Gauge equivalence of Lax Pairs of Nonlinear Partial Difference Equations

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Outline

- Origins of nonlinear PΔEs
- Classification of integrable nonlinear PΔEs in 2D
- Lax pair of nonlinear PΔEs & gauge equivalence
- Algorithmic computation of Lax pairs (Nijhoff 2001, Bobenko & Suris 2001)
- New results: Lax pair for Hietarinta's generalized Boussinesq lattices & their gauge equivalence
- Software demonstration
- Conclusions and future work

Origins of nonlinear PΔEs

- full discretizations of completely integrable PDEs (Ablowitz, Ladik, Taha)
- fully discretized bilinear equations (Hirota)
- direct linearization of completely integrable PDEs (Quispel, Nijhoff)
- superposition principle (Bianchi permutability) for Bäcklund transformations between solutions of a completely integrable PDE
- classification of multi-dimensionally consistent PΔEs (Adler, Bobenko, Suris)

- **Example: discrete potential Korteweg-de Vries (pKdV) equation**

$$\boxed{(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0}$$

- u is dependent variable or field (scalar case)

  n and m are lattice points

  p and q are parameters

- **Notation:**

  $(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (x, x_1, x_2, x_{12})$

- Alternate notations (in the literature):

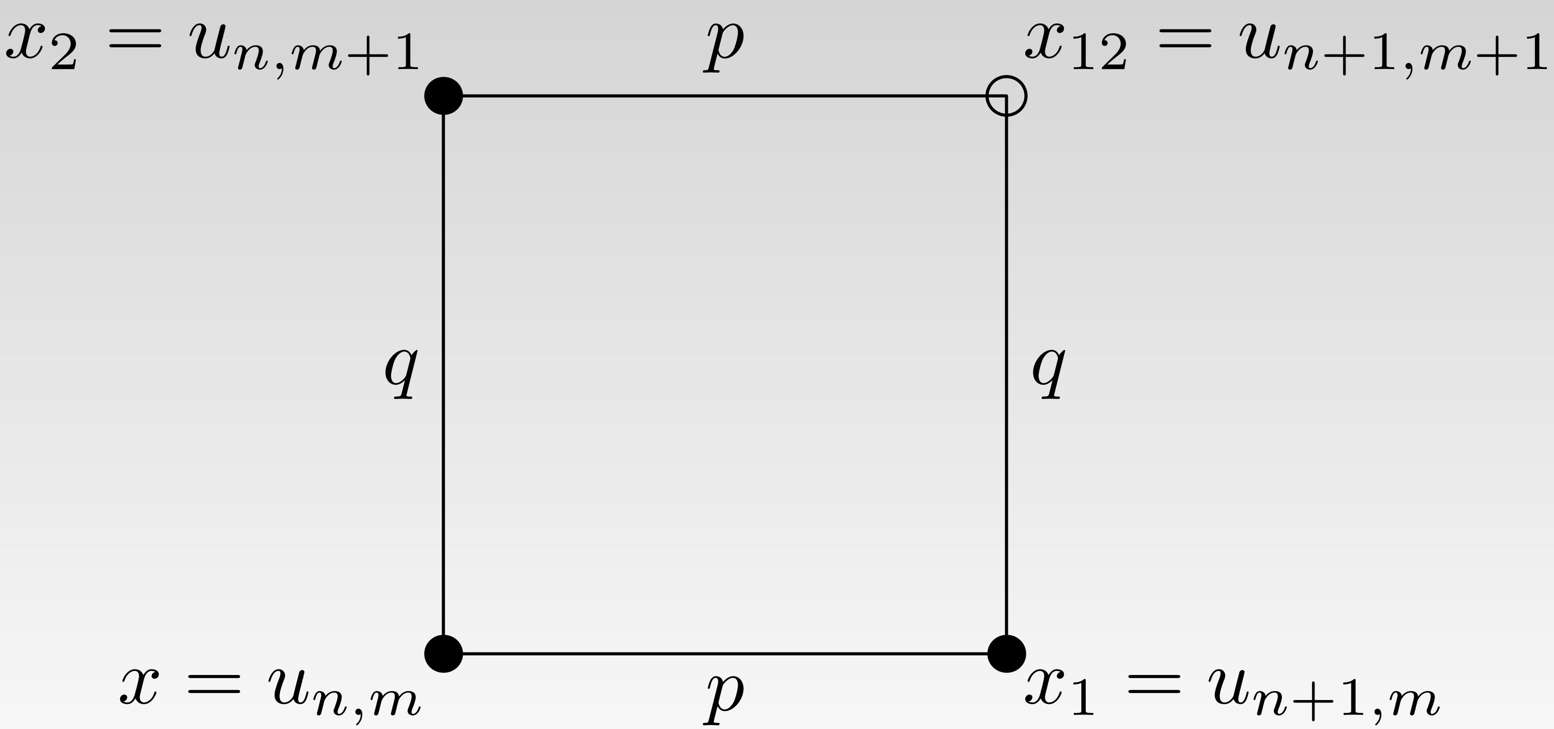
  $(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (u, \tilde{u}, \hat{u}, \hat{\tilde{u}})$

  $(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (u_{00}, u_{10}, u_{01}, u_{11})$

- **Example: discrete pKdV equation**

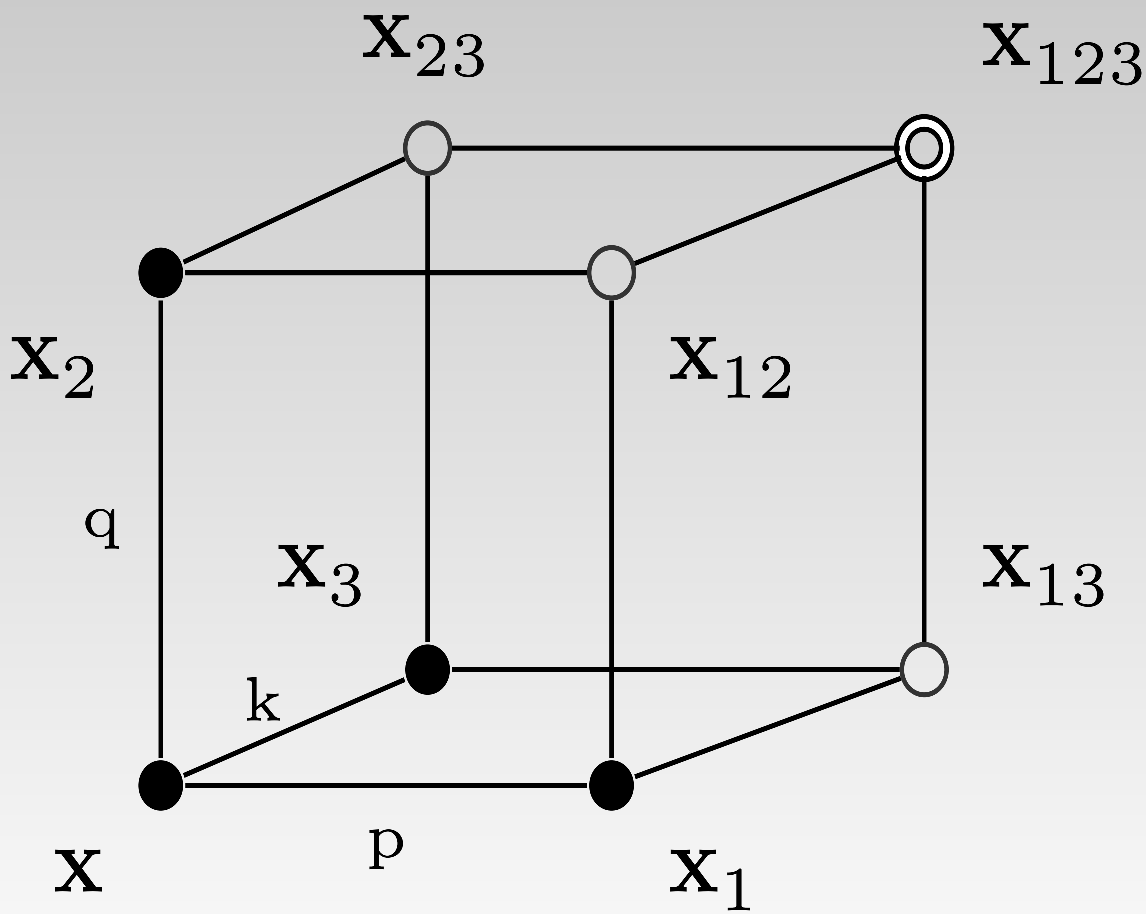
$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0$$

Short: $$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$



Concept of Consistency Around the Cube

Superposition of Bäcklund transformations between 4 solutions x, x_1, x_2, x_3 (3 parameters: p, q, k)

