Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations

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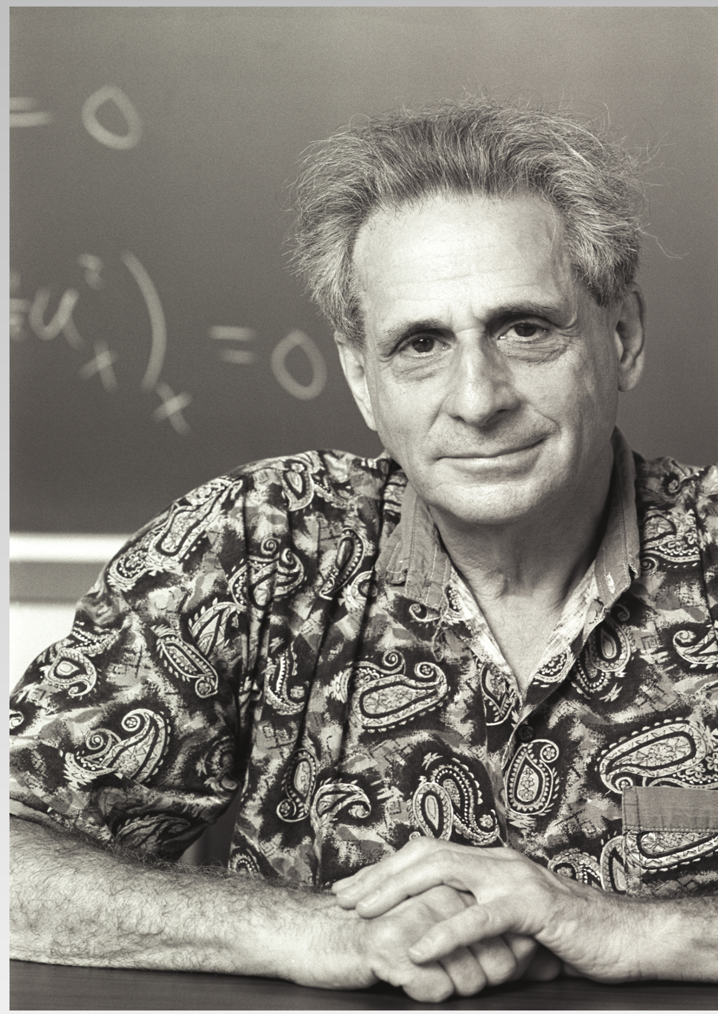
In Memory of



Martin D. Kruskal (1925-2006)

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Outline

- Conservation laws of nonlinear PDEs
- Famous examples in historical perspective
- Example: Shallow water wave equations (Dellar)
- Algorithmic method for conservation laws
- Computer demonstration
- Tools:
  - The variational derivative (testing exactness)
  - The homotopy operator (inverting D_x and Div)
- Application to shallow water wave equations
- Additional examples
- Conclusions and future work

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Conservation Laws

- Conservation law in (1+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \mathrm{D}_x J = 0} \quad \text{(on PDE)}$$

conserved density ρ and flux J

- Conservation law in (3+1)-dimensions

$$\boxed{\mathrm{D}_t \rho + \nabla \cdot \mathbf{J} = \mathrm{D}_t \rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 + \mathrm{D}_z J_3 = 0} \text{ (on PDE)}$$

conserved density ρ and flux $\mathbf{J} = (J_1, J_2, J_3)$

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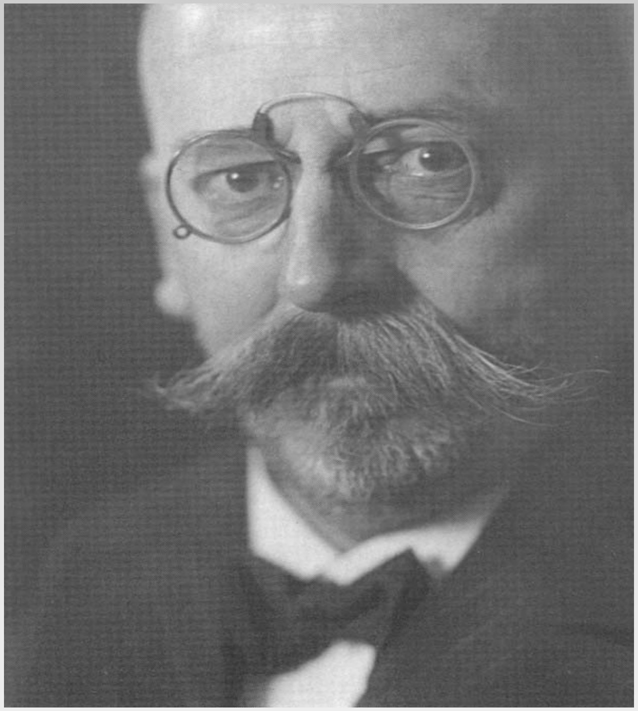
Famous Examples in Historical Perspective

- Example: Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Diederik Korteweg



Gustav de Vries

- First six (of ∞ many) densities-flux pairs:

$$\mathrm{D}_t\left(u\right)+\mathrm{D}_x\left(\frac{u^2}{2}+u_{2x}\right)=0$$

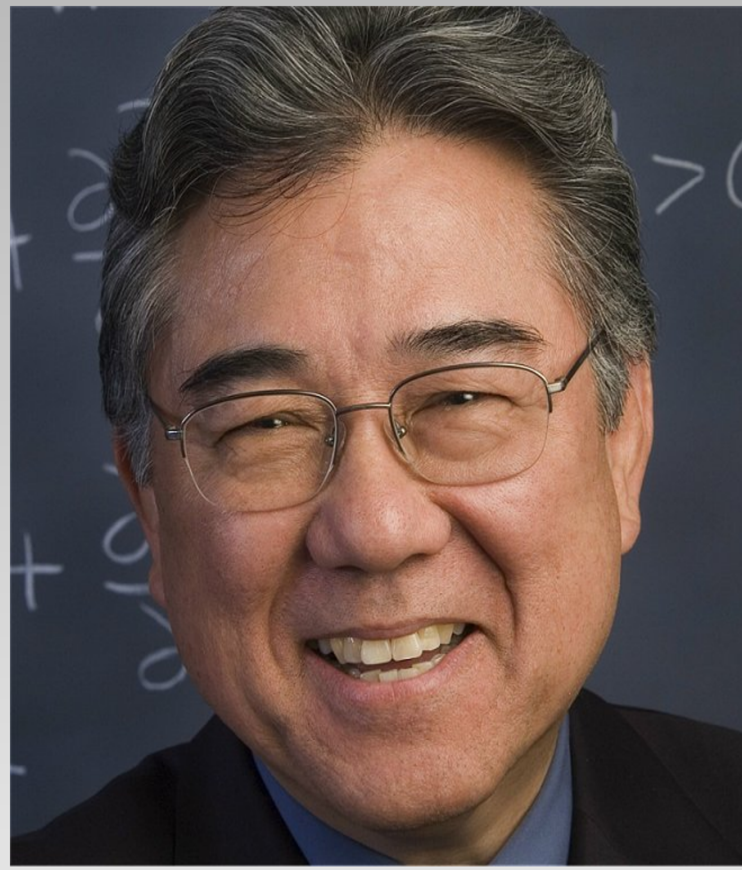
$$\mathrm{D}_t\left(u^2\right)+\mathrm{D}_x\left(\frac{2}{3}u^3-{u_x}^2+2uu_{2x}\right)=0$$

