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Symbolic Computation of Conserved Densities, Generalized Symmetries, and Recursion Operators for Nonlinear Differential-Difference Equations

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Outline

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- Generalized symmetries for DDEs
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Part I Purpose, Motivation, Strategy, Demo

- **Purpose**

  Design and implement algorithms to compute polynomial conservation laws, generalized symmetries, and recursion operators for nonlinear systems of differential-difference equations (DDEs).

- **Motivation**

  – Conservation laws describe the conservation of physical quantities (linear momentum, energy, etc.).

    Compare with constants of motion (linear momentum, energy) in mechanics.

  – Conservation laws help in the study of quantitative and qualitative properties of DDEs and their solutions.

  – Conserved densities can be used to test numerical integrators.

  – The existence of a sufficiently large (in principal infinite) number of conservation laws or symmetries assures complete **integrability**.

  – Conserved densities and symmetries aid in finding the recursion operator (which guarantees the existence of infinitely many symmetries).

Definitions and Examples for DDEs (lattices)

- **Nonlinear system of DDEs**

  (continuous in time, discretized in space)

$$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...),$$

  $\mathbf{u}_n = (u_{1,n}, u_{2,n}, \cdots, u_{m,n})$ and $\mathbf{F} = (F_1, F_2, \cdots, F_m)$ are vector dynamical variables.

  In practice: denote components of $\mathbf{u}_n$ by $(u_n, v_n, w_n, \cdots)$.

  $\mathbf{F}$ is polynomial with constant coefficients (parameters).

  No restrictions on the level of the shifts or the degree of nonlinearity.

- **Typical Examples**

  ⋆ The Kac-van Moerbeke lattice

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}).$$

  ⋆ The (quadratic) Volterra lattice

$$\dot{u}_n = u_n^2(u_{n+1} - u_{n-1}).$$

  ⋆ One-dimensional Toda lattice

$$\ddot{y}_n = \exp{(y_{n-1} - y_n)} - \exp{(y_n - y_{n+1})}.$$

  y_n is the displacement from equilibrium of the nth particle with unit mass under an exponentially decaying interaction force between nearest neighbors.

  Change of variables:

$$u_n = \dot{y}_n, \qquad v_n = \exp{(y_n - y_{n+1})}$$

  yields

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}).$$

⋆ The Ablowitz and Ladik lattice

$$i\,\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \kappa u_n^* u_n (u_{n+1} + u_{n-1}),$$

is an integrable discretization of the NLS equation:

$$iu_t + u_{xx} + \kappa u^2 u^* = 0$$

u_n^* is the complex conjugate of u_n.

Treat u_n and $v_n = u_n^*$ as independent variables and add the complex conjugate equation. Set $\kappa = 1$ (scaling) and absorb i in scale on t :

$$\begin{aligned} \dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}), \\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

⋆ The Taha-Herbst lattice

$$\begin{aligned} \dot{u}_n = & -(1 + \alpha h^2 u_n + \beta h^2 u_n^2) \left\{ \tfrac{1}{h^3} \left(\tfrac{1}{2} u_{n+2} - u_{n+1} + u_{n-1} - \tfrac{1}{2} u_{n-2} \right) \right. \\ & + \tfrac{\alpha}{2h} [u_{n+1}^2 - u_{n-1}^2 + u_n (u_{n+1} - u_{n-1}) + u_{n+1} u_{n+2} - u_{n-1} u_{n-2}] \\ & \left. + \tfrac{\beta}{2h} [u_{n+1}^2 (u_{n+2} + u_n) - u_{n-1}^2 (u_{n-2} + u_n)] \right\}, \end{aligned}$$

is an integrable discretization of a combined KdV-mKdV equation

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx} = 0.$$

Discretizations the KdV and mKdV equations are special cases.