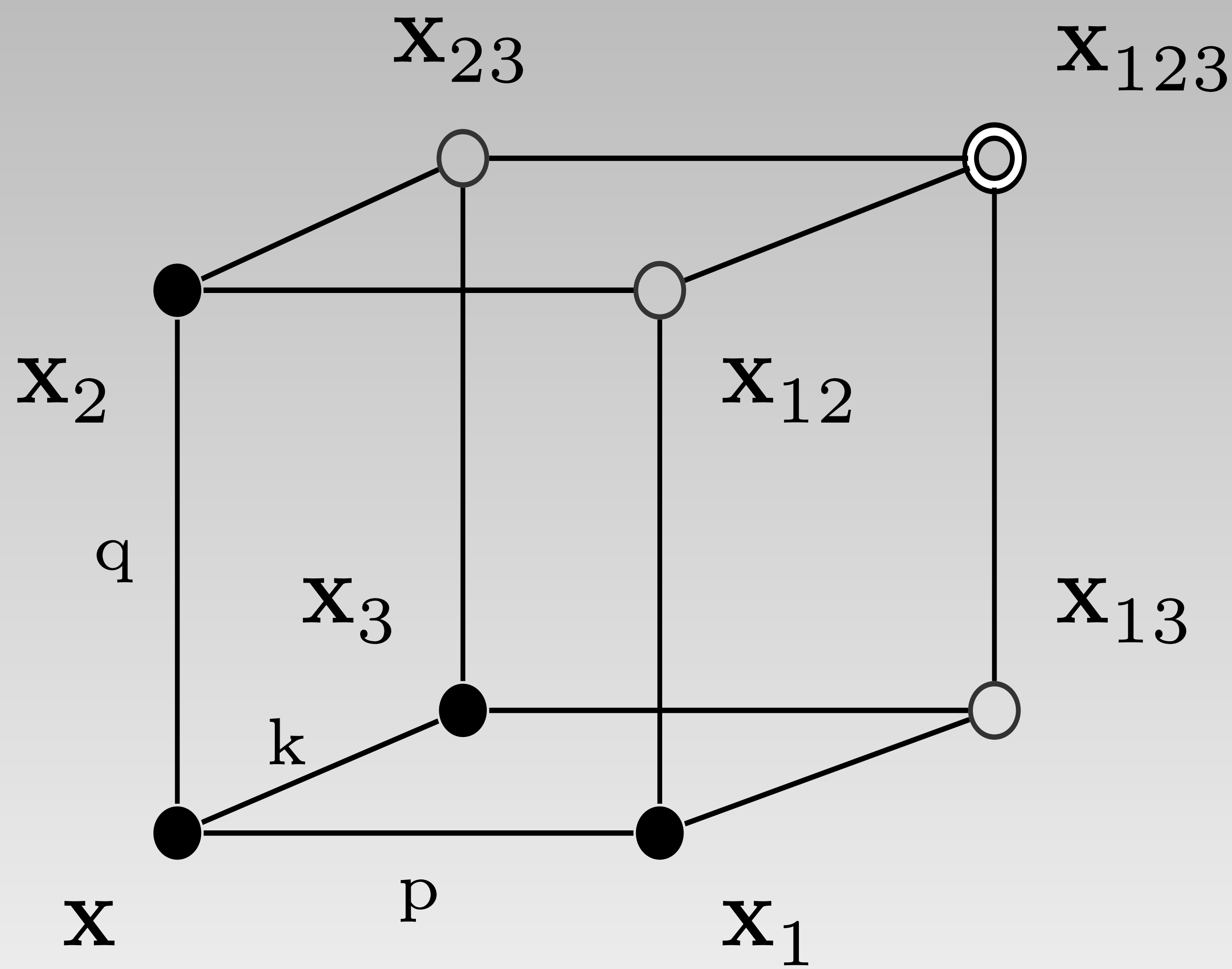


- Introduce a third lattice variable ℓ
- View u as dependent on three lattice points: n, m, ℓ. So, $x = u_{n,m} \longrightarrow x = u_{n,m,\ell}$
- Move in three directions:

  $n \to n+1$ over distance p

  $m \to m+1$ over distance q

  $\ell \to \ell+1$ over distance k (spectral parameter)
- Require that the same $\mathrm{P\Delta E}$ holds on the front, bottom, and left faces of the cube
- Require consistency for the computation of $x_{123} = u_{n+1,m+1,\ell+1}$ (3 ways $\to$ same answer)



$\mathbf{x}_{23}$
$\mathbf{x}_{123}$
$\mathbf{x}_2$
$\mathbf{x}_{12}$
q
$\mathbf{x}_3$
$\mathbf{x}_{13}$
k
$\mathbf{x}$
p
$\mathbf{x}_1$

Classification of 2D scalar nonlinear PΔEs

Adler, Bobenko, Suris (ABS) 2003, 2007

- Consider a family PΔEs: $Q(x, x_1, x_2, x_{12}; p, q) = 0$
- Assumptions (ABS 2003):

1. Affine linear

$$Q(x, x_1, x_2, x_{12}; p, q) = a_1 x x_1 x_2 x_{12} + a_2 x x_1 x_2 + \ldots + a_{14} x_2 + a_{15} x_{12} + a_{16}$$

2. Invariant under D_4 (symmetries of square)

$$\begin{aligned} Q(x, x_1, x_2, x_{12}; p, q) &= \epsilon Q(x, x_2, x_1, x_{12}; q, p) \\ &= \sigma Q(x_1, x, x_{12}, x_2; p, q) \end{aligned}$$

$\epsilon, \sigma = \pm 1$

3. Consistency around the cube

Result of the ABS Classification

- List H
  - H1

$$(x - x_{12})(x_1 - x_2) + q - p = 0$$

  - H2

$$(x - x_{12})(x_1 - x_2) + (q - p)(x + x_1 + x_2 + x_{12}) + q^2 - p^2 = 0$$

  - H3

$$p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) + \delta(p^2 - q^2) = 0$$

- List A
  - A1

$$p(x+x_2)(x_1+x_{12})-q(x+x_1)(x_2+x_{12})-\delta^2 pq(p-q)=0$$

  - A2

$$(q^2-p^2)(xx_1x_2x_{12}+1)+q(p^2-1)(xx_2+x_1x_{12})$$
$$-p(q^2-1)(xx_1+x_2x_{12})=0$$

- List Q
  - Q1

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta^2 pq(p-q) = 0$$

  - Q2

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+pq(p-q)$$
$$(x+x_1+x_2+x_{12})-pq(p-q)(p^2-pq+q^2)=0$$

  - Q3

$$(q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12})$$
$$-p(q^2-1)(xx_2+x_1x_{12})-\frac{\delta^2}{4pq}(p^2-q^2)(p^2-1)(q^2-1)=0$$

- Q4 (mother) Hietarinta's Parametrization

$$\operatorname{sn}(\alpha+\beta;k)\,(x_1x_2+xx_{12})$$
$$-\operatorname{sn}(\alpha;k)\,(xx_1+x_2x_{12})-\operatorname{sn}(\beta;k)\,(xx_2+x_1x_{12})$$
$$+\operatorname{sn}(\alpha;k)\operatorname{sn}(\beta;k)\operatorname{sn}(\alpha+\beta;k)(1+k^2xx_1x_2x_{12})=0$$

where $\operatorname{sn}(\alpha;k)$ is the Jacobi elliptic sine function with modulus k.

- Other parameterizations (Adler, Nijhoff, Viallet) are given in the literature.

Systems of PΔEs of Boussinesq-type

Example 1: Schwarzian-Boussinesq System

$$
\begin{aligned}
y\,x_1 - z_1 + z &= 0 \\
y\,x_2 - z_2 + z &= 0 \\
x\,y_{12}(y_1 - y_2) - y\,(p\,x_1\,y_2 - q\;x_2\,y_1) &= 0
\end{aligned}
$$

- System has three dependent variable x, y, and z. Thus, $\mathbf{x} = (x, y, z)$.
- System has two single-edge equations and one full-face equation.
- System is consistent around the cube (CAC).

Example 2: Hietarinta's A-2 System (2011)

$$z\,x_1 - y_1 - x = 0$$

$$z\,x_2 - y_2 - x = 0$$

$$y - x\,z_{12} + b_0 x + \frac{\mathbb{G}(-p,-a)x_1 - \mathbb{G}(-q,-a)x_2}{z_2 - z_1} = 0$$

with $\mathbb{G}(\omega,\kappa) := \omega^3 - \kappa^3 + \alpha_2(\omega^2 - \kappa^2) + \alpha_1(\omega - \kappa) = 0$

- System has three dependent variable x, y, and z. Thus, $\mathbf{x} = (x, y, z)$.
- System has two single-edge equations and one full-face equation.
- System is CAC without any condition for $\mathbb{G}$.

Refresher: Lax Pairs of Nonlinear PDEs

- Historical example: Korteweg-de Vries equation

$$\boxed{u_t + \alpha\, u u_x + u_{xxx} = 0} \qquad \alpha \in \mathbb{R}$$

- Key idea: Replace the nonlinear PDE with a compatible linear system (Lax pair):

$$\psi_{xx} + \left(\tfrac{1}{6}\alpha u - \lambda\right)\psi = 0$$

$$\psi_t + 4\psi_{xxx} + \alpha u \psi_x + \tfrac{1}{2}\alpha u_x \psi = 0$$

ψ is eigenfunction; λ is constant eigenvalue $(\lambda_t = 0)$ (isospectral)

Lax Pairs in Matrix Form (AKNS Scheme)

- Express compatibility of

$$\begin{aligned} \mathrm{D}_x \mathbf{\Psi} &= \mathbf{X}\,\mathbf{\Psi} \\ \mathrm{D}_t \mathbf{\Psi} &= \mathbf{T}\,\mathbf{\Psi} \end{aligned}$$

where $\mathbf{\Psi} = \begin{bmatrix} \psi \\ \psi_x \end{bmatrix}$

- Lax equation (zero-curvature equation):

$$\boxed{\mathrm{D}_t \mathbf{X} - \mathrm{D}_x \mathbf{T} + [\mathbf{X}, \mathbf{T}] \doteq \mathbf{0}}$$

with commutator $[\mathbf{X}, \mathbf{T}] = \mathbf{X}\mathbf{T} - \mathbf{T}\mathbf{X}$

and where $\doteq$ means “evaluated on the PDE”

- Example: Lax pair for the KdV equation

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6}\alpha u & 0 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \frac{1}{6}\alpha u_x & -4\lambda - \frac{1}{3}\alpha u \\ -4\lambda^2 + \frac{1}{3}\alpha\lambda u + \frac{1}{18}\alpha^2 u^2 + \frac{1}{6}\alpha u_{2x} & -\frac{1}{6}\alpha u_x \end{bmatrix}$$

Substitution into the Lax equation yields

$$\mathrm{D}_t\mathbf{X} - \mathrm{D}_x\mathbf{T} + [\mathbf{X}, \mathbf{T}] = -\frac{1}{6}\alpha \begin{bmatrix} 0 & 0 \\ u_t + \alpha u u_x + u_{3x} & 0 \end{bmatrix}$$

Equivalence under Gauge Transformations

- Lax pairs are equivalent under a gauge transformation:

If $(\mathbf{X}, \mathbf{T})$ is a Lax pair then so is $(\mathbb{X}, \mathbb{T})$ with

$$
\begin{aligned}
\mathbb{X} &= \mathbf{G}\mathbf{X}\mathbf{G}^{-1} + \mathsf{D}_x(\mathbf{G})\mathbf{G}^{-1} \\
\mathbb{T} &= \mathbf{G}\mathbf{T}\mathbf{G}^{-1} + \mathsf{D}_t(\mathbf{G})\mathbf{G}^{-1}
\end{aligned}
$$

$\mathbf{G}$ is arbitrary invertible matrix and $\mathbf{\Phi} = \mathbf{G}\mathbf{\Psi}$ where $\mathbf{\Phi}$ goes with $(\mathbb{X}, \mathbb{T})$, i.e., $\mathsf{D}_x\mathbf{\Phi} = \mathbb{X}\,\mathbf{\Phi}$ and $\mathsf{D}_t\mathbf{\Phi} = \mathbb{T}\,\mathbf{\Phi}$.