$$\mathrm{D}_t\left(u^3-3{u_x}^2\right)+$$
$$\mathrm{D}_x\left(\frac{3}{4}u^4-6u{u_x}^2+3u^2u_{2x}+3{u_{2x}}^2-6u_xu_{3x}\right)=0$$

$$\mathrm{D}_t\left(u^5-30\,u^2{u_x}^2+36\,u{u_{2x}}^2-\frac{108}{7}u_{3x}^2\right)+$$
$$\mathrm{D}_x\left(\frac{5}{6}u^6-40u^3u_x^2-\ldots-\frac{216}{7}u_{3x}u_{5x}\right)=0$$

$$\mathrm{D}_t\left(u^6 - 60\,u^3{u_x}^2 - 30\,{u_x}^4 + 108\,u^2{u_{2x}}^2 + \frac{720}{7}\,{u_{2x}}^3 - \frac{648}{7}\,u{u_{3x}}^2 + \frac{216}{7}\,{u_{4x}}^2\right) + \mathrm{D}_x\left(\frac{6}{7}u^7 - 75u^4u_x^2 - \ldots + \frac{432}{7}u_{4x}u_{6x}\right) = 0$$

- Third conservation law: Gerald Whitham, 1965
- Fourth and fifth: Norman Zabusky, 1965-66
- Sixth: algebraic mistake, 1966
- Seventh (sixth thru tenth): Robert Miura, 1966



Robert Miura

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- First five: IBM 7094 computer with FORMAC (1966) — storage space problem!
- First eleven densities: AEC CDC-6600 computer (2.2 seconds) — large integers problem!
- 2006 Leroy P. Steele Prize (AMS): Gardner, Greene, Kruskal, and Miura
- National Medal of Science, von Neuman Prize (SIAM), ...: Martin Kruskal

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- Key property: Dilation invariance
- Example: KdV equation and its density-flux pairs are invariant under the scaling symmetry

$$(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u)$$

  λ is arbitrary parameter
- Examples of conservation laws

$$\mathrm{D}_t \left(u^2 \right) + \mathrm{D}_x \left(\frac{2}{3} u^3 - {u_x}^2 + 2 u u_{2x} \right) = 0$$

$$\mathrm{D}_t \left(u^3 - 3 {u_x}^2 \right) +$$

$$\mathrm{D}_x \left(\frac{3}{4} u^4 - 6 u {u_x}^2 + 3 u^2 u_{2x} + 3 {u_{2x}}^2 - 6 u_x u_{3x} \right) = 0$$

Transcendental Equations in (1+1)-Dimensions

- Example: sine-Gordon equation

$$U_{XT} = \sin U$$

or

$$u_{tt} - u_{xx} = \sin u$$

Written as a first order system:

$$\begin{aligned} u_t &= v \\ v_t &= u_{xx} + \alpha \sin u \end{aligned}$$

- Scaling invariance (trick!)

$$(x, t, u, v, \alpha) \to (\frac{x}{\lambda}, \frac{t}{\lambda}, \lambda^0 u, \lambda v, \lambda^2 \alpha)$$

λ is arbitrary parameter

.

- First few densities-flux pairs

$$\rho_{(1)} = 2\alpha\cos u + v^2 + {u_x}^2 \qquad J_{(1)} = -2u_x v$$

$$\rho_{(2)} = 2u_x v \qquad J_{(2)} = 2\alpha\cos u - v^2 - {u_x}^2$$

$$\rho_{(3)} = 12vu_x\cos u + 2v^3u_x + 2v{u_x}^3 - 16v_xu_{2x}$$

$$\rho_{(4)} = 2\cos^2 u - 2\sin^2 u + v^4 + 6v^2{u_x}^2 + {u_x}^4 + 4v^2\cos u + 20{u_x}^2\cos u - 16{v_x}^2 - 16{u_{2x}}^2$$

$J_{(3)}$ and $J_{(4)}$ are not shown (too long)

.

An Example in (2+1)-Dimensions

- Example: Shallow water wave (SWW) equations

  [P. Dellar, Phys. Fluids **15** (2003) 292-297]

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\,\mathbf{\Omega}\times\mathbf{u} + \nabla(\theta h) - \frac{1}{2}h\nabla\theta = \mathbf{0}$$

$$\theta_t + \mathbf{u}\cdot(\nabla\theta) = 0$$

$$h_t + \nabla\cdot(\mathbf{u}h) = 0$$

where $\mathbf{u}(x, y, t), \theta(x, y, t)$ and $h(x, y, t)$

.

- In components:

$$u_t + uu_x + vu_y - 2\,\Omega v + \frac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\,\Omega u + \frac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- SWW equations are invariant under

$$(x, y, t, u, v, h, \theta, \Omega) \longrightarrow$$

$$(\lambda^{-1}x, \lambda^{-1}y, \lambda^{-b}t, \lambda^{b-1}u, \lambda^{b-1}v, \lambda^{a}h, \lambda^{2b-a-2}\theta, \lambda^{b}\Omega)$$

where $W(h) = a$ and $W(\Omega) = b \quad (a, b \in \mathbb{Q})$

- First few densities-flux pairs of SWW system:

$$\rho_{(1)} = h \qquad \mathbf{J}^{(1)} = \begin{pmatrix} uh \\ vh \end{pmatrix}$$

$$\rho_{(2)} = h\theta \qquad \mathbf{J}^{(2)} = \begin{pmatrix} uh\theta \\ vh\theta \end{pmatrix}$$

$$\rho_{(3)} = h\theta^2 \qquad \mathbf{J}^{(3)} = \begin{pmatrix} uh\theta^2 \\ vh\theta^2 \end{pmatrix}$$

$$\rho_{(4)} = (u^2 + v^2)h + h^2\theta \qquad \mathbf{J}^{(4)} = \begin{pmatrix} u^3h + uv^2h + 2uh^2\theta \\ v^3h + u^2vh + 2vh^2\theta \end{pmatrix}$$

$$\rho_{(5)} = (2\Omega + v_x - u_y)\theta$$

$$\mathbf{J}^{(5)} = \frac{1}{2} \begin{pmatrix} (4\Omega u - 2uu_y + 2uv_x - h\theta_y)\theta \\ (4\Omega v + 2vv_x - 2vu_y + h\theta_x)\theta \end{pmatrix}$$

Generalizations:

$$\mathrm{D}_t(f(\theta)h) + \mathrm{D}_x(f(\theta)uh) + \mathrm{D}_y(f(\theta)vh) = 0$$

$$\begin{aligned} &\mathrm{D}_t\left(g(\theta)(2\Omega + v_x - u_x)\right) \\ &+ \mathrm{D}_x\left(\frac{1}{2}g(\theta)(4\Omega u - 2uu_y + 2uv_x - h\theta_y)\right) \\ &+ \mathrm{D}_y\left(\frac{1}{2}g(\theta)(4\Omega v - 2u_y v + 2vv_x + h\theta_x)\right) = 0 \end{aligned}$$

for any functions $f(\theta)$ and $g(\theta)$

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Reasons to Compute Conservation Laws

- Conservation of physical quantities (linear momentum, mass, energy)
- Verify the closure of a model
- Testing of complete integrability and application of Inverse Scattering Transform
- Testing of numerical integrators
- Study of quantitative and qualitative properties of PDEs (Hamiltonian structure, recursion operators)

.

