm{a}^2 - \mathrm{c}}{2(\mathrm{a} - \mathrm{c})}}\mathrm{x} + \Delta], \end{aligned}$$

provided $b = \sqrt{\frac{a^2-c}{2(a-c)}}$.

Typical Examples of DDEs (lattices)

- The Toda lattice:

$$\ddot{u}_n = (1 + \dot{u}_n)\,(u_{n-1} - 2u_n + u_{n+1})\,.$$

Solitary wave solution:

$$u_n(t) = a_0 \pm \sinh(c_1)\tanh\left[\mathrm{c_1 n \pm \sinh(c_1)\, t + \Delta}\right].$$

- The Volterra lattice:

$$\begin{aligned}\dot{u}_n &= u_n(v_n - v_{n-1}),\\ \dot{v}_n &= v_n(u_{n+1} - u_n).\end{aligned}$$

Solitary wave solution:

$$\begin{aligned}u_n(t) &= -c_2\ \coth(c_1) + c_2 \tanh\left[\mathrm{c_1 n + c_2 t + \Delta}\right],\\ v_n(t) &= -c_2\ \coth(c_1) - c_2 \tanh\left[\mathrm{c_1 n + c_2 t + \Delta}\right].\end{aligned}$$

- The Relativistic Toda lattice:

$$\begin{aligned}\dot{u}_n &= (1 + \alpha u_n)(v_n - v_{n-1}),\\ \dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).\end{aligned}$$

Solitary wave solution:

$$\begin{aligned}u_n(t) &= -c_2\ \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh\left[\mathrm{c_1 n + c_2 t + \Delta}\right],\\ v_n(t) &= \frac{c_2\ \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh\left[\mathrm{c_1 n + c_2 t + \Delta}\right].\end{aligned}$$

Algorithm for Tanh Solutions for system of PDEs

Given: System of nonlinear PDEs of order n

$$\Delta(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \cdots \mathbf{u}^{(n)}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable $\mathbf{u}$ has M components u_i (or $u, v, w, ...$).
Independent variable $\mathbf{x}$ has N components x_j (or $x, y, z, ..., t$).

Step T1:

- Seek solution $\mathbf{u}(\mathbf{x}) = \mathbf{U}(T)$, with

$$T = \tanh \xi = \tanh \left[\sum_{\mathrm{j}}^{\mathrm{N}} \mathrm{c_j x_j} + \Delta \right].$$

- Observe $\tanh' \xi = 1 - \tanh^2 \xi$ or $T' = 1 - T^2$. Hence, *all* derivative of T are polynomial in T. For example, $T'' = -2T(1 - T^2)$, etc.

- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{dT} \frac{\partial T}{\partial x_j} = c_j (1 - T^2) \frac{d \bullet}{dT}.$$

  Produces a nonlinear system of ODEs

$$\boldsymbol{\Delta}(T, \mathbf{U}(T), \mathbf{U}'(T), \mathbf{U}''(T), \ldots, \mathbf{U}^{(m)}(T)) = \mathbf{0}.$$

  Note: Compare with the ultraspherical (linear) ODE:

$$(1 - x^2) y''(x) - (2\alpha + 1) x y'(x) + n(n + 2\alpha) y(x) = 0$$

  with integer $n \geq 0$ and α real. Includes:

  * Legendre equation ($\alpha = \frac{1}{2}$),
  * ODE for Chebeyshev polynomials of type I ($\alpha = 0$),
  * ODE for Chebeyshev polynomials of type II ($\alpha = 1$).

- Example: For the Boussinesq system

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0, \end{aligned}$$

after cancelling common factors $1 - T^2$,

$$\begin{aligned} &c_2 U' + c_1 V' = 0, \\ &c_2 V' + c_1 U' - 3c_1 UU' \\ &\quad + \alpha c_1^3 \left[2(1-3T^2)U' + 6T(1-T^2)U'' - (1-T^2)^2 U''' \right] = 0. \end{aligned}$$

Step T2:

- Seek polynomial solutions

$$U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j.$$

Determine the highest exponents $M_i \geq 1$.

Substitute $U_i(T) = T^{M_i}$ into the LHS of ODE.

Gives polynomial $\mathbf{P}(T)$.

For every P_i consider all possible balances of the highest exponents in T.

Solve the resulting linear system(s) for the unknowns M_i.