Thus,

$$
\boxed{\mathsf{D}_t\mathbb{X} - \mathsf{D}_x\mathbb{T} + [\mathbb{X}, \mathbb{T}] \doteq \mathbf{0}}
$$

- Example: For the KdV equation

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6}\alpha u & 0 \end{bmatrix} \quad \text{and} \quad \mathbb{X} = \begin{bmatrix} -ik & \frac{1}{6}\alpha u \\ -1 & ik \end{bmatrix}$$

Here,

$$\mathbb{X} = \mathbf{G}\mathbf{X}\mathbf{G}^{-1} \quad \text{and} \quad \mathbb{T} = \mathbf{G}\mathbf{T}\mathbf{G}^{-1}$$

with

$$\mathbf{G} = \begin{bmatrix} -i\,k & 1 \\ -1 & 0 \end{bmatrix}$$

where $\lambda = -k^2$

Lax Pair of Nonlinear PΔEs

- Replace the nonlinear PΔE by

$$\begin{aligned} \psi_1 &= \mathrm{L}\,\psi \quad (\text{recall } \psi_1 = \psi_{n+1,m}) \\ \psi_2 &= \mathrm{M}\,\psi \quad (\text{recall } \psi_2 = \psi_{n,m+1}) \end{aligned}$$

For scalar PΔEs, L, M are 2×2 matrices;

$x_3 = \frac{f}{F}$ and $\psi = \begin{bmatrix} F \\ f \end{bmatrix}$

For systems of PΔEs, L, M are $N \times N$ matrices;

$x_3 = \frac{f}{F}, y_3 = \frac{g}{F}, z_3 = \frac{h}{F}$, etc., $\psi = [F \;\; f \;\; g \;\; h \;\; \ldots]^{\mathrm{T}}$

where T is transpose.

- Express compatibility:

$$
\begin{aligned}
\psi_{12} &= \mathrm{L}_2\,\psi_2 = \mathrm{L}_2\,\mathrm{M}\,\psi \\
\psi_{12} &= \mathrm{M}_1\,\psi_1 = \mathrm{M}_1\,\mathrm{L}\,\psi
\end{aligned}
$$

$$
\begin{array}{ccc}
\psi_2 & \xrightarrow{\;L_2\;} & \psi_{12} \\
\uparrow{\scriptstyle M} & & \uparrow{\scriptstyle M_1} \\
\psi & \xrightarrow{\;L\;} & \psi_1
\end{array}
$$

- Lax equation: $\boxed{\mathrm{L}_2\,\mathrm{M} - \mathrm{M}_1\,\mathrm{L} \doteq 0}$

Equivalence under Gauge Transformations

- Lax pairs of the same size are equivalent under a gauge transformation

If (L, M) is a Lax pair then so is $(\mathbb{L}, \mathbb{M})$ with

$$\begin{aligned} \mathbb{L} &= \mathcal{G}_1 \, \mathrm{L} \, \mathcal{G}^{-1} \\ \mathbb{M} &= \mathcal{G}_2 \, \mathrm{M} \, \mathcal{G}^{-1} \end{aligned}$$

where $\mathcal{G}$ is an invertible matrix, $\phi = \mathcal{G}\psi$ goes with $(\mathbb{L}, \mathbb{M})$ i.e., $\phi_1 = \mathbb{L}\,\phi$, $\phi_2 = \mathbb{M}\,\phi$.

Proof: Trivial verification that

$$(\mathbb{L}_2 \, \mathbb{M} - \mathbb{M}_1 \, \mathbb{L}) \, \phi \doteq 0 \leftrightarrow (\mathrm{L}_2 \, \mathrm{M} - \mathrm{M}_1 \, \mathrm{L}) \, \psi \doteq 0$$

- Lax pairs of different sizes are equivalent under a gauge-like transformations

For example, if (L, M) is a 3×3 Lax pair and $(\mathcal{L}, \mathcal{M})$ is a 4×4 Lax pair then

$$\begin{aligned} \mathcal{L} &= \mathcal{H}_1 \, \mathrm{L} \, \mathcal{H}_{\mathrm{Left}}^{-1} \\ \mathcal{M} &= \mathcal{H}_2 \, \mathrm{M} \, \mathcal{H}_{\mathrm{Left}}^{-1} \end{aligned}$$

where $\mathcal{H}$ is 4×3 matrix with $\mathrm{rank}\, \mathcal{H} = 3$ and $\phi = \mathcal{H}\psi$, where ϕ goes with $(\mathcal{L}, \mathcal{M})$.

Proof:

$$\phi_1 = \mathcal{L}\phi = \mathcal{H}_1 L \mathcal{H}_{\mathrm{Left}}^{-1} \mathcal{H}\psi = \mathcal{H}_1 L \psi = \mathcal{H}_1 \psi_1 = (\mathcal{H}\psi)_1$$

- Alternate Way:

For example, if (L, M) is a 3×3 Lax pair and $(\mathcal{L}, \mathcal{M})$ is a 4×4 Lax pair, then

$$L = \widetilde{\mathcal{H}}_1 \, \mathcal{L} \, \widetilde{\mathcal{H}}^{-1}_{\mathrm{Right}}$$
$$M = \widetilde{\mathcal{H}}_2 \, \mathcal{M} \, \widetilde{\mathcal{H}}^{-1}_{\mathrm{Right}}$$

where $\widetilde{\mathcal{H}}$ is 3×4 matrix with $\mathrm{rank}\, \widetilde{\mathcal{H}} = 3$ and $\psi = \widetilde{\mathcal{H}} \phi$, where ϕ goes with $(\mathcal{L}, \mathcal{M})$.

Proof:

$$\psi_1 = L\psi = \widetilde{\mathcal{H}}_1 \mathcal{L} \widetilde{\mathcal{H}}^{-1}_{\mathrm{Right}} \psi = \widetilde{\mathcal{H}}_1 \mathcal{L} \phi = \widetilde{\mathcal{H}}_1 \phi_1 = (\widetilde{\mathcal{H}} \phi)_1$$

Algorithmic Computation of Lax Pairs

(Nijhoff 2001, Bobenko and Suris 2001)

Applies to systems of $\mathrm{P\Delta Es}$ that are

multi-dimensionally consistent

Example: Schwarzian-Boussinesq System

$$
\begin{aligned}
y\,x_1 - z_1 + z &= 0 \\
y\,x_2 - z_2 + z &= 0 \\
x\,y_{12}(y_1 - y_2) - y\,(p\,x_1\,y_2 - q\,x_2\,y_1) &= 0
\end{aligned}
$$

- System has two single-edge equations and one full-face equation

- Edge equations require augmentation of system with additional shifted, edge equations

$$
\begin{aligned}
y_2\, x_{12} - z_{12} + z_2 &= 0 \\
y_1\, x_{12} - z_{12} + z_1 &= 0
\end{aligned}
$$

- Edge equations will provide additional constraints during homogenization (Step 2).

  The way you handle edge equations leads to gauge-equivalent Lax pairs!

- Step 1: Verify the consistency around the cube
  - ⋆ System on the front face:

$$y\,x_1 - z_1 + z = 0$$

$$y\,x_2 - z_2 + z = 0$$

$$x\,y_{12}(y_1 - y_2) - y\,(p x_1 y_2 - q x_2 y_1) = 0$$

$$y_2\,x_{12} - z_{12} + z_2 = 0$$

$$y_1\,x_{12} - z_{12} + z_1 = 0$$

Solve for x_{12}, y_{12}, and z_{12}:

$$x_{12} = \frac{z_2 - z_1}{y_1 - y_2}$$

$$y_{12} = \frac{y(p x_1 y_2 - q x_2 y_1)}{x(y_1 - y_2)}$$

$$z_{12} = \frac{y_1 z_2 - y_2 z_1}{y_1 - y_2}$$

Compute x_{123}, y_{123}, and z_{123}:

$$x_{123} = \frac{z_{23} - z_{13}}{y_{13} - y_{23}}$$

$$y_{123} = \frac{y_3\big(px_{13}y_{23} - qx_{23}y_{13}\big)}{x_3\big(y_{13} - y_{23}\big)}$$

$$z_{123} = \frac{y_{13}z_{23} - y_{23}z_{13}}{y_{13} - y_{23}}$$

⋆ System on the bottom face:

$$y\,x_1 - z_1 + z = 0$$

$$y\,x_3 - z_3 + z = 0$$

$$x\,y_{13}(y_1 - y_3) - y\,(p x_1 y_3 - k x_3 y_1) = 0$$

$$y_3\,x_{13} - z_{13} + z_3 = 0$$

$$y_1\,x_{13} - z_{13} + z_1 = 0$$

Solve for x_{13}, y_{13}, and z_{13}:

$$x_{13} = \frac{z_3 - z_1}{y_1 - y_3}$$

$$y_{13} = \frac{y(p x_1 y_3 - k x_3 y_1)}{x(y_1 - y_3)}$$

$$z_{13} = \frac{y_1 z_3 - y_3 z_1}{y_1 - y_3}$$

Compute x_{123}, y_{123}, and z_{123}:

$$x_{123} = \frac{z_{23} - z_{12}}{y_{12} - y_{23}}$$

$$y_{123} = \frac{y_2(px_{12}y_{23} - kx_{23}y_{12})}{x_2(y_{12} - y_{23})}$$

$$z_{123} = \frac{y_{12}z_{23} - y_{23}z_{12}}{y_{12} - y_{23}}$$

⋆ System on the left face:

$$y\,x_3 - z_3 + z = 0$$

$$y\,x_2 - z_2 + z = 0$$

$$x\,y_{23}(y_3 - y_2) - y\,(px_3y_2 - qx_2y_3) = 0$$

$$y_2\,x_{23} - z_{23} + z_2 = 0$$

$$y_1\,x_{23} - z_{23} + z_1 = 0$$

Solve for x_{23}, y_{23}, and z_{23}:

$$x_{23} = \frac{z_3 - z_2}{y_2 - y_3}$$

$$y_{23} = \frac{y(qx_2y_3 - kx_3y_2)}{x(y_2 - y_3)}$$

$$z_{23} = \frac{y_2z_3 - y_3z_2}{y_2 - y_3}$$

Compute x_{123}, y_{123}, and z_{123}:

$$x_{123} = \frac{z_{13} - z_{12}}{y_{12} - y_{13}}$$

$$y_{123} = \frac{y_1(qx_{12}y_{13} - kx_{13}y_{12})}{x_1(y_{12} - y_{13})}$$

$$z_{123} = \frac{y_{12}z_{13} - y_{13}z_{12}}{y_{12} - y_{13}}$$

Substitute x_{12}, y_{12}, y_{12}, x_{13}, y_{13}, z_{13}, x_{23}, y_{23}, z_{23} into the above to get

$$x_{123} = \frac{x(x_1 - x_2)\big(y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2)\big)}{(z_1 - z_2)\big(px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1)\big)}$$

$$y_{123} = \frac{q(z_2 - z_1)(kx_3y_1 - px_1y_3) + k(z_3 - z_1)(px_1y_2 - qx_2y_1)}{x_1\big(px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1)\big)}$$

$$z_{123} = \frac{px_1(y_3z_2 - y_2z_3) + qx_2(y_1z_3 - y_3z_1) + kx_3(y_2z_1 - y_1z_2)}{px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1)}$$

Answer is unique and independent of x and y.