Notation – Computations on the Jet Space

- Independent variables $\mathbf{x} = (x, y, z)$
- Dependent variables

  $\mathbf{u} = (u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(N)})$

  In examples: $\mathbf{u} = (u, v, \theta, h, \ldots)$
- Partial derivatives $u_{kx} = \frac{\partial^k u}{\partial x^k}$, $u_{kx\,ly} = \frac{\partial^{k+l} u}{\partial x^k y^l}$, etc.
- *Differential functions*

  Example: $f = uvv_x + x^2 u_x^3 v_x + u_x v_{xx}$

.

- Total derivative (with respect to x)

$$\begin{aligned} \mathrm{D}_x &= \frac{\partial}{\partial x} + \sum_{k=0}^{M_x^{(1)}} u_{(k+1)x} \frac{\partial}{\partial u_{kx}} + \sum_{k=0}^{M_x^{(2)}} v_{(k+1)x} \frac{\partial}{\partial v_{kx}} \\ &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{3x} \frac{\partial}{\partial u_{xx}} + \ldots \\ &\quad + v_x \frac{\partial}{\partial v} + v_{xx} \frac{\partial}{\partial v_x} + v_{3x} \frac{\partial}{\partial v_{xx}} + \ldots \end{aligned}$$

$M_x^{(1)}$ is the order of f in u (with respect to x)
$M_x^{(2)}$ is the order of f in v (with respect to x)

.

- Example: $f = uvv_x + x^2u_x^3v_x + u_xv_{xx}$
  Here, $M_x^{(1)} = 1$ and $M_x^{(2)} = 2$
- Total derivative with respect to x:

$$
\begin{aligned}
\mathrm{D}_x f &= \frac{\partial f}{\partial x} + \sum_{k=0}^{1} u_{(k+1)x} \frac{\partial f}{\partial u_{kx}} + \sum_{k=0}^{2} v_{(k+1)x} \frac{\partial f}{\partial v_{kx}} \\
&= \frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u} + u_{2x} \frac{\partial f}{\partial u_x} \\
&\quad + v_x \frac{\partial f}{\partial v} + v_{2x} \frac{\partial f}{\partial v_x} + v_{3x} \frac{\partial f}{\partial v_{2x}} \\
&= 2xu_x^3v_x + u_x(vv_x) + u_{xx}(3x^2u_x^2v_x + v_{xx}) \\
&\quad + v_x(uv_x) + v_{xx}(uv + x^2u_x^3) + v_{xxx}(u_x)
\end{aligned}
$$

A Method to Compute Conservation Laws

- Density is linear combination of scaling invariant terms with undetermined coefficients
- Compute $\mathrm{D}_t \rho$ with total derivative operator
- Use variational derivative (Euler operator) to compute the undetermined coefficients
- Use homotopy operator to compute flux $\mathbf{J}$ (invert D_x or Div)
- Use linear algebra and variational calculus (algorithmic)
- Work with linearly independent pieces in finite dimensional spaces

Algorithm for PDEs in (1+1)-dimensions

- **Example:** Density of **rank** 6 for the KdV equation

$$u_t + uu_x + u_{3x} = 0$$

- **Step 1**: **Compute the dilation symmetry**

Set $W(\mathrm{D}_x) = 1$ and solve

$$W(u) + W(\mathrm{D}_t) = 2W(u) + 1 = W(u) + 3$$

Hence,

$$W(u) = 2, \quad W(\mathrm{D}_t) = 3$$

Thus,

$$(x, t, u) \rightarrow (\frac{x}{\lambda}, \frac{t}{\lambda^3}, \lambda^2 u)$$

.

- **Step 2**: **Determine the form of the density**

  List powers of u, up to rank $6:$ $\quad [u, u^2, u^3]$

  Introduce x derivatives to 'adjust' the rank

  u has weight 2, introduce D_x^4

  u^2 has weight 4, introduce D_x^2

  u^3 has weight 6, no derivative needed

.

Apply the D_x derivatives

Remove total and highest derivative terms:

$$[u_{4x}] \;\rightarrow\; [\;] \quad \text{empty list}$$

$$[{u_x}^2, u u_{2x}] \;\rightarrow\; [{u_x}^2] \quad \text{since } u u_{2x} = (u u_x)_x - {u_x}^2$$

$$[u^3] \;\rightarrow\; [u^3]$$

Linearly combine the "building blocks"

Candidate density:

$$\boxed{\rho = c_1 u^3 + c_2 {u_x}^2}$$

.

- **Step 3: Compute the coefficients** c_i

Compute

$$\begin{aligned} \mathrm{D}_t\rho &= \frac{\partial\rho}{\partial t} + \rho'(u)[u_t] \\ &= \frac{\partial\rho}{\partial t} + \sum_{k=0}^{M} \frac{\partial\rho}{\partial u_{kx}} \mathrm{D}_x^k u_t \\ &= (3c_1 u^2 \mathrm{I} + 2c_2 u_x \mathrm{D}_x) u_t \end{aligned}$$

Substitute u_t by $-(uu_x + u_{3x})$

$$\begin{aligned} E &= -\mathrm{D}_t\rho = (3c_1 u^2 \mathrm{I} + 2c_2 u_x \mathrm{D}_x)(uu_x + u_{3x}) \\ &= 3c_1 u^3 u_x + 2c_2 u_x^3 + 2c_2 u u_x u_{2x} \\ &\quad +3\,c_1 u^2 u_{3x} + 2c_2 u_x u_{4x} \end{aligned}$$

.

Apply the Euler operator (variational derivative)

$$\mathcal{L}_u^{(0)} = \sum_{k=0}^{m} (-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}}$$

Here, E has order $m = 4$, thus

$$\begin{aligned} \mathcal{L}_u^{(0)} E &= \frac{\partial E}{\partial u} - \mathrm{D}_x \frac{\partial E}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial E}{\partial u_{2x}} - \mathrm{D}_x^3 \frac{\partial E}{\partial u_{3x}} + \mathrm{D}_x^4 \frac{\partial E}{\partial u_{4x}} \\ &= -6(3c_1 + c_2) u_x u_{2x} \end{aligned}$$

This term must vanish!