⋆ The Belov-Chaltikian lattice:

$$\begin{aligned} \dot{u}_n &= u_n (u_{n+1} - u_{n-1}) + v_{n-1} - v_n, \\ \dot{v}_n &= v_n (u_{n+2} - u_{n-1}). \end{aligned}$$

⋆ The Blaszak-Marciniak three field lattice:

$$\begin{aligned}\dot{u}_n &= w_{n+1} - w_{n-1},\\ \dot{v}_n &= u_{n-1}w_{n-1} - u_n w_n,\\ \dot{w}_n &= w_n(v_n - v_{n+1}).\end{aligned}$$

⋆ The Blaszak-Marciniak four field lattice:

$$\begin{aligned}\dot{u}_n &= v_{n-1}z_n - v_n z_{n+1},\\ \dot{v}_n &= w_{n-1}z_n - w_n z_{n+2},\\ \dot{w}_n &= z_{n+3} - z_n,\\ \dot{z}_n &= z_n(u_{n-1} - u_n).\end{aligned}$$

⋆ The relativistic Toda lattice:

$$\begin{aligned}\dot{u}_n &= (1 + \alpha u_n)(v_n - v_{n-1}),\\ \dot{v}_n &= v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).\end{aligned}$$

• Dilation Invariance of DDEs

$\star$ The Kac-van Moerbeke lattice

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}).$$

is invariant under the scaling symmetry

$$(t, u_n) \rightarrow (\lambda^{-1} t, \lambda u_n).$$

Weight $w(u_n)$ is defined in terms of t-derivatives.

Using $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$ and $w(u_{n\pm p}) = w(u_n)$,

$$w(u_n) + 1 = 2w(u_n).$$

Hence, $w(u_n) = 1$.

$\star$ The Toda lattice

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}).$$

is invariant under the scaling symmetry

$$(t, u_n, v_n) \rightarrow (\lambda^{-1} t, \lambda u_n, \lambda^2 v_n).$$

Weights $w(u_n), w(v_n)$ are defined in terms of t-derivatives.

Using $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$, $w(u_{n\pm p}) = w(u_n)$, $w(v_{n\pm p}) = w(v_n)$

$$\begin{aligned} w(u_n) + 1 &= w(v_n), \\ w(v_n) + 1 &= w(v_n) + w(u_n). \end{aligned}$$

Hence,

$$w(u_n) = 1, \quad w(v_n) = 2.$$

The **rank** of a monomial is its total weight in terms of t-derivatives.

- **Conservation Law for DDEs**:

$$\dot{\rho}_n = J_n - J_{n+1} \quad \text{on DDE},$$

density ρ_n, flux J_n.

$$\frac{\mathrm{d}}{\mathrm{dt}}(\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1})$$

if J_n is bounded for all n.

Subject to suitable boundary or periodicity conditions

$$\sum_n \rho_n = \text{constant}.$$

First three density-flux pairs (computed by hand) for Toda lattice:

$$\begin{array}{ll} \rho_n^{(0)} = \ln(v_n) & \qquad J_n^{(0)} = u_n \\ \rho_n^{(1)} = u_n & \qquad J_n^{(1)} = v_{n-1} \\ \rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n & \qquad J_n^{(2)} = u_n v_{n-1} \end{array}$$

- **Generalized Symmetries of DDEs**

A vector function $\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ is a *symmetry* iff

$$\mathbf{u}_n \rightarrow \mathbf{u}_n + \epsilon \mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

leaves the DDE system invariant within order ϵ.
$\mathbf{G}$ must satisfy the linearized equation

$$\mathrm{D}_t \mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}] = \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u}_n + \epsilon \mathbf{G})|_{\epsilon=0} = \sum_{k=-q}^{p} (\mathrm{D}^k \mathbf{G}) \frac{\partial \mathbf{F}}{\partial u_{n+k}},$$

where $\mathbf{F}'$ is the Fréchet derivative of $\mathbf{F}$ in direction of $\mathbf{G}$.

D is **up-shift operator**, D^{-1} is **down-shift operator**,
and $\mathrm{D}^i = \mathrm{D} \circ \mathrm{D} \circ \cdots \circ \mathrm{D}$ (i times).

- **Examples**

$\star$ Kac-van Moerbeke lattice

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}).$$

Higher order symmetries of rank (2,3)

$$\begin{aligned} G^{(1)} &= u_n(u_{n+1} - u_{n-1}), \\ G^{(2)} &= u_n u_{n+1}(u_n + u_{n+1} + u_{n+2}) - u_{n-1} u_n (u_{n-2} + u_{n-1} + u_n). \end{aligned}$$