Consistency around the cube is satisfied!

- Step 2: Homogenization

⋆ Observed that x_3, y_3 and z_3 appear linearly in numerators and denominators of

$$
\begin{aligned}
x_{13} &= \frac{z_3 - z_1}{y_1 - y_3} \\
y_{13} &= \frac{y(px_1y_3 - kx_3y_1)}{x(y_1 - y_3)} \\
z_{13} &= \frac{y_1z_3 - y_3z_1}{y_1 - y_3}
\end{aligned}
$$

⋆ Substitute

$$x_3 = \frac{f}{F}, \;\; y_3 = \frac{g}{G}, \;\; \text{and} \;\; z_3 = \frac{h}{H}.$$

⋆ Use constraints (from left face edges)

$$y\,x_1 \;-\; z_1 + z = 0, \; y\,x_2 - z_2 + z = 0$$

$$\Rightarrow y\,x_3 - z_3 + z = 0$$

Solve for $x_3 = \frac{z_3 - z}{y}$

Thus, $x_3 = \frac{f}{F} = \frac{h - zH}{yH}$, $y_3 = \frac{g}{G}$, and $z_3 = \frac{h}{H}$

⋆ Substitute $x_3,\ y_3,\ z_3$ into $x_{13},\ y_{13},\ z_{13}$:

$$x_{13} = \frac{G(h - z_1 H)}{H(y_1 G - g)}$$

$$y_{13} = \frac{y(px_1 gF - ky_1 fG)}{Fx(y_1 G - g)}$$

$$z_{13} = \frac{y_1 hG - z_1 gH}{H(y_1 G - g)}$$

Require that numerators and denominators are linear in f, g, h, F, G, and H. That forces $H = G = F$.

Hence, $x_3 = \frac{h - zF}{yF}$, $y_3 = \frac{g}{F}$, and $z_3 = \frac{h}{F}$.

⋆ Compute

$$
\begin{aligned}
x_3 &= \frac{h - zF}{yF} \longrightarrow x_{13} = \frac{h_1 - z_1 F_1}{y_1 F_1} \\
y_3 &= \frac{g}{F} \longrightarrow y_{13} = \frac{g_1}{F_1} \\
z_3 &= \frac{h}{F} \longrightarrow z_{13} = \frac{h_1}{F_1}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
x_{13} &= \frac{h - z_1 F}{y_1 F - g} = \frac{h_1 - z_1 F_1}{y_1 F_1} \\
y_{13} &= \frac{ypx_1 g - ky_1 h + kzy_1 F}{x(y_1 F - g)} = \frac{g_1}{F_1} \\
z_{13} &= \frac{y_1 h - z_1 g}{y_1 F - g} = \frac{h_1}{F_1}
\end{aligned}
$$

Note that

$$x_{13} = \frac{h - z_1 F}{y_1 F - g} = \frac{h_1 - z_1 F_1}{y_1 F_1}$$

is automatically satisfied as a result of the relation $x_3 = \frac{z_3 - z}{y}$.

⋆ Write in matrix form:

$$\psi_1 = \begin{bmatrix} F_1 \\ g_1 \\ h_1 \end{bmatrix} = t \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kzy_1}{x} & \frac{pyx_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix} \begin{bmatrix} F \\ g \\ h \end{bmatrix} = L\,\psi$$

⋆ Repeat the same steps for x_{23}, y_{23}, z_{23} to obtain

$$\psi_2 = \begin{bmatrix} F_2 \\ g_2 \\ h_2 \end{bmatrix} = s \begin{bmatrix} y_2 & -1 & 0 \\ \frac{kzy_2}{x} & \frac{qyx_2}{x} & -\frac{ky_2}{x} \\ 0 & -z_2 & y_2 \end{bmatrix} \begin{bmatrix} F \\ g \\ h \end{bmatrix} = M\,\psi$$

⋆ Therefore,

$$L = tL_{\text{core}} = t \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kzy_1}{x} & \frac{pyx_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix}$$

$$M = sM_{\text{core}} = s \begin{bmatrix} y_2 & -1 & 0 \\ \frac{kzy_2}{x} & \frac{qyx_2}{x} & -\frac{ky_2}{x} \\ 0 & -z_2 & y_2 \end{bmatrix}$$

- Step 3: Determine t and s

$\star$ Substitute $L = t\, L_{\text{core}}, M = s\, M_{\text{core}}$ into

$\mathrm{L}_2\, \mathrm{M} - \mathrm{M}_1\, \mathrm{L} = 0$

$\longrightarrow t_2\, s\, (L_{\text{core}})_2\, M_{\text{core}} - s_1\, t\, (M_{\text{core}})_1\, L_{\text{core}} = 0$

$\star$ Solve the equation from the (2-1)-element:

$\frac{t_2}{t}\, \frac{s}{s_1} = \frac{y_1}{y_2}.$

Thus, $t = s = \frac{1}{y}$, or $t = \frac{1}{y_1}$ and $s = \frac{1}{y_2}$,

or (from determinant method) $t = \sqrt[3]{\frac{x}{y_1^2\, y\, x_1}}$ and $s = \sqrt[3]{\frac{x}{y_2^2\, y\, x_2}}$.

Summary: Lax Pair for Schwarzian-BSQ System

- Option a: Solving the edge equation for $x_3 = \frac{z_3 - z}{y}$ yields

$$x_3 = \frac{h - zF}{yF}, \quad y_3 = \frac{g}{F}, \text{ and } z_3 = \frac{h}{F}, \quad \psi_{\mathrm{a}} = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{a}} = \frac{1}{y} \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kzy_1}{x} & \frac{pyx_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix}$$

- Option b: Solving the edge equation for $z_3 = x_3 y + z$ yields

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \text{ and } z_3 = \frac{zF+yf}{F}, \quad \psi_{\mathrm{b}} = \begin{bmatrix} F \\ f \\ g \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{b}} = \frac{1}{y} \begin{bmatrix} y_1 & 0 & -1 \\ z - z_1 & y & 0 \\ 0 & -\frac{k y y_1}{x_1} & \frac{p y z_1}{x} \end{bmatrix}$$

- Gauge Equivalences between these Lax Matrices
- $L_{\rm b} = \mathcal{G}_1 L_{\rm a} \mathcal{G}^{-1}, \quad \psi_{\rm b} = \mathcal{G}\psi_{\rm a}$

  with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{z}{y} & 0 & \frac{1}{y} \\ 0 & 1 & 0 \end{bmatrix}$$

Gauge and Gauge-like Equivalences

- Use of a left or right inverse depends on the edge equations!
- Lax pairs for generalized Boussinesq lattices due to Hietarinta:

  Lax pairs were computed by Zhao, Zhang, & Nijhoff (2012) and Bridgman & Hereman (2012). They are all valid but differ in size. Zhao *et al.* have 4×4 matrices. We obtained 3×3 matrices.

New Results

Investigation of Gauge Equivalences of Boussinesq-type Systems due to Hietarinta (2011)

- Four systems: A-2, B-2, C-3, and C-4.
- Only the B-2 system requires that $\mathbb{G}(a,c) + \mathbb{G}(c,b) = \mathbb{G}(a,b)$ is satisfied.

  General solution: $\mathbb{G}(a,b) = f(a) - f(b)$, where f is an arbitrary function.
- Results for the A-2 and B-2 systems are similar.
- Results for the C-3 and C-4 systems are similar.

- Example 1: Hietarinta's A-2 System (2011)

$$\begin{aligned} z\,x_1 - y_1 - x &= 0 \\ z\,x_2 - y_2 - x &= 0 \end{aligned}$$

$$y - x\,z_{12} + b_0 x + \frac{\mathbb{G}(-p,-a)x_1 - \mathbb{G}(-q,-a)x_2}{z_2 - z_1} = 0$$

with $\mathbb{G}(\omega,\kappa) := \omega^3 - \kappa^3 + \alpha_2(\omega^2 - \kappa^2) + \alpha_1(\omega - \kappa) = 0$

System is CAC without any condition for $\mathbb{G}$.