So, $c_1 = -\frac{1}{3} c_2$. Set $c_2 = -3$, then $c_1 = 1$

Hence, the final form density is

$$\boxed{\rho = u^3 - 3{u_x}^2}$$

.

- **Step 4: Compute the flux J**

Method 1: Integrate by parts (simple cases)

Now,

$$E = 3u^3 u_x + 3u^2 u_{3x} - 6u_x^3 - 6u u_x u_{2x} - 6u_x u_{4x}$$

Integration of $\mathrm{D}_x J = E$ yields final form of the flux

$$\boxed{J = \frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x}}$$

.

Method 2: Build the form of J (cumbersome)

Note: Rank $J =$ Rank ρ + Rank $\mathrm{D}_t - 1$

Build up form of J. Compute

$$\mathrm{D}_x J = \frac{\partial J}{\partial x} + \sum_{k=0}^{m} \frac{\partial J}{\partial u_{kx}} u_{(k+1)x}$$

m is the order of J. Match $\mathrm{D}_x J = E$

.

Method 3: Use the homotopy operator

(most powerful)

$$J = \mathrm{D}_x^{-1} E = \int E \, dx = \mathcal{H}_{\mathbf{u}(x)} E = \int_0^1 (I_u E)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

with integrand

$$I_u E = \sum_{k=1}^{M} \left(\sum_{i=0}^{k-1} u_{ix} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial E}{\partial u_{kx}}$$

Here $M = 4$, thus

$$
\begin{aligned}
I_u E &= (u\mathrm{I})(\frac{\partial E}{\partial u_x}) + (u_x\mathrm{I} - u\mathrm{D}_x)(\frac{\partial E}{\partial u_{2x}}) \\
&\quad +(u_{2x}\mathrm{I} - u_x\mathrm{D}_x + u\mathrm{D}_x^2)(\frac{\partial E}{\partial u_{3x}}) \\
&\quad +(u_{3x}\mathrm{I} - u_{2x}\mathrm{D}_x + u_x\mathrm{D}_x^2 - u\mathrm{D}_x^3)(\frac{\partial E}{\partial u_{4x}}) \\
&= (u\mathrm{I})(3u^3 + 18u_x^2 - 6uu_{2x} - 6u_{4x}) \\
&\quad +(u_x\mathrm{I} - u\mathrm{D}_x)(-6uu_x) \\
&\quad +(u_{2x}\mathrm{I} - u_x\mathrm{D}_x + u\mathrm{D}_x^2)(3u^2) \\
&\quad +(u_{3x}\mathrm{I} - u_{2x}\mathrm{D}_x + u_x\mathrm{D}_x^2 - u\mathrm{D}_x^3)(-6u_x) \\
&= 3u^4 - 18uu_x^2 + 9u^2u_{2x} + 6u_{2x}^2 - 12u_xu_{3x}
\end{aligned}
$$

Note: correct terms but incorrect coefficients!

Finally,

$$
\begin{aligned}
J = \mathcal{H}_{u(x)}E &= \int_0^1 (I_u E)[\lambda u] \, \frac{d\lambda}{\lambda} \\
&= \int_0^1 \Big(3\lambda^3 u^4 - 18\lambda^2 u u_x^2 + 9\lambda^2 u^2 u_{2x} + 6\lambda u_{2x}^2 \\
&\quad -12\lambda u_x u_{3x} \Big) \, d\lambda \\
&= \frac{3}{4} u^4 - 6 u u_x^2 + 3 u^2 u_{2x} + 3 u_{2x}^2 - 6 u_x u_{3x}
\end{aligned}
$$

Final form of the flux:

$$
\boxed{J = \frac{3}{4} u^4 - 6 u {u_x}^2 + 3 u^2 u_{2x} + 3 {u_{2x}}^2 - 6 u_x u_{3x}}
$$

Computer Demonstration

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Review of Vector Calculus

- Definition: $\mathbf{F}$ is *conservative* if $\mathbf{F} = \boldsymbol{\nabla} f$
- Definition: $\mathbf{F}$ is *irrotational* or *curl free* if $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$
- Theorem: $\mathbf{F} = \boldsymbol{\nabla} f$ iff $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$

  The curl annihilates gradients!
- Definition: $\mathbf{F}$ is *incompressible* or *divergence free* if $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$
- Theorem: $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{G}$ iff $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$

  The divergence annihilates curls!

Question: How can one test that $f = \boldsymbol{\nabla} \cdot \mathbf{F}$?

.

Review of Vector Calculus

- Definition: $\mathbf{F}$ is *conservative* if $\mathbf{F} = \boldsymbol{\nabla} f$
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  The divergence annihilates curls!

Question: How can one test that $f = \boldsymbol{\nabla} \cdot \mathbf{F}$?

No theorem from vector calculus!

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Tools from the Calculus of Variations

- Definition:

  A differential function f is *exact* iff $f = \mathrm{D}_x F$

- Theorem (exactness test):

  $f = \mathrm{D}_x F$ iff $\mathcal{L}^{(0)}_{u^{(j)}(x)} f \equiv 0, \quad j = 1, 2, \ldots, N$

- Definition:

  A differential function f is a *divergence* if

  $f = \mathrm{Div}\,\mathbf{F}$

- Theorem (exactness test):

  $f = \mathrm{Div}\,\mathbf{F}$ iff $\mathcal{L}^{(0)}_{u^{(j)}(\mathbf{x})} f \equiv 0, \quad j = 1, 2, \ldots, N$

The Euler operator annihilates divergences!

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Formula for Euler operator in 1D:

$$\mathcal{L}^{(0)}_{u^{(j)}(x)} = \sum_{k=0}^{M_x^{(j)}} (-\mathrm{D}_x)^k \frac{\partial}{\partial u^{(j)}_{kx}}$$
$$= \frac{\partial}{\partial u^{(j)}} - \mathrm{D}_x \frac{\partial}{\partial u^{(j)}_{x}} + \mathrm{D}_x^2 \frac{\partial}{\partial u^{(j)}_{2x}} - \mathrm{D}_x^3 \frac{\partial}{\partial u^{(j)}_{3x}} + \cdots$$

where $j = 1, 2, \ldots, N.$

.

Formula for Euler operator in 2D:

$$
\begin{aligned}
\mathcal{L}^{(0,0)}_{u^{(j)}(x,y)} &= \sum_{k_x=0}^{M_x^{(j)}} \sum_{k_y=0}^{M_y^{(j)}} (-\mathrm{D}_x)^{k_x} (-\mathrm{D}_y)^{k_y} \frac{\partial}{\partial u^{(j)}_{k_x x\, k_y y}} \\
&= \frac{\partial}{\partial u^{(j)}} - \mathrm{D}_x \frac{\partial}{\partial u^{(j)}_x} - \mathrm{D}_y \frac{\partial}{\partial u^{(j)}_y} \\
&\quad + \mathrm{D}_x^2 \frac{\partial}{\partial u^{(j)}_{2x}} + \mathrm{D}_x \mathrm{D}_y \frac{\partial}{\partial u^{(j)}_{xy}} + \mathrm{D}_y^2 \frac{\partial}{\partial u^{(j)}_{2y}} - \mathrm{D}_x^3 \frac{\partial}{\partial u^{(j)}_{3x}} \cdots
\end{aligned}
$$

where $j = 1, 2, \ldots, N.$

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Application: Testing Exactness

Example:

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x}$$

where $u(x)$ and $v(x)$

- f is exact
- After integration by parts (by hand):

$$F = \int f \, dx = 4\, v_x^2 + u_x^2 \, \cos u - 3\, v^2 \, \cos u$$

.