$\star$ Toda lattice

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}).$$

First three higher-order symmetries:

$$\begin{aligned} \mathbf{G}^{(1)} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathbf{G}^{(2)} &= \begin{pmatrix} v_n - v_{n-1} \\ v_n(u_n - u_{n+1}) \end{pmatrix} \\ \mathbf{G}^{(3)} &= \begin{pmatrix} v_n(u_n + u_{n+1}) - v_{n-1}(u_{n-1} + u_n) \\ v_n(u_{n+1}^2 - u_n^2 + v_{n+1} - v_{n-1}) \end{pmatrix} \end{aligned}$$

- **Recursion Operators of DDEs.**

A *recursion operator* $\mathcal{R}$ connects symmetries

$$\mathbf{G}^{(j+s)} = \mathcal{R}\mathbf{G}^{(j)}, \ j = 1, 2, ...,$$

s is seed. For r-component systems, $\mathcal{R}$ is an $r \times r$ matrix.

Defining equation for $\mathcal{R}$:

$$\mathrm{D}_t\mathcal{R} + [\mathcal{R}, \mathbf{F}'(\mathbf{u}_n)] = \frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[\mathbf{F}] + \mathcal{R} \circ \mathbf{F}'(\mathbf{u}_n) - \mathbf{F}'(\mathbf{u}_n) \circ \mathcal{R} = 0,$$

where [,] means commutator, $\circ$ stands for composition, and

$$\mathbf{F}'(\mathbf{u}_n) = \sum_{k=-q}^{p} (\frac{\partial \mathbf{F}}{\partial u_{n+k}})\, \mathrm{D}^k$$

p, q are bounds of the shifts, D is up-shift operator and
$\mathrm{D}^k = \mathrm{D} \circ \mathrm{D} \circ \cdots \circ \mathrm{D}$ (k times).

$\mathcal{R}'[\mathbf{F}]$ is the Fréchet derivative of $\mathcal{R}$ in direction of $\mathbf{F}$:

$$\mathcal{R}'[\mathbf{F}] = \sum_{k=-q}^{p} (\mathrm{D}^k\mathbf{F})\, \frac{\partial \mathcal{R}}{\partial u_{n+k}}$$

Example 1

The Kac-van Moerbeke lattice

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}),$$

has recursion operator

$$\begin{aligned} \mathcal{R} &= u_n\mathrm{D} + u_n\mathrm{D}^{-1} + (u_n + u_{n+1})\mathrm{I} + u_n(u_{n+1} - u_{n-1})(\mathrm{D} - \mathrm{I})^{-1}\frac{1}{u_n}\mathrm{I} \\ &= u_n(\mathrm{I} + \mathrm{D})(u_n\mathrm{D} - \mathrm{D}^{-1}u_n)(\mathrm{D} - \mathrm{I})^{-1}\frac{1}{u_n}\mathrm{I} \end{aligned}$$

Note: $\rho_n^{(0)} = \ln(u_n)$ and $J_n^{(0)} = -(u_n + u_{n-1})$ are density-flux pair.

Example 2

The (quadratic) Volterra equation

$$\dot{u}_n = u_n^2(u_{n+1} - u_{n-1})$$

has recursion operator

$$\mathcal{R} = u_n^2 \mathrm{D} + u_n^2 \mathrm{D}^{-1} + 2u_n u_{n+1} \mathrm{I} + 2u_n^2(u_{n+1} - u_{n-1})(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{u_n} \mathrm{I}$$

Example 3

The Toda lattice

$$\dot{u}_n = v_{n-1} - v_n \qquad \dot{v}_n = v_n(u_n - u_{n+1})$$

has recursion operator

$$\mathcal{R} = \begin{pmatrix} -u_n \mathrm{I} & -\mathrm{D}^{-1} - \mathrm{I} + (v_{n-1} - v_n)(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \\ -v_n \mathrm{I} - v_n \mathrm{D} & u_{n+1} \mathrm{I} + v_n(u_n - u_{n+1})(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \end{pmatrix}$$

The recursion operator can be factored as

$$\mathcal{R} = \mathcal{H}\mathcal{S}$$

with Hamiltonian (symplectic) operator

$$\mathcal{H} \;=\; \begin{pmatrix} \mathrm{D}^{-1} v_n \mathrm{I} - v_n \mathrm{D} & -u_n v_n \mathrm{I} + u_n \mathrm{D}^{-1} v_n \mathrm{I} \\ -v_n \mathrm{D} u_n \mathrm{I} + u_n v_n \mathrm{I} & -v_n \mathrm{D} v_n \mathrm{I} + v_n \mathrm{D}^{-1} v_n \mathrm{I} \end{pmatrix}$$

and co-symplectic operator

$$\mathcal{S} \;=\; \begin{pmatrix} 0 & (\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \\ \frac{1}{v_n} \mathrm{D}(\mathrm{D} - \mathrm{I})^{-1} & 0 \end{pmatrix}$$

- **Key Observation**

  Conserved densities, generalized symmetries, and recursion operators are invariant under the dilation (scaling) symmetry of the given DDE.