- Option a: Solving for $x_3 = \frac{y_3+x}{z}$ yields

$$x_3 = \frac{xF+g}{zF}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \text{ and } \psi_{\mathrm{a}} = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{a}} = \frac{1}{z} \begin{bmatrix} -zz_1 & 0 & z \\ xz_1 & z_1 & -zx_1 \\ \ell_{31} & -\frac{\mathbb{G}(-k,-a)}{x} & \frac{z(y+b_0x)}{x} \end{bmatrix}$$

$$\ell_{31} = \frac{1}{x}\left(\mathbb{G}(-p,-a)zx_1 - \mathbb{G}(-k,-a)x - zz_1(y+b_0x)\right)$$

- Option b: Solving for $y_3 = z\,x_3 - x$ yields

$x_3 = \frac{f}{F}, \quad y_3 = \frac{-xF+zf}{F}, \quad z_3 = \frac{h}{F},$ and $\psi_{\mathrm{b}} = \begin{bmatrix} F \\ f \\ h \end{bmatrix}$

- Corresponding Lax matrix:

$$L_{\mathrm{b}} = \begin{bmatrix} -z_1 & 0 & 1 \\ -x_1 & 1 & 0 \\ \ell_{31} & -\frac{\mathbb{G}(-k,-a)}{x} & \frac{y+b_0x}{x} \end{bmatrix}$$

$$\ell_{31} = \frac{1}{x}\Big(\mathbb{G}(-p,-a)x_1 - z_1(y+b_0x)\Big)$$

- Option c: Substituting

$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F},$ and setting $\psi_c = \begin{bmatrix} F \\ f \\ g \\ h \end{bmatrix}$

- Corresponding Lax matrix:

$$L_c = \begin{bmatrix} -z_1 & 0 & 0 & 1 \\ -x_1 & 1 & 0 & 0 \\ 0 & z_1 & 0 & -x_1 \\ \ell_{41} & -\frac{\mathbb{G}(-k,-a)}{x} & 0 & \frac{y+b_0 x}{x} \end{bmatrix}$$

$$\ell_{41} = \frac{1}{x}\Big(\mathbb{G}(-p,-a)x_1 - z_1(y + b_0 x)\Big)$$

- Gauge Equivalences between these Lax Matrices

- $L_{\rm b} = \mathcal{G}_1 L_{\rm a} \mathcal{G}^{-1}, \quad \psi_{\rm b} = \mathcal{G}\psi_{\rm a}$

  with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{x}{z} & \frac{1}{z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_{\mathrm{c}} = \mathcal{H}_1 L_{\mathrm{a}} \mathcal{H}_{\mathrm{Left}}^{-1}, \quad \psi_{\mathrm{c}} = \mathcal{H}\psi_{\mathrm{a}}$

$$\mathcal{H} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{x}{z} & \frac{1}{z} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathcal{H}_{\mathrm{Left}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -x & z & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $L_{\rm c} = \mathcal{J}_1 L_{\rm b} \mathcal{J}_{\rm Left}^{-1}, \quad \psi_{\rm c} = \mathcal{J} \psi_{\rm b}$

$$\mathcal{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x & z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathcal{J}_{\rm Left}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Example 2: Hietarinta's B-2 System (2011)

$$
\begin{aligned}
x\,x_1 - z_1 - y &= 0 \\
x\,x_2 - z_2 - y &= 0
\end{aligned}
$$

$$
y_{12} + \alpha_1 + z + \alpha_2(x_{12} - x) - x\,x_{12} + \frac{\mathbb{G}(-p,-q)}{x_2 - x_1} = 0
$$

with $\mathbb{G}(\omega,\kappa) := \omega^3 - \kappa^3 + \alpha_2(\omega^2 - \kappa^2) + \alpha_1(\omega - \kappa) = 0$

System is 3D consistent if $\mathbb{G}(a,c) + \mathbb{G}(c,b) = \mathbb{G}(a,b)$ holds, i.e., when $\mathbb{G}(a,b) = f(a) - f(b)$, where f is an arbitrary function.

- Option a: Solving for $x_3 = \frac{z_3+y}{x}$ yields

$$x_3 = \frac{yF+h}{xF}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \text{ and } \psi_{\mathrm{a}} = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{a}} = \frac{1}{x} \begin{bmatrix} y - xx_1 & 0 & 1 \\ \ell_{21} & x(x-\alpha_2) & \alpha_2 x - \alpha_1 - z \\ -yy_1 & xx_1 & -y_1 \end{bmatrix}$$

$$\ell_{21} = (\alpha_2 x - \alpha_1 - z)(y - xx_1) - \mathbb{G}(-p, -k)x - xy_1(x - \alpha_2)$$

- Option b: Solving for $z_3 = x\, x_3 - y$ yields

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, z_3 = \frac{-yF + xf}{F}, \quad \text{and } \psi_{\mathrm{b}} = \begin{bmatrix} F \\ f \\ g \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{b}} = \begin{bmatrix} -x_1 & 1 & 0 \\ -y_1 & 0 & 1 \\ \ell_{31} & \alpha_2 x - \alpha_1 - z & x - \alpha_2 \end{bmatrix}$$

$$\ell_{31} = -\Big(x_1(\alpha_2 x - \alpha_1 - z) + y_1(x - \alpha_2) + \mathbb{G}(-p, -k)\Big)$$

- Option c: Substituting

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and setting } \psi_{\mathrm{c}} = \begin{bmatrix} F \\ f \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{c}} = \begin{bmatrix} -x_1 & 1 & 0 & 0 \\ -y_1 & 0 & 1 & 0 \\ \ell_{31} & \alpha_2 x - \alpha_1 - z & x - \alpha_2 & 0 \\ 0 & -y_1 & x_1 & 0 \end{bmatrix}$$

$$\ell_{31} = -\Big(x_1(\alpha_2 x - \alpha_1 - z) + y_1(x - \alpha_2) + \mathbb{G}(-p, -k)\Big)$$

- Gauge Equivalences between these Lax Matrices
- $L_{\rm b} = \mathcal{G}_1 L_{\rm a} \mathcal{G}^{-1}, \quad \psi_{\rm b} = \mathcal{G} \psi_{\rm a}$

  with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{y}{x} & 0 & \frac{1}{x} \\ 0 & 1 & 0 \end{bmatrix}$$

- $L_{\mathrm{c}} = \mathcal{H}_1 L_{\mathrm{a}} \mathcal{H}_{\mathrm{Left}}^{-1}, \quad \psi_{\mathrm{c}} = \mathcal{H}\psi_{\mathrm{a}}$

$$\mathcal{H} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{y}{x} & 0 & \frac{1}{x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathcal{H}_{\mathrm{Left}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -y & x & 0 & 0 \end{bmatrix}$$

- $L_{\mathrm{c}} = \mathcal{J}_1 L_{\mathrm{b}} \mathcal{J}_{\mathrm{Left}}^{-1}, \quad \psi_{\mathrm{c}} = \mathcal{J} \psi_{\mathrm{b}}$

$$\mathcal{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -y & x & 0 \end{bmatrix}$$

and

$$\mathcal{J}_{\mathrm{Left}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Example 3: Hietarinta's C-3 System (2011)

$$
\begin{aligned}
z\,y_1 + x_1 - x = 0 &= 0 \\
z\,y_2 + x_2 - x &= 0
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{G}(-a,-b)x_{12} - y\,z_{12} \\
&+z\left(\frac{\mathbb{G}(-q,-b)z_1y_2 - \mathbb{G}(-p,-b)z_2y_1}{z_1 - z_2}\right) = 0
\end{aligned}
$$

with $\mathbb{G}(\omega,\kappa) := \omega^3 - \kappa^3 + \alpha_2(\omega^2 - \kappa^2) + \alpha_1(\omega - \kappa) = 0$

System is CAC without any condition on $\mathbb{G}$.

- Solving for $x_3 = x - z\, y_3$ yields

$$x_3 = \tfrac{xF - zg}{F}, \quad y_3 = \tfrac{g}{F}, \quad z_3 = \tfrac{h}{F}, \quad \text{and} \quad \psi_{\mathrm{a}} = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{a}} = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 1 \\ z y_1 & -z & 0 \\ -\mathbb{G}(-a,-b)\,\frac{x z_1}{y} & \ell_{32} & \ell_{33} \end{bmatrix}$$

$$\ell_{32} = \frac{z z_1}{y}\Big(\mathbb{G}(-a,-b) - \mathbb{G}(-k,-b)\Big)$$

$$\ell_{33} = \frac{1}{y}\Big(\mathbb{G}(-a,-b)(x - y_1 z) + \mathbb{G}(-p,-b) z y_1\Big)$$

- Solving for $y_3 = \frac{x - x_3}{z}$ yields

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{xF - f}{zF}, \quad z_3 = \frac{h}{F}, \text{ and } \psi_{\mathrm{b}} = \begin{bmatrix} F \\ f \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{b}} = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 1 \\ 0 & -z_1 & x_1 \\ -\mathbb{G}(-k, -b)\frac{xz_1}{y} & \ell_{32} & \ell_{33} \end{bmatrix}$$

$$\ell_{32} = \frac{z_1}{y}\Big(\mathbb{G}(-k, -b) - \mathbb{G}(-a, -b)\Big)$$

$$\ell_{33} = \frac{1}{y}\Big(x_1 \mathbb{G}(-a, -b) + y_1 z \mathbb{G}(-p, -b)\Big)$$

- Substituting

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F},$$ and setting $$\psi_c = \begin{bmatrix} F \\ f \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_c = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 0 & 1 \\ 0 & -z_1 & 0 & x_1 \\ z y_1 & 0 & -z & 0 \\ 0 & -\mathbb{G}(-a,-b)\frac{z_1}{y} & \ell_{34} & \ell_{44} \end{bmatrix}$$

$$\ell_{34} = -\mathbb{G}(-k,-b)\frac{z z_1}{y}$$

$$\ell_{44} = \frac{1}{y}\Big(\mathbb{G}(-a,-b)x_1 + \mathbb{G}(-p,-b)z y_1\Big)$$

- Gauge Equivalences between these Lax Matrices
- $L_{\rm b} = \mathcal{G}_1 L_{\rm a} \mathcal{G}^{-1}, \quad \psi_{\rm b} = \mathcal{G}\psi_{\rm a}$

  with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ x & -z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_{\mathrm{a}} = \widetilde{\mathcal{H}}_1 L_{\mathrm{c}} \widetilde{\mathcal{H}}^{-1}_{\mathrm{Right}}, \quad \psi_{\mathrm{a}} = \widetilde{\mathcal{H}} \psi_{\mathrm{c}}$

$$\widetilde{\mathcal{H}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{z} & -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\widetilde{\mathcal{H}}^{-1}_{\mathrm{Right}} = \begin{bmatrix} 1 & 0 & 0 \\ x & -z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_{\rm b} = \widetilde{\mathcal{J}}_1 L_{\rm c} \widetilde{\mathcal{J}}^{-1}_{\rm Right}, \quad \psi_{\rm b} = \widetilde{\mathcal{J}} \psi_{\rm c}$

$$\widetilde{\mathcal{J}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 0 & -z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\widetilde{\mathcal{J}}^{-1}_{\rm Right} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{x}{z} & -\frac{1}{z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Example 4: Hietarinta's C-4 System (2011)

$$z\, y_1 + x_1 - x = 0 \;=\; 0$$

$$z\, y_2 + x_2 - x \;=\; 0$$

$$y\, z_{12} - z \left(\frac{z_2 y_1 \mathcal{G}(p) - z_1 y_2 \mathcal{G}(q)}{z_1 - z_2} \right) - x\, x_{12} + \tfrac{1}{4} \mathbb{G}(-a, -b)^2 \;=\; 0$$

with $\mathbb{G}(\omega, \kappa) := \omega^3 - \kappa^3 + \alpha_2(\omega^2 - \kappa^2) + \alpha_1(\omega - \kappa) = 0$

System is CAC without any condition on $\mathbb{G}$.