- Example: Balance highest exponents for the Boussinesq system

$$M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1.$$

So, $M_1 = M_2 = 2$.

Hence,

$$\begin{aligned} U(T) &= a_{10} + a_{11} T + a_{12} T^2, \\ V(T) &= a_{20} + a_{21} T + a_{22} T^2. \end{aligned}$$

Step T3:

- Derive algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the power terms in T.
- Example: Algebraic system for Boussinesq case

$$
\begin{aligned}
&a_{11}\, c_1\, (3a_{12} + 2\alpha\, c_1^2) = 0,\\
&a_{12}\, c_1\, (a_{12} + 4\alpha\, c_1^2) = 0,\\
&a_{21}\, c_1 + a_{11}\, c_2 = 0,\\
&a_{22}\, c_1 + a_{12}\, c_2 = 0,\\
&a_{11}\, c_1 - 3a_{10}\, a_{11}\, c_1 + 2\alpha a_{11}\, c_1^3 + a_{21}\, c_2 = 0,\\
&-3a_{11}^2\, c_1 + 2\, a_{12}\, c_1 - 6a_{10}\, a_{12}\, c_1 + 16\alpha\, a_{12}\, c_1^3 + 2a_{22}\, c_2 = 0.
\end{aligned}
$$

Step T4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for Boussinesq system

$$
\begin{aligned}
a_{10} &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2}, \quad a_{11} = 0,\\
a_{12} &= -4\alpha c_1^2, \quad a_{20} = \text{free},\\
a_{21} &= 0, \quad a_{22} = 4\alpha c_1 c_2.
\end{aligned}
$$

Step T5:

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.
- Example: Solitary wave solution for Boussinesq system:

$$
\begin{aligned}
u(x,t) &= \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \Delta\right],\\
v(x,t) &= a_{20} + 4\alpha c_1 c_2 \tanh^2\left[\mathrm{c_1 x + c_2 t} + \Delta\right].
\end{aligned}
$$

Algorithm for Sech Solutions for system of PDEs

Given: System of PDEs of order n

$$\Delta(\mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}), \mathbf{u}''(\mathbf{x}), \cdots \mathbf{u}^{(n)}(\mathbf{x})) = \mathbf{0}.$$

Dependent variable $\mathbf{u}$ has M components u_i (or $u, v, w, ...$).
Independent variable $\mathbf{x}$ has N components x_j (or $x, y, z, ..., t$).

Step S1:

- Seek solution $u_i(\mathbf{x}) = U_i(S)$, with

$$S = \operatorname{sech}\xi = \operatorname{sech}\left[\sum_{\mathrm{j}}^{\mathrm{N}} \mathrm{c_j x_j} + \Delta\right].$$

- Observe $(\operatorname{sech}\xi)' = -\tanh\xi \operatorname{sech}\xi$ or $S' = -TS = -\sqrt{1-S^2}\, S$.

- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d\bullet}{dS}\frac{\partial S}{\partial x_j} = -c_j S\sqrt{1-S^2}\frac{d\bullet}{dS}.$$

  Leads to coupled system of nonlinear ODEs

$$\mathbf{\Gamma}(S, \mathbf{U}(S), \mathbf{U}'(S), \ldots) + \sqrt{1-S^2}\, \mathbf{\Pi}(S, \mathbf{U}(S), \mathbf{U}'(S), \ldots) = \mathbf{0}.$$

  All components of $\mathbf{\Gamma}$ and $\mathbf{\Pi}$ are polynomial ODEs.

  First case: $\mathbf{\Gamma} = \mathbf{0}$ or $\mathbf{\Pi} = \mathbf{0}$.

$$\mathbf{\Delta}(S, \mathbf{U}(S), \mathbf{U}'(S), \ldots) = \mathbf{0}.$$

  $\mathbf{\Delta}$ stands for either $\mathbf{\Gamma}$ or $\mathbf{\Pi}$.

  Note: All terms in the given system of PDE must be of even or odd order.

- Example: For the 3D mKdV equation

$$u_t + 6u^2 u_x + u_{xyz} = 0,$$

after cancelling a common factor $-\sqrt{1-S^2}\,S$,

$$c_4 U' + 6c_1 U^2 U' + c_1 c_2 c_3 [(1-6S^2)U' + 3S(1-2S^2)U'' + S^2(1-S^2)U'''] = 0.$$

Step S2:

- Seek polynomial solutions

$$U_i(S) = \sum_{j=0}^{M_i} a_{ij} S^j.$$

Substitute $U_i(S) = S^{M_i}$ and balance the highest power terms in S to determine M_i.