- **Overall Strategy**

  Exploit dilation symmetry as much as possible.

  Keep the computations as simple as possible.

  Use linear algebra

  * solve linear systems

  * construct basis vectors (building blocks)

  * use linear independence

  * work in finite dimensional spaces

  Use calculus and differential equations

  * derivatives

  * integrals (as little as possible)

  * solve systems of linear ODEs

  Use tools from variational calculus

  * variational derivative (Euler operator)

  * higher Euler operators and homotopy operator

  * Fréchet derivative

  * calculus with operators

  Use analogy between continuous and semi-discrete cases

Analogy PDEs and DDEs

	Continuous Case (PDEs)	Semi-discrete Case (DDEs)
System	$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$	$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$
Conservation Law	$\mathrm{D}_t \rho + \mathrm{D}_x J = 0$	$\dot{\rho}_n + J_{n+1} - J_n = 0$
Symmetry	$\mathrm{D}_t \mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u} + \epsilon \mathbf{G})\|_{\epsilon=0}$	$\mathrm{D}_t \mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}] = \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u}_n + \epsilon \mathbf{G})\|_{\epsilon=0}$
Recursion Operator	$\mathrm{D}_t \mathcal{R} + [\mathcal{R}, \mathbf{F}'(u)] = 0$	$\mathrm{D}_t \mathcal{R} + [\mathcal{R}, \mathbf{F}'(\mathbf{u}_n)] = 0$

Table 1: Conservation Laws and Symmetries

	KdV Equation	Volterra Lattice
Equation	$u_t = 6 u u_x + u_{3x}$	$\dot{u}_n = u_n \left(u_{n+1} - u_{n-1}\right)$
Densities	$\rho = u, \quad \rho = u^2$ $\rho = u^3 - \frac{1}{2} u_x^2$	$\rho_n = u_n, \quad \rho_n = u_n(\frac{1}{2} u_n + u_{n+1})$ $\rho_n = \frac{1}{3} u_n^3 + u_n u_{n+1}(u_n + u_{n+1} + u_{n+2})$
Symmetries	$G = u_x, \quad G = 6 u u_x + u_{3x}$ $G = 30 u^2 u_x + 20 u_x u_{2x} + 10 u u_{3x} + u_{5x}$	$G = u_n u_{n+1} \left(u_n + u_{n+1} + u_{n+2}\right) - u_{n-1} u_n (u_{n-2} + u_{n-1} + u_n)$
Recursion Operator	$\mathcal{R} = \mathrm{D}_x^2 + 4u + 2 u_x \mathrm{D}_x^{-1}$	$\mathcal{R} = u_n (\mathrm{I} + \mathrm{D})(u_n \mathrm{D} - \mathrm{D}^{-1} u_n)(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{u_n}$

Table 2: Prototypical Examples

Review of Algorithm for Conserved Densities of PDEs

(i) Determine weights (scaling properties) of variables and auxiliary parameters.

(ii) Construct the form of the density (find monomial building blocks).

(iii) Determine the constant coefficients (parameters).

(iv) Compute the flux with the homotopy operator.

Example: Density of **rank** 6 for the KdV equation

$$u_t + uu_x + u_{3x} = 0$$

Step 1: **Compute the weights (dilation symmetry).**

Solve

$$w(u) + w(\mathrm{D}_t) = 2w(u) + 1 = w(u) + 3.$$

Hence,

$$w(u) = 2, \quad w(\mathrm{D}_t) = 3.$$

Step 2: **Determine the form of the density.**

List all possible powers of u, up to rank 6 : $\quad [u, u^2, u^3]$.

Introduce x derivatives to 'complete' the rank.

u has weight 2, introduce D_x^4.

u^2 has weight 4, introduce D_x^2.

u^3 has weight 6, no derivative needed.

Apply the D_x derivatives.