- Solving for $x_3 = x - z\, y_3$ yields

$$x_3 = \tfrac{xF - zg}{F}, \quad y_3 = \tfrac{g}{F}, \quad z_3 = \tfrac{h}{F}, \quad \text{and} \quad \psi_{\mathrm{a}} = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{a}} = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 1 \\ x - x_1 & -z & 0 \\ \ell_{31} & \tfrac{z z_1}{y}\Big(x + \mathcal{G}(k)\Big) & \ell_{33} \end{bmatrix}$$

$$\ell_{31} = -\,\frac{z_1}{y}\Big(x^2 - \frac{1}{4}\mathbb{G}(-a,-b)^2\Big)$$

$$\ell_{33} = \frac{1}{y}\Big(x x_1 - \frac{1}{4}\mathbb{G}(-a,-b)^2 - y_1 z\, \mathcal{G}(p)\Big)$$

- Solving for $y_3 = \frac{x - x_3}{z}$ yields

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{xF - f}{zF}, \quad z_3 = \frac{h}{F}, \text{ and } \psi_{\mathrm{b}} = \begin{bmatrix} F \\ f \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_{\mathrm{b}} = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 1 \\ 0 & -z_1 & x_1 \\ \ell_{31} & -\frac{z_1}{y}\Big(x + \mathcal{G}(k)\Big) & \ell_{33} \end{bmatrix}$$

$$\ell_{31} = \frac{z_1}{y}\Big(\frac{1}{4}\mathbb{G}(-a, -b)^2 + x\mathcal{G}(k)\Big)$$

$$\ell_{33} = \frac{1}{y}\Big(xx_1 - \frac{1}{4}\mathbb{G}(-a, -b)^2 - y_1 z \mathcal{G}(p)\Big)$$

- Substituting

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F},$$ and setting $$\psi_c = \begin{bmatrix} F \\ f \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_c = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 0 & 1 \\ 0 & -z_1 & 0 & x_1 \\ zy_1 & 0 & -z & 0 \\ \ell_{11} & -\frac{xz_1}{y} & \frac{zz_1}{y}\mathcal{G}(k) & \ell_{44} \end{bmatrix}$$

$$\ell_{11} = \frac{1}{4}\mathbb{G}(-a,-b)^2\frac{z_1}{y}$$

$$\ell_{44} = \frac{1}{y}\Big(xx_1 - \frac{1}{4}\mathbb{G}(-a,-b)^2 - y_1 z\,\mathcal{G}(p)\Big)$$

- Gauge Equivalences between these Lax Matrices
- $L_{\mathrm{b}} = \mathcal{G}_1 L_{\mathrm{a}} \mathcal{G}^{-1}, \quad \psi_{\mathrm{b}} = \mathcal{G} \psi_{\mathrm{a}}$

  with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ x & -z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_{\mathrm{a}} = \widetilde{\mathcal{H}}_1 L_{\mathrm{c}} \widetilde{\mathcal{H}}^{-1}_{\mathrm{Right}}, \quad \psi_{\mathrm{a}} = \widetilde{\mathcal{H}} \psi_{\mathrm{c}}$

$$\widetilde{\mathcal{H}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{z} & -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\widetilde{\mathcal{H}}^{-1}_{\mathrm{Right}} = \begin{bmatrix} 1 & 0 & 0 \\ x & -z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_{\rm b} = \widetilde{\mathcal{J}}_1 L_{\rm c} \widetilde{\mathcal{J}}^{-1}_{\rm Right}, \quad \psi_{\rm b} = \widetilde{\mathcal{J}} \psi_{\rm c}$

$$\widetilde{\mathcal{J}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 0 & -z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\widetilde{\mathcal{J}}^{-1}_{\rm Right} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{x}{z} & -\frac{1}{z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Software Demonstration

Conclusions and Future Work

- *Mathematica* code works for scalar $\mathbf{P\Delta E}s$ in 2D defined on quad-graphs (quadrilateral faces).
- *Mathematica* code has been extended to systems of $\mathbf{P\Delta E}s$ in 2D defined on quad-graphs.
- Code can be used to test (i) consistency around the cube and compute or test (ii) Lax pairs.
- Consistency around cube $\Longrightarrow$ $\mathrm{P\Delta E}$ has Lax pair.
- $\mathrm{P\Delta E}$ has Lax pair $\not\Rightarrow$ consistency around cube. Indeed, there are $\mathrm{P\Delta Es}$ with a Lax pair that are not consistent around the cube.
  Example: discrete sine-Gordon equation.

- Avoid the determinant method to avoid square roots! Factorization plays an essential role!
- Future Work: Extension to more complicated systems of $\mathbf{P}\Delta\mathrm{E}s$.

Thank You

Additional Examples

- Example: Discrete pKdV Equation

$$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

- Lax pair:

$$L = t L_{\text{core}} = t \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = s M_{\text{core}} = s \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{\text{Det} L_{\text{core}}}} = \frac{1}{\sqrt{k^2 - p^2}}$

and $s = \frac{1}{\sqrt{\text{Det} M_{\text{core}}}} = \frac{1}{\sqrt{k^2 - q^2}}$.

Here, $x_3 = \frac{f}{F}$, $\psi = [f \;\; F]^{\text{T}}$, and $\frac{t_2}{t} \frac{s}{s_1} = 1$.

- Example: (H1) Equation (ABS classification)

$$(x - x_{12})(x_1 - x_2) + q - p = 0$$

- Lax pair:

$$L = t \begin{bmatrix} x & p - k - x x_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} x & q - k - x x_2 \\ 1 & -x_2 \end{bmatrix}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{k-p}}$ and $s = \frac{1}{\sqrt{k-q}}$

Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and $\frac{t_2}{t} \frac{s}{s_1} = 1$.

- Example: (H2) Equation (ABS 2003)

$$(x-x_{12})(x_1-x_2)+(q-p)(x+x_1+x_2+x_{12})+q^2-p^2=0$$

- Lax pair:

$$L=t\begin{bmatrix} p-k+x & p^2-k^2+(p-k)(x+x_1)-xx_1 \\ 1 & -(p-k+x_1) \end{bmatrix}$$

$$M=s\begin{bmatrix} q-k+x & q^2-k^2+(q-k)(x+x_2)-xx_2 \\ 1 & -(q-k+x_2) \end{bmatrix}$$

with $t=\frac{1}{\sqrt{2(k-p)(p+x+x_1)}}$ and $s=\frac{1}{\sqrt{2(k-q)(q+x+x_2)}}$

Here, $x_3=\frac{f}{F}$, $\psi=\begin{bmatrix} f \\ F \end{bmatrix}$, and $\frac{t_2}{t}\,\frac{s}{s_1}=\frac{p+x+x_1}{q+x+x_2}$.

- Example: (H3) Equation (ABS 2003)

$$p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) + \delta(p^2 - q^2) = 0$$

- Lax pair:

$$L = t \begin{bmatrix} kx & -\left(\delta(p^2 - k^2) + pxx_1\right) \\ p & -kx_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} kx & -\left(\delta(q^2 - k^2) + qxx_2\right) \\ q & -kx_2 \end{bmatrix}$$

with $t = \frac{1}{\sqrt{(p^2-k^2)(\delta p+xx_1)}}$ and $s = \frac{1}{\sqrt{(q^2-k^2)(\delta q+xx_2)}}$

Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{\delta p+xx_1}{\delta q+xx_2}$.

- Example: (H3) Equation ($\delta = 0$) (ABS 2003)

$$\boxed{p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) = 0}$$

- Lax pair:

$$\boxed{L = t \begin{bmatrix} kx & -pxx_1 \\ p & -kx_1 \end{bmatrix}}$$

$$\boxed{M = s \begin{bmatrix} kx & -qxx_2 \\ q & -kx_2 \end{bmatrix}}$$

with $t = s = \frac{1}{x}$ or $t = \frac{1}{x_1}$ and $s = \frac{1}{x_2}$

Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and $\frac{t_2}{t}\frac{s}{s_1} = \frac{x\,x_1}{x\,x_2} = \frac{x_1}{x_2}$.

- Example: (A1) Equation (ABS 2003)

$$p(x+x_2)(x_1+x_{12})-q(x+x_1)(x_2+x_{12})-\delta^2pq(p-q)=0$$

(Q1) if $x_1 \to -x_1$ and $x_2 \to -x_2$

- Lax pair:

$$L = t\begin{bmatrix} (k-p)x_1+kx & -p\left(\delta^2k(k-p)+xx_1\right) \\ p & -\left((k-p)x+kx_1\right) \end{bmatrix}$$

$$M = s\begin{bmatrix} (k-q)x_2+kx & -q\left(\delta^2k(k-q)+xx_2\right) \\ q & -\left((k-q)x+kx_2\right) \end{bmatrix}$$

- with $t = \frac{1}{\sqrt{k(k-p)\big((\delta p+x+x_1)(\delta p-x-x_1)\big)}}$ and

$s = \frac{1}{\sqrt{k(k-q)\big((\delta q+x+x_2)(\delta q-x-x_2)\big)}}$

Here $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and

$\frac{t_2}{t}\,\frac{s}{s_1} = \frac{q\big(\delta p+(x+x_1)\big)\big(\delta p-(x+x_1)\big)}{p\big(\delta q+(x+x_2)\big)\big(\delta q-(x+x_2)\big)}.$

Question: Rational choice for t and s?