- Example: Balance of exponents for the 3D mKdV case

$$3M_1 - 1 = M_1 + 1.$$

So, $M_1 = 1$. Hence,

$$U(S) = a_{10} + a_{11} S.$$

Step S3:

- Derive algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the power terms in S.

- Example: Algebraic system for 3D mKdV case

$$\begin{aligned} &a_{11} c_1 \, (a_{11}^2 - c_2 \, c_3) = 0, \\ &a_{11} \, (6a_{10}^2 \, c_1 + c_1 \, c_2 \, c_3 + c_4) = 0, \\ &a_{10} \, a_{11}^2 \, c_1 = 0. \end{aligned}$$

Step S4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for 3D mKdV case

$$\begin{aligned} a_{10} &= 0, \\ a_{11} &= \pm\sqrt{c_1\, c_3}, \\ c_4 &= -c_1\, c_2\, c_3. \end{aligned}$$

Step S5:

- Return to the original variables. Test the final solution(s). Reject trivial solutions.
- Example: Solitary wave solution for the 3D mKdV equation

$$u(x,y,z,t) = \pm\sqrt{c_2\, c_3}\,\mathrm{sech}(\mathrm{c_1\, x + c_2\, y + c_3\, z - c_1 c_2 c_3\, t}).$$

Second case: $\mathbf{\Gamma} \neq \mathbf{0}$ and $\mathbf{\Pi} \neq \mathbf{0}$.

$$\mathbf{\Gamma}(S, \mathbf{U}(S), \mathbf{U}'(S), \ldots) + \sqrt{1-S^2}\,\mathbf{\Pi}(S, \mathbf{U}(S), \mathbf{U}'(S), \ldots) = \mathbf{0}.$$

Most general solution

$$U_i(S) = \sum_{j=0}^{\tilde{M}_i} \sum_{k=0}^{\tilde{N}_i} \tilde{a}_{i,jk} S^j T^k.$$

Double series is not necessary! Solution can be rearranged as

$$U_i(S) = \sum_{j=0}^{M_i} a_{ij} S^j + T \sum_{j=0}^{N_i} b_{ij} S^j.$$

Algorithm for Mixed Tanh/Sech Solutions for PDEs

Step ST1:

- Seek solution in $u_i(\mathbf{x}) = U_i(S)$, with

$$S = \text{ sech}\xi = \text{ sech}\left[\sum_{\rm j}^{\rm N} \rm c_j x_j + \Delta\right].$$

Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d\bullet}{dS}\frac{\partial S}{\partial x_j} = -c_j S\sqrt{1-S^2}\frac{d\bullet}{dS}.$$

- Example: Coupled system due to Gao and Tian (2001)

$$\begin{aligned} u_t - u_x - 2v &= 0, \\ v_t + 2uw &= 0, \\ w_t + 2uv &= 0, \end{aligned}$$

transforms into

$$(c_1 - c_2)S\sqrt{1-S^2}U' - 2V = 0,$$

$$\begin{aligned} c_2 S\sqrt{1-S^2}V' - 2UW &= 0, \\ c_2 S\sqrt{1-S^2}W' - 2UV &= 0. \end{aligned}$$

Step ST2:

- Seek solution

$$U_i(S) = \sum_{j=0}^{M_i} a_{ij}S^j + \sqrt{1-S^2}\ \sum_{j=0}^{N_i} b_{ij}S^j.$$

First, determine the leading exponents M_i, N_i. Substitute

$$U_i(S) = a_{i0} + a_{i\,M_i}S^{M_i} + \sqrt{1-S^2}\,(b_{i0} + b_{i\,N_i}\,S^{N_i})$$

to get

$$\mathbf{P}(S) + \sqrt{1-S^2}\,\mathbf{Q}(S) = \mathbf{0}.$$

P and **Q** are polynomials.

Consider possible balances of the highest exponents in P_i and Q_i.

Get a linear system of $2M$ (or less) equations for the $2M$ unknown M_i and N_i.

No longer assume $M_i \geq 1, N_i \geq 1$ (some M_i or N_i may be zero).