- Exactness test with Euler operator:

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6vv_x \cos u + 2u_x u_{2x} \cos u + 8v_x v_{2x}$$

$$\mathcal{L}_{u(x)}^{(0)} f = \frac{\partial f}{\partial u} - \mathrm{D}_x \frac{\partial f}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial f}{\partial u_{2x}} \equiv 0$$

$$\mathcal{L}_{v(x)}^{(0)} f = \frac{\partial f}{\partial v} - \mathrm{D}_x \frac{\partial f}{\partial v_x} + \mathrm{D}_x^2 \frac{\partial f}{\partial v_{2x}} \equiv 0$$

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Inverting D_x and Div

Problem Statement in 1D

- Example:

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x}$$

- Find $F = \int f \, dx$. So, $f = D_x F$
- Result (by hand):

$$F = 4\, v_x^2 + u_x^2 \, \cos u - 3\, v^2 \, \cos u$$

Mathematica cannot compute this integral!

Problem Statement in 2D

- Example:

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

  where $u(x, y)$ and $v(x, y)$

- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ so, $f = \mathrm{Div}\,\mathbf{F}$

- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Problem Statement in 2D

- Example:

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

where $u(x, y)$ and $v(x, y)$

- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ so, $f = \mathrm{Div}\,\mathbf{F}$

- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Mathematica cannot do this!

Problem Statement in 2D

- Example:

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

where $u(x, y)$ and $v(x, y)$

- Find $\mathbf{F} = \mathrm{Div}^{-1} f$ so, $f = \mathrm{Div}\,\mathbf{F}$

- Result (by hand):

$$\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$$

Mathematica cannot do this!

Can this be done without integration by parts?

Problem Statement in 2D

- Example:

$$f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$$

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Mathematica cannot do this!

Can this be done without integration by parts?

Can this be reduced to single integral in one variable?

Yes! With the Homotopy operator

.

Using the Homotopy Operator

- Theorem (integration with homotopy operator):

- In 1D: If f is exact then

$$F = \mathrm{D}_x^{-1} f = \int f \, dx = \mathcal{H}_{\mathbf{u}(x)} f$$

- In 2D: If f is a divergence then

$$\mathbf{F} = \mathrm{Div}^{-1} f = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f)$$

- Homotopy Operator in 1D (variable x):

$$\boxed{\mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 \sum_{j=1}^{N} (I_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}}$$

with integrand

$$I_{u^{(j)}} f = \sum_{k=1}^{M_x^{(j)}} \left(\sum_{i=0}^{k-1} u_{ix}^{(j)} \, (-\mathcal{D}_x)^{k-(i+1)} \right) \frac{\partial f}{\partial u_{kx}^{(j)}}$$

N is the number of dependent variables and $(I_{u^{(j)}} f)[\lambda \mathbf{u}]$ means that in $I_{u^{(j)}} f$ one replaces $\mathbf{u} \rightarrow \lambda \mathbf{u}, \; \mathbf{u}_x \rightarrow \lambda \mathbf{u}_x,$ *etc.*

More general: $\mathbf{u} \rightarrow \lambda(\mathbf{u} - \mathbf{u}_0) + \mathbf{u}_0$

$\mathbf{u}_x \rightarrow \lambda(\mathbf{u}_x - \mathbf{u}_{x0}) + \mathbf{u}_{x0}$ etc.

Application of Homotopy Operator in 1D

Example:

$$f = 3u_x v^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x}$$

- Compute

$$\begin{aligned} I_u f &= u \frac{\partial f}{\partial u_x} + \left(u_x \mathrm{I} - u \mathrm{D}_x \right) \frac{\partial f}{\partial u_{2x}} \\ &= 3uv^2 \sin u - u u_x^2 \sin u + 2 u_x^2 \cos u \end{aligned}$$

- Similarly,

$$I_v f = v \frac{\partial f}{\partial v_x} + (v_x \mathrm{I} - v \mathrm{D}_x) \frac{\partial f}{\partial v_{2x}}$$
$$= -6v^2 \cos u + 8v_x^2$$

- Finally,

$$F = \mathcal{H}_{\mathbf{u}(x)} f = \int_0^1 (I_u f + I_v f)[\lambda \mathbf{u}] \frac{d\lambda}{\lambda}$$
$$= \int_0^1 \Big(3\lambda^2 u v^2 \sin(\lambda u) - \lambda^2 u u_x^2 \sin(\lambda u)$$
$$+ 2\lambda u_x^2 \cos(\lambda u) - 6\lambda v^2 \cos(\lambda u) + 8\lambda v_x^2 \Big) \, d\lambda$$
$$= 4v_x^2 + u_x^2 \cos u - 3v^2 \cos u$$

- Homotopy Operator in 2D (variables x and y):

$$\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(x)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

$$\mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f = \int_0^1 \sum_{j=1}^{N} (I^{(y)}_{u^{(j)}} f)[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}$$

where

$$\mathcal{I}^{(x)}_{u^{(j)}} f = \sum_{k_x=1}^{M_x^{(j)}} \sum_{k_y=0}^{M_y^{(j)}} \left(\sum_{i_x=0}^{k_x-1} \sum_{i_y=0}^{k_y} u_{i_x x \, i_y y} \binom{i_x+i_y}{i_x} \frac{\binom{k_x+k_y-i_x-i_y-1}{k_x-i_x-1}}{\binom{k_x+k_y}{k_x}} \right.$$

$$\left. (-\mathrm{D}_x)^{k_x-i_x-1} \, (-\mathrm{D}_y)^{k_y-i_y} \right) \frac{\partial f}{\partial u^{(j)}_{k_x x \, k_y y}}$$

.

Application of Homotopy Operator in 2D

- Example: $f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$
- By hand: $\tilde{\mathbf{F}} = (u v_y - u_x v_y, -u v_x + u_x v_x)$
- Compute

$$\begin{aligned} I_u^{(x)} f &= u \frac{\partial f}{\partial u_x} + (u_x \mathrm{I} - u \mathrm{D}_x) \frac{\partial f}{\partial u_{2x}} \\ &\quad + \left(\frac{1}{2} u_y \mathrm{I} - \frac{1}{2} u \mathrm{D}_y \right) \frac{\partial f}{\partial u_{xy}} \\ &= u v_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy} \end{aligned}$$

.