- Example: (A2) Equation (ABS 2003)

$$(q^2-p^2)(xx_1x_2x_{12}+1)+q(p^2-1)(xx_2+x_1x_{12})$$
$$-p(q^2-1)(xx_1+x_2x_{12})=0$$

(Q3) with $\delta=0$ via Möbius transformation:

$$x\to x, x_1\to\frac{1}{x_1}, x_2\to\frac{1}{x_2}, x_{12}\to x_{12}, p\to p, q\to q$$

- Lax pair:

$$L=t\begin{bmatrix} k(p^2-1)x & -\left(p^2-k^2+p(k^2-1)xx_1\right) \\ p(k^2-1)+(p^2-k^2)xx_1 & -k(p^2-1)x_1 \end{bmatrix}$$

$$M=s\begin{bmatrix} k(q^2-1)x & -\left(q^2-k^2+q(k^2-1)xx_2\right) \\ q(k^2-1)+(q^2-k^2)xx_2 & -k(q^2-1)x_2 \end{bmatrix}$$

- with $t = \frac{1}{\sqrt{(k^2-1)(k^2-p^2)(p-xx_1)(pxx_1-1)}}$

  and $s = \frac{1}{\sqrt{(k^2-1)(k^2-q^2)(q-xx_2)(qxx_2-1)}}$

  Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and

  $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{(q^2-1)(p-xx_1)(pxx_1-1)}{(p^2-1)(q-xx_2)(qxx_2-1)}.$

  Question: Rational choice for t and s?

- Example: (Q1) Equation (ABS 2003)

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta^2 pq(p-q)=0$$

- Lax pair:

$$L = t\begin{bmatrix} (p-k)x_1+kx & -p\left(\delta^2 k(p-k)+xx_1\right) \\ p & -\left((p-k)x+kx_1\right) \end{bmatrix}$$

$$M = s\begin{bmatrix} (q-k)x_2+kx & -q\left(\delta^2 k(q-k)+xx_2\right) \\ q & -\left((q-k)x+kx_2\right) \end{bmatrix}$$

with $t=\frac{1}{\delta p\pm(x-x_1)}$ and $s=\frac{1}{\delta q\pm(x-x_2)}$,
or $t=\frac{1}{\sqrt{k(p-k)\big((\delta p+x-x_1)(\delta p-x+x_1)\big)}}$ and

$$s=\frac{1}{\sqrt{k(q-k)\big((\delta q+x-x_2)(\delta q-x+x_2)\big)}}$$

Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and

$$\frac{t_2}{t}\,\frac{s}{s_1} = \frac{q\Big(\delta p+(x-x_1)\Big)\Big(\delta p-(x-x_1)\Big)}{p\Big(\delta q+(x-x_2)\Big)\Big(\delta q-(x-x_2)\Big)}.$$

- Example: (Q1) Equation $(\delta = 0)$ (ABS 2003)

$$p(x-x_2)(x_1-x_{12}) - q(x-x_1)(x_2-x_{12}) = 0$$

which is the cross-ratio equation

$$\frac{(x-x_1)(x_{12}-x_2)}{(x_1-x_{12})(x_2-x)} = \frac{p}{q}$$

- Lax pair:

$$L = t\begin{bmatrix} (p-k)x_1 + kx & -pxx_1 \\ p & -\left((p-k)x + kx_1\right) \end{bmatrix}$$

$$M = s\begin{bmatrix} (q-k)x_2 + kx & -qxx_2 \\ q & -\left((q-k)x + kx_2\right) \end{bmatrix}$$

$t = \frac{1}{x - x_1}$ and $s = \frac{1}{x - x_2}$

or $t = \frac{1}{\sqrt{k(k-p)}(x - x_1)}$ and $s = \frac{1}{\sqrt{k(k-q)}(x - x_2)}$.

Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and $\frac{t_2}{t} \frac{s}{s_1} = \frac{q(x-x_1)^2}{p(x-x_2)^2}$.

- Example: (Q2) Equation (ABS 2003)

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+pq(p-q)$$
$$(x+x_1+x_2+x_{12})-pq(p-q)(p^2-pq+q^2)=0$$

- Lax pair:

$$L=t\begin{bmatrix}(k-p)(kp-x_1)+kx & \\ & -p\left(k(k-p)(k^2-kp+p^2-x-x_1)+xx_1\right) \\ p & -\left((k-p)(kp-x)+kx_1\right)\end{bmatrix}$$

$$M=s\begin{bmatrix}(k-q)(kq-x_2)+kx & \\ & -q\left(k(k-q)(k^2-kq+q^2-x-x_2)+xx_2\right) \\ q & -\left((k-q)(kq-x)+kx_2\right)\end{bmatrix}$$

- with

$$t = \frac{1}{\sqrt{k(k-p)\big((x-x_1)^2-2p^2(x+x_1)+p^4\big)}}$$

and

$$s = \frac{1}{\sqrt{k(k-q)\big((x-x_2)^2-2q^2(x+x_2)+q^4\big)}}$$

Here, $x_3 = \frac{f}{F}$, $\psi = [f \quad F]^{\mathrm{T}}$, and

$$\begin{aligned} \frac{t_2}{t}\,\frac{s}{s_1} &= \frac{q\left((x-x_1)^2 - 2p^2(x+x_1) + p^4\right)}{p\left((x-x_2)^2 - 2q^2(x+x_2) + q^4\right)} \\ &= \frac{p\left((X+X_1)^2 - p^2\right)\left((X-X_1)^2 - p^2\right)}{q\left((X+X_2)^2 - q^2\right)\left((X-X_2)^2 - q^2\right)} \end{aligned}$$

with $x = X^2$, and, consequently, $x_1 = X_1^2$, $x_2 = X_2^2$.

- Example: (Q3) Equation (ABS 2003)

$$(q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12})$$
$$-p(q^2-1)(xx_2+x_1x_{12})-\frac{\delta^2}{4pq}(p^2-q^2)(p^2-1)(q^2-1)=0$$

- Lax pair:

$$L=t\begin{bmatrix} -4kp\left(p(k^2-1)x+(p^2-k^2)x_1\right) & \\ & -(p^2-1)(\delta^2k^2-\delta^2k^4-\delta^2p^2+\delta^2k^2p^2-4k^2pxx_1) \\ -4k^2p(p^2-1) & 4kp\left(p(k^2-1)x_1+(p^2-k^2)x\right) \end{bmatrix}$$

$$M=s\begin{bmatrix} -4kq\left(q(k^2-1)x+(q^2-k^2)x_2\right) & \\ & -(q^2-1)(\delta^2k^2-\delta^2k^4-\delta^2q^2+\delta^2k^2q^2-4k^2qxx_2) \\ -4k^2q(q^2-1) & 4kq\left(q(k^2-1)x_2+(q^2-k^2)x\right) \end{bmatrix}$$

- with

$$t = \frac{1}{2k\sqrt{p(k^2-1)(k^2-p^2)\left(4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2\right)}}$$

and

$$s = \frac{1}{2k\sqrt{q(k^2-1)(k^2-q^2)\left(4q^2(x^2+x_2^2)-4q(1+q^2)xx_2+\delta^2(1-q^2)^2\right)}}.$$

Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and

$$\frac{t_2}{t}\frac{s}{s_1}$$

$$= \frac{q(q^2-1)\left(4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x^2+x_2^2)-4q(1+q^2)xx_2+\delta^2(1-q^2)^2\right)}$$

$$= \frac{q(q^2-1)\left(4p^2(x-x_1)^2-4p(p-1)^2xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x-x_2)^2-4q(q-1)^2xx_2+\delta^2(1-q^2)^2\right)}$$

$$= \frac{q(q^2-1)\left(4p^2(x+x_1)^2-4p(p+1)^2xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x+x_2)^2-4q(q+1)^2xx_2+\delta^2(1-q^2)^2\right)}$$

where

$$4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2$$
$$=\delta^2(p-\mathrm{e}^{X+X_1})(p-\mathrm{e}^{-(X+X_1)})(p-\mathrm{e}^{X-X_1})(p-\mathrm{e}^{-(X-X_1)})$$
$$=\delta^2(p-\cosh(X+X_1)+\sinh(X+X_1))$$
$$(p-\cosh(X+X_1)-\sinh(X+X_1))$$
$$(p-\cosh(X-X_1)+\sinh(X-X_1))$$
$$(p-\cosh(X-X_1)-\sinh(X-X_1))$$

with $x = \delta\cosh(X)$, and, consequently,

$x_1 = \delta\cosh(X_1),\ x_2 = \delta\cosh(X_2)$.

- Example: (Q3) Equation ($\delta = 0$) (ABS 2003)

$$(q^2 - p^2)(xx_{12} + x_1x_2) + q(p^2 - 1)(xx_1 + x_2x_{12})$$
$$-p(q^2 - 1)(xx_2 + x_1x_{12}) = 0$$

- Lax pair:

$$L = t \begin{bmatrix} (p^2 - k^2)x_1 + p(k^2 - 1)x & -k(p^2 - 1)xx_1 \\ (p^2 - 1)k & -((p^2 - k^2)x + p(k^2 - 1)x_1) \end{bmatrix}$$

$$M = s \begin{bmatrix} (q^2 - k^2)x_2 + q(k^2 - 1)x & -k(q^2 - 1)xx_2 \\ (q^2 - 1)k & -((q^2 - k^2)x + q(k^2 - 1)x_2) \end{bmatrix}$$

- with $t = \frac{1}{px - x_1}$ and $s = \frac{1}{qx - x_2}$

  or $t = \frac{1}{px_1 - x}$ and $s = \frac{1}{qx_2 - x}$

  or $t = \frac{1}{\sqrt{(k^2-1)(p^2-k^2)(px-x_1)(px_1-x)}}$

  and $s = \frac{1}{\sqrt{(k^2-1)(q^2-k^2)(qx-x_2)(qx_2-x)}}$.

  Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and

  $\frac{t_2}{t} \frac{s}{s_1} = \frac{(q^2-1)(px-x_1)(px_1-x)}{(p^2-1)(qx-x_2)(qx_2-x)}$.