Trouble. Strongly underdetermined problem. Set *all* $M_i = 2$ and $N_i = 1$.

- Example: Quadratic solutions in S and T only.

  Substitute

$$\begin{aligned} U(S) &= a_{10} + a_{11}S + a_{12}S^2 + \sqrt{1-S^2}\,(b_{10} + b_{11}S), \\ V(S) &= a_{20} + a_{21}S + a_{22}S^2 + \sqrt{1-S^2}\,(b_{20} + b_{21}S), \\ W(S) &= a_{30} + a_{31}S + a_{32}S^2 + \sqrt{1-S^2}\,(b_{30} + b_{31}S). \end{aligned}$$

  leads to

$$\mathbf{P}(S) + \sqrt{1-S^2}\,\mathbf{Q}(S) = \mathbf{0},$$

  P and **Q** are polynomials.

Step ST3:

- Derive the algebraic system for the coefficients a_{ij}, b_{ij} by setting to zero the coefficients of power terms in S in $\mathbf{P} = \mathbf{0}$ and $\mathbf{Q} = \mathbf{0}$ separately.

- Example: Algebraic system has 25 equations (not shown).

Step ST4:

- Solve the nonlinear algebraic system with parameters.
- Example: 11 solutions in total: 3 are trivial (U_i = constant), 8 are nontrivial.

Step ST5:

- Return to the original variables. Test the final solution(s). Reject trivial (constant) solutions.
- Example: Solitary wave solutions:

$$\begin{aligned} u(x,t) &= \pm c_2 \tanh\xi, \\ v(x,t) &= \mp\frac{1}{2}c_2(c_1-c_2)\,\text{sech}^2\xi, \\ w(x,t) &= -\frac{1}{2}c_2(c_1-c_2)\,\text{sech}^2\xi, \end{aligned}$$

(could have been obtained with tanh-method), and

$$\begin{aligned} u(x,t) &= \pm i c_2\,\text{sech}\xi, \\ v(x,t) &= \pm\frac{1}{2}i c_2(c_1-c_2)\,\tanh\xi\,\text{sech}\xi, \\ w(x,t) &= \frac{1}{4}c_2(c_1-c_2)\left(1-2\,\text{sech}^2\xi\right), \end{aligned}$$

and also

$$\begin{aligned} u(x,t) &= \pm\frac{1}{2}i c_2\left(\text{sech}\xi+\mathrm{i}\tanh\xi\right), \\ v(x,t) &= \pm\frac{1}{4}c_2(c_1-c_2)\,\text{sech}\xi\left(\text{sech}\xi+\mathrm{i}\tanh\xi\right), \\ w(x,t) &= -\frac{1}{4}c_2(c_1-c_2)\,\text{sech}\xi\left(\text{sech}\xi+\mathrm{i}\tanh\xi\right). \end{aligned}$$

plus the c.c. solutions.

In all solutions $\xi = c_1x + c_2t + \Delta$.

Algorithm for Tanh Solutions for system of DDEs

Given: System of nonlinear differential-difference equations (DDEs) of order m

$$\Delta(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ..., \dot{\mathbf{u}}_n, ..., \mathbf{u}_n^{(m)}) = 0.$$

Dependent variable $\mathbf{u}_n$ has M components $u_{i,n}$ (or $u_n, v_n, w_n, ...$)

Independent variable $\mathbf{x}$ has 2 components x_i (or n, t).

No derivatives on shifted variables!

Step D1:

- Seek solution $\mathbf{u}_n(t) = \mathbf{U}_n(T)$, with

$$T = T_n(t) = \tanh\left[\mathrm{c_1 n + c_2 t} + \Delta\right].$$

- Note: The argument of T depends on n.

- Repeatedly apply the operator rule

$$\frac{d\bullet}{dt} = \frac{d\bullet}{dT}\frac{dT}{dt} = c_2(1-T^2)\frac{d\bullet}{dT}.$$

  Produces a nonlinear system of type

$$\boldsymbol{\Delta}(T, \cdots, \mathbf{U}_{n-1}, \mathbf{U}_n, \mathbf{U}_{n+1}, \cdots, \mathbf{U}_n', \mathbf{U}_n'', \cdots, \mathbf{U}_n^{(m)}) = \mathbf{0}.$$