- Similarly,

$$I_v^{(x)} f = v\frac{\partial f}{\partial v_x} = -u_y v + u_{xy} v$$

- Hence,

$$\begin{aligned} F_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)} f = \int_0^1 \left(I_u^{(x)} f + I_v^{(x)} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\ &= \int_0^1 \lambda \left(u v_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy} - u_y v + u_{xy} v \right) d\lambda \\ &= \frac{1}{2} u v_y + \frac{1}{4} u_y v_x - \frac{1}{2} u_x v_y + \frac{1}{4} u v_{xy} - \frac{1}{2} u_y v + \frac{1}{2} u_{xy} v \end{aligned}$$

.

- Analogously,

$$
\begin{aligned}
F_2 &= \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} f = \int_0^1 \left(I^{(y)}_u f + I^{(y)}_v f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
&= \int_0^1 \left(\lambda \left(-u v_x - \frac{1}{2} u v_{2x} + \frac{1}{2} u_x v_x \right) + \lambda \left(u_x v - u_{2x} v \right) \right) d\lambda \\
&= -\frac{1}{2} u v_x - \frac{1}{4} u v_{2x} + \frac{1}{4} u_x v_x + \frac{1}{2} u_x v - \frac{1}{2} u_{2x} v
\end{aligned}
$$

- So,

$$
\mathbf{F} = \begin{pmatrix} \frac{1}{2} u v_y + \frac{1}{4} u_y v_x - \frac{1}{2} u_x v_y + \frac{1}{4} u v_{xy} - \frac{1}{2} u_y v + \frac{1}{2} u_{xy} v \\ -\frac{1}{2} u v_x - \frac{1}{4} u v_{2x} + \frac{1}{4} u_x v_x + \frac{1}{2} u_x v - \frac{1}{2} u_{2x} v \end{pmatrix}
$$

.

Let $\mathbf{K} = \tilde{\mathbf{F}} - \mathbf{F}$ then

$$\mathbf{K} = \begin{pmatrix} \frac{1}{2}uv_y - \frac{1}{4}u_y v_x - \frac{1}{2}u_x v_y - \frac{1}{4}uv_{xy} + \frac{1}{2}u_y v - \frac{1}{2}u_{xy} v \\ -\frac{1}{2}uv_x + \frac{1}{4}uv_{2x} + \frac{3}{4}u_x v_x - \frac{1}{2}u_x v + \frac{1}{2}u_{2x} v \end{pmatrix}$$

then $\mathrm{Div}\,\mathbf{K} = 0$

- Also, $\mathbf{K} = (\mathrm{D}_y\theta, -\mathrm{D}_x\theta)$ with $\theta = \frac{1}{2}uv - \frac{1}{4}uv_x - \frac{1}{2}u_x v$

  ($curl$ in 2D)

  Needed: Fast algorithm to remove curl terms
  (and strategy to avoid curl terms)

.

Computation of Conservation Laws for SWW

Quick Recapitulation

- Conservation law in (2+1) dimensions

$$\boxed{\mathrm{D}_t\rho + \nabla\cdot\mathbf{J} = \mathrm{D}_t\rho + \mathrm{D}_x J_1 + \mathrm{D}_y J_2 = 0} \quad \text{(on PDE)}$$

conserved density ρ and flux $\mathbf{J} = (J_1, J_2)$

- Example: Shallow water wave (SWW) equations

$$u_t + uu_x + vu_y - 2\,\Omega v + \frac{1}{2}h\theta_x + \theta h_x = 0$$

$$v_t + uv_x + vv_y + 2\,\Omega u + \frac{1}{2}h\theta_y + \theta h_y = 0$$

$$\theta_t + u\theta_x + v\theta_y = 0$$

$$h_t + hu_x + uh_x + hv_y + vh_y = 0$$

- Typical density-flux pair:

$$\rho_{(5)} = v_x\theta - u_y\theta + 2\Omega\theta$$

$$\mathbf{J}^{(5)} = \frac{1}{2}\begin{pmatrix} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{pmatrix}$$

Algorithm for PDEs in (2+1)-dimensions

- **Step 1: Construct the form of the density**

  The SWW equations are invariant under the scaling symmetries

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda\theta, \lambda h, \lambda^2\Omega)$$

and

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda^2\theta, \lambda^0 h, \lambda^2\Omega)$$

  Construct a candidate density, for example,

$$\rho = c_1\Omega\theta + c_2 u_y\theta + c_3 v_y\theta + c_4 u_x\theta + c_5 v_x\theta$$

  which is scaling invariant under *both* symmetries.

.

- **Step 2: Determine the constants** c_i

Compute $E = -\mathrm{D}_t \rho$ and remove time derivatives

$$
\begin{aligned}
E &= -(\frac{\partial \rho}{\partial u_x} u_{tx} + \frac{\partial \rho}{\partial u_y} u_{ty} + \frac{\partial \rho}{\partial v_x} v_{tx} + \frac{\partial \rho}{\partial v_y} v_{ty} + \frac{\partial \rho}{\partial \theta} \theta_t) \\
&= c_4 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_x \\
&\quad + c_2 \theta (u u_x + v u_y - 2\Omega v + \tfrac{1}{2} h \theta_x + \theta h_x)_y \\
&\quad + c_5 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_x \\
&\quad + c_3 \theta (u v_x + v v_y + 2\Omega u + \tfrac{1}{2} h \theta_y + \theta h_y)_y \\
&\quad + (c_1 \Omega + c_2 u_y + c_3 v_y + c_4 u_x + c_5 v_x)(u \theta_x + v \theta_y)
\end{aligned}
$$

Require that

$$
\mathcal{L}^{(0,0)}_{u(x,y)} E = \mathcal{L}^{(0,0)}_{v(x,y)} E = \mathcal{L}^{(0,0)}_{\theta(x,y)} E = \mathcal{L}^{(0,0)}_{h(x,y)} E \equiv 0.
$$

.

- Solution: $c_1 = 2,\ c_2 = -1,\ c_3 = c_4 = 0,\ c_5 = 1$ gives

$$\boxed{\rho = 2\Omega\theta - u_y\theta + v_x\theta}$$

- **Step 3: Compute the flux $\mathbf{J}$**

$$\begin{aligned} E \;=\; & \theta(u_x v_x + u v_{2x} + v_x v_y + v v_{xy} + 2\Omega u_x \\ & + \tfrac{1}{2}\theta_x h_y - u_x u_y - u u_{xy} - u_y v_y - u_{2y} v \\ & + 2\Omega v_y - \tfrac{1}{2}\theta_y h_x) \\ & + 2\Omega u\theta_x + 2\Omega v\theta_y - u u_y \theta_x \\ & - u_y v \theta_y + u v_x \theta_x + v v_x \theta_y \end{aligned}$$

Apply the 2D homotopy operator:

$$\mathbf{J} = (J_1, J_2) = \mathrm{Div}^{-1} E = (\mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E, \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E)$$

.