- Example: (α,β)-equation (Quispel 1983)

$$\big((p-\alpha)x-(p+\beta)x_1\big)\big((p-\beta)x_2-(p+\alpha)x_{12}\big)$$
$$-\big((q-\alpha)x-(q+\beta)x_2\big)\big((q-\beta)x_1-(q+\alpha)x_{12}\big)=0$$

- Lax pair:

$$L=t\begin{bmatrix}(p-\alpha)(p-\beta)x+(k^2-p^2)x_1 & -(k-\alpha)(k-\beta)xx_1 \\ (k+\alpha)(k+\beta) & -\big((p+\alpha)(p+\beta)x_1+(k^2-p^2)x\big)\end{bmatrix}$$

$$M=s\begin{bmatrix}(q-\alpha)(q-\beta)x+(k^2-q^2)x_2 & -(k-\alpha)(k-\beta)xx_2 \\ (k+\alpha)(k+\beta) & -\big((q+\alpha)(q+\beta)x_2+(k^2-q^2)x\big)\end{bmatrix}$$

- with $t = \frac{1}{(\alpha-p)x+(\beta+p)x_1}$ and $s = \frac{1}{(\alpha-q)x+(\beta+q)x_2}$

  or $t = \frac{1}{(\beta-p)x+(\alpha+p)x_1}$ and $s = \frac{1}{(\beta-q)x+(\alpha+q)x_2}$

  or $t = \frac{1}{\sqrt{(p^2-k^2)\big((\beta-p)x+(\alpha+p)x_1\big)\big((\alpha-p)x+(\beta+p)x_1\big)}}$

  and $s = \frac{1}{\sqrt{(q^2-k^2)\big((\beta-q)x+(\alpha+q)x_2\big)\big((\alpha-q)x+(\beta+q)x_2\big)}}$

  Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and

  $$\frac{t_2}{t}\,\frac{s}{s_1} = \frac{\big((\beta-p)x+(\alpha+p)x_1\big)\big((\alpha-p)x+(\beta+p)x_1\big)}{\big((\beta-q)x+(\alpha+q)x_2\big)\big((\alpha-q)x+(\beta+q)x_2\big)}.$$

- Example: Discrete sine-Gordon Equation

$$\boxed{xx_1x_2x_{12} - pq(xx_{12} - x_1x_2) - 1 = 0}$$

(H3) with $\delta = 0$ via extended Möbius transformation:

$x \to x, x_1 \to x_1, x_2 \to \frac{1}{x_2}, x_{12} \to -\frac{1}{x_{12}}, p \to \frac{1}{p}, q \to q$

Discrete sine-Gordon equation is NOT consistent around the cube, but has a Lax pair!

- Lax pair:

$$\boxed{L = \begin{bmatrix} p & -kx_1 \\ -\frac{k}{x} & \frac{px_1}{x} \end{bmatrix}} \qquad \boxed{M = \begin{bmatrix} \frac{qx_2}{x} & -\frac{1}{kx} \\ -\frac{x_2}{k} & q \end{bmatrix}}$$

- Example: Lattice due to Hietarinta (2011)

$$z\,x_1 - y_1 - x = 0$$

$$z\,x_2 - y_2 - x = 0$$

$$z_{12} - \frac{y}{x} - \frac{1}{x}\left(\frac{px_1 - qx_2}{z_1 - z_2}\right) = 0$$

System has two single-edge equations and one full-face equation.

- Lax pair:

$$L = t\begin{bmatrix} \frac{yz}{x} & \frac{k}{x} & \frac{kx - px_1 z - yzz_1}{x} \\ -x_1 z & z_1 & xz_1 \\ z & 0 & -zz_1 \end{bmatrix}$$

and

$$M = s \begin{bmatrix} \frac{yz}{x} & \frac{k}{x} & \frac{kx - qx_2z - yzz_2}{x} \\ -x_2z & z_2 & xz_2 \\ z & 0 & -zz_2 \end{bmatrix},$$

where $t = s = \frac{1}{z}$, or $t = \frac{1}{z_1}$, $s = \frac{1}{z_2}$,

or $t = \sqrt[3]{\frac{x}{x_1 z^2 z_1}}$, $s = \sqrt[3]{\frac{x}{x_2 z^2 z_2}}$.

Here, $x_3 = \frac{h}{G}, y_3 = \frac{g}{G}, z_3 = \frac{f}{G}$, $\psi = \begin{bmatrix} f \\ g \\ G \end{bmatrix}$, and

$\frac{t_2}{t} \frac{s}{s_1} = \frac{z_1}{z_2}$.

- Example: Discrete Boussinesq System (Tongas and Nijhoff 2005)

$$
\begin{aligned}
z_1 - x\,x_1 + y &= 0 \\
z_2 - x\,x_2 + y &= 0 \\
(x_2 - x_1)(z - x\,x_{12} + y_{12}) - p + q &= 0
\end{aligned}
$$

- Lax pair:

$$
\boxed{L = t \begin{bmatrix} -x_1 & 1 & 0 \\ -y_1 & 0 & 1 \\ p - k - xy_1 + x_1 z & -z & x \end{bmatrix}}
$$

$$
M = s \begin{bmatrix} -x_2 & 1 & 0 \\ -y_2 & 0 & 1 \\ q - k - xy_2 + x_2 z & -z & x \end{bmatrix}
$$

with $t = s = 1$, or $t = \frac{1}{\sqrt[3]{p-k}}$ and $s = \frac{1}{\sqrt[3]{q-k}}$.

Here, $x_3 = \frac{f}{F}$, $y_3 = \frac{g}{F}$, $\psi = \begin{bmatrix} f \\ F \\ g \end{bmatrix}$, and $\frac{t_2}{t} \frac{s}{s_1} = 1$.

- Example: System of pKdV Lattices

  (Xenitidis and Mikhailov 2009)

$$\begin{aligned}(x - x_{12})(y_1 - y_2) - p^2 + q^2 &= 0\\ (y - y_{12})(x_1 - x_2) - p^2 + q^2 &= 0\end{aligned}$$

- Lax pair:

$$L = \begin{bmatrix} 0 & 0 & tx & t(p^2 - k^2 - xy_1) \\ 0 & 0 & t & -ty_1 \\ Ty & T(p^2 - k^2 - x_1 y) & 0 & 0 \\ T & -Tx_1 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & sx & s(q^2 - k^2 - xy_2) \\ 0 & 0 & s & -sy_2 \\ Sy & S(q^2 - k^2 - x_2 y) & 0 & 0 \\ S & -Sx_2 & 0 & 0 \end{bmatrix}$$

with $t = s = T = S = 1$,

or $t\,T = \frac{1}{\sqrt{\mathrm{Det} L_c}} = \frac{1}{p-k}$ and $s\,S = \frac{1}{\sqrt{\mathrm{Det} M_c}} = \frac{1}{q-k}$.

Here, $x_3 = \frac{f}{F}$, $y_3 = \frac{g}{G}$, $\psi = [f \;\; F \;\; g \;\; G]^{\mathrm{T}}$, and $\frac{t_2}{T}\,\frac{S}{s_1} = 1$ and $\frac{T_2}{t}\,\frac{s}{S_1} = 1$,

or $\frac{\mathcal{T}_2}{\mathcal{T}}\,\frac{\mathcal{S}}{\mathcal{S}_1} = 1$, with $\mathcal{T} = t\,T$, $\mathcal{S} = s\,S$.

- Example: Discrete NLS System (Xenitidis and Mikhailov 2009)

$$\begin{aligned} y_1 - y_2 - y\left((x_1 - x_2)y + p - q\right) &= 0 \\ x_1 - x_2 + x_{12}\left((x_1 - x_2)y + p - q\right) &= 0 \end{aligned}$$

- Lax pair:

$$\boxed{L = t \begin{bmatrix} -1 & x_1 \\ y & k - p - yx_1 \end{bmatrix}}$$

$$\boxed{M = s \begin{bmatrix} -1 & x_2 \\ y & k - q - yx_2 \end{bmatrix}}$$

with $t = s = 1$, or $t = \frac{1}{\sqrt{\mathrm{Det} L_c}} = \frac{1}{\sqrt{\alpha - k}}$ and $s = \frac{1}{\sqrt{\beta - k}}$.

Here, $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and $\frac{t_2}{t} \frac{s}{s_1} = 1$.

- Example: Schwarzian-Boussinesq Lattice (Nijhoff 1999)

$$
\begin{aligned}
y\, z_1 - x_1 + x &= 0 \\
y\, z_2 - x_2 + x &= 0 \\
z\, y_{12}\, (y_1 - y_2) - y(p\, y_2\, z_1 - q\, y_1\, z_2) &= 0
\end{aligned}
$$

- Lax pair:

$$
\boxed{L = t \begin{bmatrix} y & 0 & -y z_1 \\ -\frac{k y y_1}{z} & \frac{p y z_1}{z} & 0 \\ 0 & -1 & y_1 \end{bmatrix}}
$$

$$M = s \begin{bmatrix} y & 0 & -yz_2 \\ -\frac{kyy_2}{z} & \frac{pyz_2}{z} & 0 \\ 0 & -1 & y_2 \end{bmatrix}$$

with $t = s = \frac{1}{y}$, or $t = \frac{1}{y_1}$ and $s = \frac{1}{y_2}$,

or $t = \sqrt[3]{\frac{z}{y^2 y_1 z_1}}$ and $s = \sqrt[3]{\frac{z}{y^2 y_2 z_2}}$.

Here, $x_3 = \frac{fy+Fx}{F}$, $y_3 = \frac{g}{F}$, $z_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ g \\ F \end{bmatrix}$,

and $\frac{t_2}{t} \frac{s}{s_1} = \frac{y_1}{y_2}$.

- Example: Toda modified Boussinesq System (Nijhoff 1992)

$$y_{12}(p-q+x_2-x_1)-(p-1)y_2+(q-1)y_1=0$$

$$y_1y_2(p-q-z_2+z_1)-(p-1)yy_2+(q-1)yy_1=0$$

$$y(p+q-z-x_{12})(p-q+x_2-x_1)-(p^2+p+1)y_1 + (q^2+q+1)y_2=0$$

- Lax pair:

$$L=t\begin{bmatrix} k+p-z & \frac{1+k+k^2}{y} & \frac{-k^2y-y_1-p^2(y_1-y)-ky(x_1-z)+yzx_1}{y} - \frac{p(y_1+yx_1+yz)}{y} \\ 0 & p-1 & (1-k)y_1 \\ 1 & 0 & p-k-x_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} k+q-z & \frac{1+k+k^2}{y} & \frac{-k^2y-y_2-q^2(y_2-y)-ky(x_2-z)+yzx_2}{y} \\ & & -\frac{q(y_2+yx_2+yz)}{y} \\ 0 & q-1 & (1-k)y_2 \\ 1 & 0 & q-k-x_2 \end{bmatrix}$$

with $t = s = 1$, or $t = \sqrt[3]{\frac{y_1}{y}}$ and $s = \sqrt[3]{\frac{y_2}{y}}$.

Here, $x_3 = \frac{f}{F}$, $y_3 = \frac{g}{F}$, $\psi = \begin{bmatrix} f \\ g \\ F \end{bmatrix}$,

and $\frac{t_2}{t} \frac{s}{s_1} = 1$.