- Example: Toda lattice

$$\ddot{u}_n = (1+\dot{u}_n)(u_{n-1} - 2u_n + u_{n+1})$$

  transforms into

$$c_2^2(1-T^2)\left[2TU_n' - (1-T^2)U_n''\right] + \left[1 + c_2(1-T^2)U_n'\right]\left[U_{n-1} - 2U_n + U_{n+1}\right] = 0.$$

Step D2:

- Seek polynomial solutions

$$U_{i,n}(T_n) = \sum_{j=0}^{M_i} a_{ij} T_n^j.$$

Use

$$\tanh(\mathrm{x}+\mathrm{y}) = \frac{\tanh \mathrm{x} + \tanh \mathrm{y}}{1 + \tanh \mathrm{x} \tanh \mathrm{y}}$$

to deal with the shift:

$$U_{i,n\pm p}\left(T(n \pm p)\right) = \sum_{j=0}^{M_i} a_{i,j} \left[T(n+p)\right]^j = \sum_{j=0}^{M_i} a_{i,j} \left[\frac{T_n \pm \tanh(\mathrm{pc}_1)}{1 \pm T_n \tanh(\mathrm{pc}_1)}\right]^j.$$

Substitute $U_{i,n} = T_n^{M_i}$, and

$$U_{i,n\pm p}\left(T(n \pm p)\right) = \left[T(n+p)\right]^{M_i} = \left[\frac{T_n \pm \tanh(\mathrm{pc}_1)}{1 \pm T_n \tanh(\mathrm{pc}_1)}\right]^{M_i},$$

and balance the potential highest exponents in T_n to determine M_i.

Note: $U_{i,n\pm p}\left(T(n \pm p)\right)$ is homogeneous of degree zero in T.

- Example: Balance of exponents for Toda lattice

$$2M_1 - 1 = M_1 + 1.$$

So, $M_1 = 1$.

Hence,

$$U_n(T_n) = a_{10} + a_{11} T_n,$$

$$U_{n\pm1}(T(n \pm 1)) = a_{10} + a_{11} T(n \pm 1) = a_{10} + a_{11} \frac{T_n \pm \tanh(\mathrm{c}_1)}{1 \pm T_n \tanh(\mathrm{c}_1)}.$$

Step D3:

- Determine the algebraic system for the unknown coefficients a_{ij} by setting to zero the coefficients of the powers in T_n.
- Example: Algebraic system for Toda lattice

$$
\begin{aligned}
&c_2^2 - \tanh^2(c_1) - a_{11} c_2 \tanh^2(c_1) = 0, \\
&c_2 - a_{11} = 0.
\end{aligned}
$$

Step D4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution of algebraic system for Toda lattice

$$
\begin{aligned}
a_{10} &= \text{free}, \\
a_{11} &= \pm \sinh(c_1), \\
c_2 &= \pm \sinh(c_1).
\end{aligned}
$$

Step D5:

- Return to the original variables. Test solution(s) of DDE. Reject trivial ones.
- Example: Solitary wave solution for Toda lattice:

$$
u_n(t) = a_0 \pm \sinh(c_1) \tanh\left[c_1 n \pm \sinh(c_1)\, t + \Delta\right].
$$

Example: System of DDEs: Relativistic Toda lattice

$$\begin{aligned}\dot{u}_n &= (1+\alpha u_n)(v_n - v_{n-1}),\\ \dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).\end{aligned}$$

Change of variables

$$u_n(t) = U_n(T_n), \quad v_n(t) = V_n(T_n),$$

with

$$T_n(t) = \tanh\left[\mathrm{c_1 n + c_2 t} + \Delta\right].$$

gives

$$\begin{aligned}&c_2(1-T^2)U_n' - (1+\alpha U_n)(V_n - V_{n-1}) = 0,\\ &c_2(1-T^2)V_n' - V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) = 0.\end{aligned}$$

Seek polynomial solutions

$$U_n(T_n) = \sum_{j=0}^{M_1} a_{1j} T_n^j, V_n(T_n) = \sum_{j=0}^{M_2} a_{2j} T_n^j.$$