Compute

$$I_u^{(x)} E = u \frac{\partial E}{\partial u_x} + \left(\frac{1}{2} u_y \mathrm{I} - \frac{1}{2} u \mathrm{D}_y \right) \frac{\partial E}{\partial u_{xy}}$$

$$= u v_x \theta + 2\Omega u \theta + \frac{1}{2} u^2 \theta_y - u u_y \theta$$

Similarly, compute

$$I_v^{(x)} E = v v_y \theta + \frac{1}{2} v^2 \theta_y + u v_x \theta$$

$$I_\theta^{(x)} E = \frac{1}{2} \theta^2 h_y + 2\Omega u \theta - u u_y \theta + u v_x \theta$$

$$I_h^{(x)} E = -\frac{1}{2} \theta \theta_y h$$

.

Next,

$$
\begin{aligned}
J_1 &= \mathcal{H}^{(x)}_{\mathbf{u}(x,y)} E \\
&= \int_0^1 \left(I^{(x)}_u E + I^{(x)}_v E + I^{(x)}_\theta E + I^{(x)}_h E \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda} \\
&= \int_0^1 \Bigg(4\lambda\Omega u\theta + \lambda^2 \bigg(3uv_x\theta + \frac{1}{2}u^2\theta_y - 2uu_y\theta + vv_y\theta \\
&\qquad\qquad + \frac{1}{2}v^2\theta_y + \frac{1}{2}\theta^2 h_y - \frac{1}{2}\theta\theta_y h \bigg) \Bigg) d\lambda \\
&= \; 2\Omega u\theta - \frac{2}{3}uu_y\theta + uv_x\theta + \frac{1}{3}vv_y\theta + \frac{1}{6}u^2\theta_y \\
&\quad + \frac{1}{6}v^2\theta_y - \frac{1}{6}h\theta\theta_y + \frac{1}{6}h_y\theta^2
\end{aligned}
$$

.

Analogously,

$$
\begin{aligned}
J_2 &= \mathcal{H}^{(y)}_{\mathbf{u}(x,y)} E \\
&= 2\Omega v\theta + \frac{2}{3} v v_x \theta - v u_y \theta - \frac{1}{3} u u_x \theta - \frac{1}{6} u^2 \theta_x - \frac{1}{6} v^2 \theta_x \\
&\qquad + \frac{1}{6} h\theta\theta_x - \frac{1}{6} h_x \theta^2
\end{aligned}
$$

Hence,

$$
\mathbf{J} = \frac{1}{6} \begin{pmatrix} 12\Omega u\theta - 4uu_y\theta + 6uv_x\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y - h\theta\theta_y + h_y\theta^2 \\ 12\Omega v\theta + 4vv_x\theta - 6vu_y\theta - 2uu_x\theta - u^2\theta_x - v^2\theta_x + h\theta\theta_x - h_x\theta^2 \end{pmatrix}
$$

.

After removing the curl term

$$\tilde{\mathbf{J}}^{(5)} = \frac{1}{2} \begin{pmatrix} 4\Omega u\theta - 2uu_y\theta + 2uv_x\theta - h\theta\theta_y \\ 4\Omega v\theta + 2vv_x\theta - 2vu_y\theta + h\theta\theta_x \end{pmatrix}$$

Needed: Fast algorithm to remove curl terms
(and strategy to avoid curl terms)

Additional Examples

- Example: Kadomtsev-Petviashvili (KP) Equation

$$(u_t + \alpha u u_x + u_{3x})_x + \sigma^2 u_{2y} = 0$$

parameter $\alpha \in \mathbb{R}$ and $\sigma^2 = \pm 1$.

The equation be written as a conservation law

$$\mathrm{D}_t(u_x) + \mathrm{D}_x(\alpha u u_x + u_{3x}) + \mathrm{D}_y(\sigma^2 u_y) = 0.$$

Exchange y and t and set $u_t = v$

$$\begin{aligned} u_t &= v \\ v_t &= -\frac{1}{\sigma^2}(u_{xy} + \alpha u_x^2 + \alpha u u_{2x} + u_{4x}) \end{aligned}$$

.

- Examples of conservation laws explicitly dependent on t, x, and y

$$\mathrm{D}_t\,(xu_x) + \mathrm{D}_x\left(3u^2 - u_{2x} - 6xuu_x + xu_{3x}\right) + \mathrm{D}_y\,(\alpha x u_y) = 0$$

$$\mathrm{D}_t(yu_x) + \mathrm{D}_x\,(y(\alpha uu_x + u_{3x})) + \mathrm{D}_y\left(\sigma^2(yu_y - u)\right) = 0$$

$$\mathrm{D}_t\left(\sqrt{t}u\right) + \mathrm{D}_x\left(\frac{1}{2}\alpha\sqrt{t}u^2 + \sqrt{t}u_{2x} + \frac{\sigma^2 y^2}{4\sqrt{t}}u_t + \frac{\sigma^2 y^2}{4\sqrt{t}}u_{3x}\right.$$

$$\left. + \frac{\alpha\sigma^2 y^2}{4\sqrt{t}}uu_x - x\sqrt{t}u_t - \alpha x\sqrt{t}uu_x - x\sqrt{t}u_{3x}\right)$$

$$+\mathrm{D}_y\left(-\frac{yu}{2\sqrt{t}} + \frac{y^2 u_y}{4\sqrt{t}} + x\sqrt{t}u_y\right) = 0$$

- More general conservation laws for KP equation:

$$
\mathrm{D}_t\Big(f(t)u\Big)+\mathrm{D}_x\Big(f(t)(\tfrac{1}{2}\alpha u^2+u_{xx})
$$
$$
+(\tfrac{1}{2}\sigma^2 f'(t)y^2-f(t)x)(u_t+\alpha u u_x+u_{3x})\Big)
$$
$$
+\mathrm{D}_y\Big(u_y(\tfrac{1}{2}f'(t)y^2-\sigma^2 f(t)x)-f'(t)yu\Big)=0
$$

$$
\mathrm{D}_t\Big(f(t)yu\Big)+\mathrm{D}_x\Big(f(t)y(\tfrac{1}{2}\alpha u^2+u_{xx})
$$
$$
+(\tfrac{1}{6}\sigma^2 f'(t)y^3-f(t)xy)(u_t+\alpha u u_x+u_{3x})\Big)
$$
$$
+\mathrm{D}_y\Big(u_y(\tfrac{1}{6}f'(t)y^3-\sigma^2 f(t)xy)-u(\tfrac{1}{2}f'(t)y^2-\sigma^2 f(t)x)\Big)=0
$$

where $f(t)$ is arbitrary function.

.

- Example: Potential KP Equation

  Replace u by u_x and integrate with respect to x.

$$u_{xt} + \alpha u_x u_{2x} + u_{4x} + \sigma^2 u_{2y} = 0$$

- Examples of conservation laws (not explicitly dependent on x, y, t)

$$\mathrm{D}_t(u_x) + \mathrm{D}_x\left(\frac{1}{2}\alpha u_x^2 + u_{3x}\right) + \mathrm{D}_y(\sigma^2 u_y) = 0$$

$$\mathrm{D}_t(u_x^2) + \mathrm{D}_x\left(\frac{2}{3}\alpha u_x^3 - u_{2x}^2 + 2u_x u_{3x} - \sigma^2 u_{2y}\right)$$

$$+\mathrm{D}_y\left(2\sigma^2 u_x u_y\right) = 0$$

$$\mathrm{D}_t\left(u_x u_y\right)+\mathrm{D}_x\left(\alpha u_x^2 u_y+u_t u_y+2u_{3x}u_y-2u_{2x}u_{xy}\right)$$
$$+\mathrm{D}_y\left(\sigma^2 u_y^2-\frac{1}{3}u_x^3-u_t u_x+u_{2x}^2\right)=0$$

$$\mathrm{D}_t\left(2\alpha u u_x u_{2x}+3uu_{4x}-3\sigma^2 u_y^2\right)+\mathrm{D}_x\left(2\alpha u_t u_x^2+3u_t^2\right.$$
$$\left.-2\alpha u u_x u_{tx}-3u_{tx}u_{2x}+3u_t u_{3x}+3u_x u_{t\,2x}-3uu_{t\,3x}\right)$$
$$+\mathrm{D}_y\left(6\sigma^2 u_t u_y\right)=0$$

Various generalizations exist

.

- Example: Khoklov-Zabolotskaya Equation (describes e.g. sound waves in nonlinear media)

$$(u_t - uu_x)_x - u_{2y} = 0$$

- Examples of conservation laws (with $f(t)$):

$$\mathrm{D}_t(u_x) + \mathrm{D}_x(-uu_x) + \mathrm{D}_y(-u_y) = 0$$

$$\mathrm{D}_t\,(fu) + \mathrm{D}_x\left(-(fx + \tfrac{1}{2}f'y^2)(u_t - uu_x) - \frac{1}{2}fu^2\right)$$

$$+\mathrm{D}_y\left((fx + \tfrac{1}{2}f'y^2)u_y - f'yu\right) = 0$$

$$\mathrm{D}_t\,(fyu) + \mathrm{D}_x\left(-(fxy + \frac{1}{6}f'y^3)(u_t - uu_x) - \frac{1}{2}fyu^2\right)$$

$$+\mathrm{D}_y\left((fxy + \frac{1}{6}f'y^3)u_y - (fx + \frac{1}{2}f'y^2)u\right) = 0$$

.

- Example: Zakharov-Kuznetsov Equation

  (describes e.g. ion acoustic solitons in magnetic plasma)

$$u_t + \alpha u u_x + \beta \nabla^2 u_x = 0$$

  in 2-D, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

- Examples of conservation laws:

$$\mathrm{D}_t\,(u) + \mathrm{D}_x\left(\frac{\alpha}{2}u^2 + \beta u_{2x}\right) + \mathrm{D}_y\,(\beta u_{xy}) = 0$$

$$\mathrm{D}_t\left(u^2\right) + \mathrm{D}_x\left(\frac{2\alpha}{3}u^3 - \beta(u_x^2 - u_y^2) + 2\beta u(u_{2x} + u_{2y})\right)$$

$$+\mathrm{D}_y\left(-2\beta u_x u_y\right) = 0$$

$$
\mathrm{D}_t \left(u^3 - \frac{3\beta}{\alpha}(u_x^2 + u_y^2) \right) + \mathrm{D}_x \left(\frac{3\alpha}{4} u^4 + 3\beta u^2 u_{2x} \right.
$$

$$
-6\beta u(u_x^2 + u_y^2) + \frac{3\beta^2}{\alpha}(u_{2x}^2 - u_{2y}^2)
$$

$$
\left. - \frac{6\beta^2}{\alpha}(u_x(u_{3x} + u_{x2y}) + u_y(u_{2xy} + u_{3y})) \right)
$$

$$
+\mathrm{D}_y \left(3\beta u^2 u_{xy} + \frac{6\beta^2}{\alpha} u_{xy}(u_{2x} + u_{2y}) \right) = 0
$$

$$\mathrm{D}_t\left(tu^2-\frac{2}{\alpha}xu\right)+\mathrm{D}_x\left(t(\frac{2\alpha}{3}u^3-\beta(u_x^2-u_y^2)\right.$$

$$\left.+2\beta u(u_{2x}+u_{2y}))-\frac{2}{\alpha}x(\frac{\alpha}{2}u^2+\beta u_{2x})+\frac{2\beta}{\alpha}u_x\right)$$

$$-\mathrm{D}_y\left(2\beta(tu_xu_y+\frac{1}{\alpha}xu_{xy})\right)=0$$

.

- Example: Navier's Equation

(describes e.g. wave motion in elastic solids)

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \Delta \mathbf{u}$$

where $\mathbf{u} = (u, v, w)$, λ and μ are Lamé's constants

In components,

$$\rho u_{2t} = (\lambda + \mu)\left(u_{2x} + v_{xy} + w_{xz}\right) + \mu\left(u_{2x} + u_{2y} + u_{2z}\right)$$
$$\rho v_{2t} = (\lambda + \mu)\left(u_{xy} + v_{2y} + w_{yz}\right) + \mu\left(v_{2x} + v_{2y} + v_{2z}\right)$$
$$\rho w_{2t} = (\lambda + \mu)\left(u_{xz} + v_{yz} + w_{2z}\right) + \mu\left(w_{2x} + w_{2y} + w_{2z}\right)$$

.

- Examples of densities (fluxes are long):

$$\rho_{(1)} = \rho u_t$$

$$\rho_{(2)} = \rho v_t$$

$$\rho_{(3)} = \rho w_t$$

$$\rho_{(4)} = u_x(v_{tz} - w_{ty}) - v_x(u_{tz} - w_{tx}) + w_x(u_{ty} - v_{tx})$$

$$\rho_{(5)} = u_y(v_{tz} - w_{ty}) + v_x w_{ty} - v_y u_{tz} - w_x v_{ty} + w_y u_{ty}$$

$$\rho_{(6)} = (u_y - v_x)w_{tz} - (u_z - w_x)v_{tz} + (v_z - w_y)u_{tz}$$

$$\rho_{(7)} = \rho(v_t u_{tz} - w_t(u_{ty} - v_{tx})) + \mu \Big(u_y w_{zz} + v_x(u_{xz} - w_{yy} - w_{zz}) + v_y u_{yz} + v_z u_{zz} + w_x(v_{xx} - u_{xy}) - w_y u_{yy}\Big)$$

and many more....

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Conclusions and Future Work

- The power of Euler and homotopy operators:
  - Testing exactness
  - Integration by parts, D_x^{-1}, and Div^{-1}
- Integration of non-exact expressions

  Example: $f = u_x v + u v_x + u^2 u_{2x}$

  $\int f \, dx = uv + \int u^2 u_{2x} \, dx$
- Treat broader class of PDEs (other than those of evolution type)
- Full implementation in *Mathematica*

Thank You

Software packages in *Mathematica*

Codes are available via the Internet:

URL: http://inside.mines.edu/~whereman/

Publications

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3. W. Hereman, Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions, Int. J. Quan. Chem. **106**(1), 278-299 (2006).

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