Balance the highest exponents in T_n to determine M_1, and M_2 :

$$M_1 + 1 = M_1 + M_2, \qquad M_2 + 1 = M_1 + M_2.$$

So, $M_1 = M_2 = 1$. Hence,

$$U_n = a_{10} + a_{11} T_n, \quad V_n = a_{20} + a_{21} T_n.$$

Algebraic system for a_{ij} :

$$\begin{aligned}-a_{11}\, c_2 + a_{21}\, \tanh(\mathrm{c_1}) + \alpha\, \mathrm{a_{10}}\, \mathrm{a_{21}}\, \tanh(\mathrm{c_1}) &= 0,\\ a_{11}\, \tanh(\mathrm{c_1})\, (\alpha\, \mathrm{a_{21}} + \mathrm{c_2}) &= 0,\\ -a_{21}\, c_2 + a_{11}\, a_{20}\, \tanh(\mathrm{c_1}) + 2\alpha\, \mathrm{a_{20}}\, \mathrm{a_{21}}\, \tanh(\mathrm{c_1}) &= 0,\\ \tanh(\mathrm{c_1})\, (\mathrm{a_{11}}\, \mathrm{a_{21}} + 2\alpha\, \mathrm{a_{21}^2} - \mathrm{a_{11}}\, \mathrm{a_{20}}\, \tanh(\mathrm{c_1})) &= 0,\\ a_{21}\, \tanh^2(\mathrm{c_1})\, (\mathrm{c_2} - \mathrm{a_{11}}) &= 0.\end{aligned}$$

Solution of the algebraic system

$$\begin{aligned} a_{10} &= -c_2 \coth(c_1) - \frac{1}{\alpha}, \\ a_{11} &= c_2, \\ a_{20} &= \frac{c_2 \ \coth(c_1)}{\alpha}, \\ a_{21} &= -\frac{c_2}{\alpha}. \end{aligned}$$

Solitary wave solution in original variables:

$$\begin{aligned} u_n(t) &= -c_2 \ \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh\left[\mathrm{c}_1\mathrm{n} + \mathrm{c}_2\mathrm{t} + \Delta\right], \\ v_n(t) &= \frac{c_2 \ \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh\left[\mathrm{c}_1\mathrm{n} + \mathrm{c}_2\mathrm{t} + \Delta\right]. \end{aligned}$$

Solving/Analyzing Systems of Algebraic Systems with Parameters

Class of fifth-order evolution equations with parameters:

$$u_t + \alpha\gamma^2 u^2 u_x + \beta\gamma u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0.$$

Well-Known Special cases

Lax case: $\alpha = \frac{3}{10}, \beta = 2, \gamma = 10$. **Two** solutions:

$$u(x,t) = 4c_1^2 - 6c_1^2 \tanh^2 \left[\mathrm{c_1 x - 56 c_1^5 t + \Delta}\right],$$

and

$$u(x,t) = a_0 - 2c_1^2 \tanh^2 \left[\mathrm{c_1 x - 2(15 a_0^2 c_1 - 40 a_0 c_1^3 + 28 c_1^5) t + \Delta}\right],$$

where a_0 is arbitrary.

Sawada-Kotera case: $\alpha = \frac{1}{5}, \beta = 1, \gamma = 5$. **Two** solutions:

$$u(x,t) = 8c_1^2 - 12c_1^2 \tanh^2 \left[\mathrm{c_1 x - 16 c_1^5 t + \Delta}\right],$$

and

$$u(x,t) = a_0 - 6c_1^2 \tanh^2 \left[\mathrm{c_1 x - (5 a_0^2 c_1 - 40 a_0 c_1^3 + 76 c_1^5) t + \Delta}\right],$$

where a_0 is arbitrary.

Kaup-Kupershmidt case: $\alpha = \frac{1}{5}, \beta = \frac{5}{2}, \gamma = 10$. **Two** solutions:

$$u(x,t) = c_1^2 - \frac{3}{2}c_1^2 \tanh^2 \left[\mathrm{c_1 x - c_1^5 t + \Delta}\right]$$

and

$$u(x,t) = 8c_1^2 - 12c^2 \tanh^2 \left[\mathrm{c_1 x - 176 c_1^5 t + \Delta}\right].$$

No free constants!

Ito case: $\alpha = \frac{2}{9}, \beta = 2, \gamma = 3$. **One** solution:

$$u(x,t) = 20c_1^2 - 30c_1^2 \tanh^2 \left[\mathrm{c_1 x - 96 c_1^5 t + \Delta}\right].$$

What about the General case?

Q1: Can we retrieve the special solutions?

Q2: What are the condition(s) on the parameters α, β, γ for solutions of tanh-type to **exist**?

Tanh solutions:

$$u(x,t) = a_0 + a_1 \tanh\left[\mathrm{c_1x + c_2t + \Delta}\right] + \mathrm{a_2} \tanh^2\left[\mathrm{c_1x + c_2t + \Delta}\right].$$

Nonlinear algebraic system must be analyzed, solved (or reduced!):

$$a_1(\alpha\gamma^2 a_2^2 + 6\gamma a_2 c_1^2 + 2\beta\gamma a_2 c_1^2 + 24c_1^4) = 0,$$

$$a_1(\alpha\gamma^2 a_1^2 + 6\alpha\gamma^2 a_0 a_2 + 6\gamma a_0 c_1^2 - 18\gamma a_2 c_1^2 - 12\beta\gamma a_2 c_1^2 - 120c_1^4) = 0,$$

$$\alpha\gamma^2 a_2^2 + 12\gamma a_2 c_1^2 + 6\beta\gamma a_2 c_1^2 + 360c_1^4 = 0,$$

$$2\alpha\gamma^2 a_1^2 a_2 + 2\alpha\gamma^2 a_0 a_2^2 + 3\gamma a_1^2 c_1^2 + \beta\gamma a_1^2 c_1^2 + 12\gamma a_0 a_2 c_1^2$$
$$-8\gamma a_2^2 c_1^2 - 8\beta\gamma a_2^2 c_1^2 - 480 a_2 c_1^4 = 0,$$

$$a_1(\alpha\gamma^2 a_0^2 c_1 - 2\gamma a_0 c_1^3 + 2\beta\gamma a_2 c_1^3 + 16c_1^5 + c_2) = 0,$$

$$\alpha\gamma^2 a_0 a_1^2 c_1 + \alpha\gamma^2 a_0^2 a_2 c_1 - \gamma a_1^2 c_1^3 - \beta\gamma a_1^2 c_1^3 - 8\gamma a_0 a_2 c_1^3 + 2\beta\gamma a_2^2 c_1^3$$
$$+136 a_2 c_1^5 + a_2 c_2 = 0.$$

Unknowns: a_0, a_1, a_2.

Parameters: $c_1, c_2, \alpha, \beta, \gamma$.

Solve and **Reduce** cannot be used on the whole system!

Strategy to Solve/Reduce Nonlinear Systems

Assumptions:

- All $c_i \neq 0$
- Parameters $(\alpha, \beta, \gamma, ...)$ are nonzero. Otherwise the maximal exponents M_i may change.
- All $M_i \geq 1$ in tanh- and sech-methods.
- All $a_{i\,M_i} \neq 0$ in tanh- and sech-methods. Highest power terms in U_i must be present, except in mixed sech-tanh-method.
- Solve for a_{ij}, then c_i, then find conditions on parameters.

Strategy followed by hand:

- Solve all linear equations in a_{ij} first (cost: branching). Start with the ones without parameters. Capture constraints in the process.
- Solve linear equations in c_i if they are free of a_{ij}.
- Solve linear equations in parameters if they free of a_{ij}, c_i.
- Solve quasi-linear equations for a_{ij}, c_i, parameters.
- Solve quadratic equations for a_{ij}, c_i, parameters.
- Eliminate cubic terms for a_{ij}, c_i, parameters, without solving.
- Show remaining equations, if any.

Alternatives:

- Use (adapted) Gröbner Basis Techniques.
- Use combinatorics on coefficients $a_{ij} = 0$ or $a_{ij} \neq 0$.

Actual Solution: Two major cases:

CASE 1: $a_1 = 0$, two subcases

Subcase 1-a:

$$\begin{aligned} a_2 &= -\frac{3}{2}a_0, \\ c_2 &= c_1^3(24c_1^2 - \beta\gamma a_0), \end{aligned}$$

where a_0 is one of the two roots of the quadratic equation:

$$\alpha\gamma^2 a_0^2 - 8\gamma a_0 c_1^2 - 4\beta\gamma a_0 c_1^2 + 160c_1^4 = 0.$$

Subcase 1-b: If $\beta = 10\alpha - 1$, then

$$\begin{aligned} a_2 &= -\frac{6}{\alpha\gamma}c_1^2, \\ c_2 &= -\frac{1}{\alpha}(\alpha^2\gamma^2 a_0^2 c_1 - 8\alpha\gamma a_0 c_1^3 + 12c_1^5 + 16\alpha c_1^5), \end{aligned}$$

where a_0 is arbitrary.

CASE 2: $a_1 \neq 0$, then

$$\alpha = \frac{1}{392}(39 + 38\beta + 8\beta^2)$$

and

$$a_2 = -\frac{168}{\gamma(3 + 2\beta)}c_1^2,$$

provided β is root of

$$(104\beta^2 + 886\beta + 1487)(520\beta^3 + 2158\beta^2 - 1103\beta - 8871) = 0.$$

Subcase 2-a: If $\beta^2 = -\frac{1}{104}(886\beta + 1487)$, then

$$\alpha = -\frac{2\beta+5}{26},$$

$$a_0 = -\frac{49c_1^2(9983+4378\beta)}{26\gamma(8+3\beta)(3+2\beta)^2},$$

$$a_1 = \pm\frac{336c_1^2}{\gamma(3+2\beta)},$$

$$a_2 = -\frac{168c_1^2}{\gamma(3+2\beta)},$$

$$c_2 = -\frac{364\,c_1^5\,(3851+1634\beta)}{6715+2946\beta}.$$

Subcase 2-b: If $\beta^3 = \frac{1}{520}(8871 + 1103\beta - 2158\beta^2)$, then

$$\alpha = \frac{39+38\beta+8\beta^2}{392},$$

$$a_0 = \frac{28\,c_1^2\,(6483+5529\beta+1066\beta^2)}{(3+2\beta)(23+6\beta)(81+26\beta)\gamma},$$

$$a_1^2 = \frac{28224\,c_1^4\,(4\beta-1)(26\beta-17)}{(3+2\beta)^2(23+6\beta)(81+26\beta)\gamma^2},$$

$$a_2 = -\frac{168c_1^2}{\gamma(3+2\beta)},$$

$$c_2 = -\frac{8\,c_1^5\,(1792261977+1161063881\beta+188900114\beta^2)}{959833473+632954969\beta+105176786\beta^2}.$$

Implementation Issues – Software Demo – Future Work

- Demonstration of Mathematica package for tanh/sech methods.
- Long term goal: Develop PDESolve for closed form solutions of nonlinear PDEs and DDEs.
- Implement various methods: Lie symmetry methods, etc.
- Look at other types of explicit solutions involving
  - hyperbolic functions $\sinh, \cosh, \tanh, ...$
  - complex exponentials combined with sech or $\tanh$.
- Seek solutions $u(x,t) = U(F(\xi))$, for special functions F, where $F'(\xi)$ is polynomial or irrational expression in F.

  Examples:

  - If $F = \tanh\xi$

    $$F'(\xi) = 1 - F^2(\xi).$$

    Chain rule:

    $$\frac{\partial \bullet}{\partial x_j} = c_j(1 - F^2)\frac{d\bullet}{dF}.$$

  - If $F = \operatorname{sech}\xi$

    $$F'(\xi) = -F(\xi)\sqrt{1 - F^2(\xi)}.$$

    Chain rule:

    $$\frac{\partial \bullet}{\partial x_j} = -c_j F\sqrt{1 - F^2}\frac{d\bullet}{dF}.$$

  - If $F = \operatorname{cn}\xi$

    $$\operatorname{cn}'\xi = -\operatorname{sn}\xi\,\operatorname{dn}\xi,$$

    $$F'(\xi) = -\sqrt{1 - F^2}\sqrt{1 - k^2 + k^2F^2}.$$

Chain rule:

$$\frac{\partial \bullet}{\partial x_j} = -c_j \sqrt{1-F^2}\sqrt{1-k^2+k^2F^2}\frac{d\bullet}{dF}.$$

The modulus (k) of the elliptic functions is added to the list of c_i.

- Add the constraining differential equations to the system of PDEs directly.
- Why are tanh and sech solutions so prevalent?
- Other applications (of the nonlinear algebraic solver):

  Computation of conservation laws, symmetries, first integrals, etc. leading to **linear** parameterized systems for unknowns coefficients (see InvariantsSymmetries by Göktaş and Hereman).

- Preprint:
  D. Baldwin, Ü. Göktaş, W. Hereman, L. Hong, R. Martino, and J. Miller, *Symbolic computation of tanh and sech solutions of nonlinear partial differential and differential-difference equations*, Journal of Symbolic Computation (2001), to be submitted.