Remove total derivative terms ($\mathrm{D}_x u_{px}$) and highest derivative terms:

$$
\begin{aligned}
[u_{4x}] &\rightarrow [\,] \quad \text{empty list.}\\
[{u_x}^2, uu_{2x}] &\rightarrow [{u_x}^2] \quad \text{since } uu_{2x} = (uu_x)_x - {u_x}^2.\\
[u^3] &\rightarrow [u^3].
\end{aligned}
$$

Linearly combine the 'building blocks':

$$\rho = c_1 u^3 + c_2 {u_x}^2.$$

Step 3: **Determine the coefficients c_i.**

Use the defining equation

$$\mathrm{D}_t \rho + \mathrm{D}_x J = 0 \quad \text{(on PDE)},$$

Compute

$$
\begin{aligned}
E &= \mathrm{D}_t \rho = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{m} \frac{\partial \rho}{\partial u_{kx}} \mathrm{D}_x^k u_t = \frac{\partial \rho}{\partial t} + \rho'(u)[F]\\
&= 3c_1 u^2 u_t + 2c_2 u_x u_{xt}\\
&= -3c_1 u^2 (uu_x + u_{3x}) - 2c_2 u_x (uu_x + u_{3x})_x.\\
&= -(3c_1 u^3 u_x + 3c_1 u^2 u_{3x} + 2c_2 u_x^3 + 2c_2 uu_x u_{2x} + 2c_2 u_x u_{4x}).
\end{aligned}
$$

Apply the Euler operator (continuous variational derivative)

$$
\begin{aligned}
\mathcal{L}_{\mathbf{u}}^{(0)} &= \sum_{k=0}^{m} (-\mathrm{D}_x)^k \frac{\partial}{\partial \mathbf{u}_{kx}}\\
&= \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial \mathbf{u}_x} + \mathrm{D}_x^2 \frac{\partial}{\partial \mathbf{u}_{2x}} + \cdots + (-1)^m \mathrm{D}_x^m \frac{\partial}{\partial \mathbf{u}_{mx}}.
\end{aligned}
$$

to E of order $m = 4$. Result:

$$\mathcal{L}_u^{(0)}(E) = -6(3c_1 + c_2) u_x u_{xx} \equiv 0$$

So, $c_1 = -\frac{1}{3} c_2$. Set $c_2 = -3$, then $c_1 = 1$.

Hence,

$$\rho = u^3 - 3{u_x}^2.$$

Step 4: **Compute the flux** J.

– Method 1: Integrate by parts (simple cases)

Integration of $\mathrm{D}_x J = -E$ yields

$$J = \frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2 u_{2x} + 3{u_{2x}}^2 - 6u_x u_{3x}.$$

– Method 2: Build the form of J (cumbersome)

Note: Rank J = Rank ρ + Rank $\mathrm{D}_t - 1$.

Build up form of J. Compute

$$\mathrm{D}_x J = \frac{\partial J}{\partial x} + \sum_{k=0}^{m} \frac{\partial J}{\partial u_{kx}} u_{(k+1)x},$$

m is the order of J. Match $\mathrm{D}_x J = -E$.

– Method 3: Use the homotopy operator (most powerful)

Higher Euler Operators:

$$\mathcal{L}_{\mathbf{u}}^{(i)} = \sum_{k=i}^{\infty} \binom{k}{i} (-D_x)^{k-i} \frac{\partial}{\partial \mathbf{u}_{kx}}.$$

Examples (scalar case, $\mathbf{u} = u_1 = u$):

$$\mathcal{L}_u^{(0)} = \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{2x}} - \mathrm{D}_x^3 \frac{\partial}{\partial u_{3x}} + \cdots$$

$$\mathcal{L}_u^{(1)} = \frac{\partial}{\partial u_x} - 2\mathrm{D}_x \frac{\partial}{\partial u_{2x}} + 3\mathrm{D}_x^2 \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x^3 \frac{\partial}{\partial u_{4x}} + \cdots$$

$$\mathcal{L}_u^{(2)} = \frac{\partial}{\partial u_{2x}} - 3\mathrm{D}_x \frac{\partial}{\partial u_{3x}} + 6\mathrm{D}_x^2 \frac{\partial}{\partial u_{4x}} - 10\mathrm{D}_x^3 \frac{\partial}{\partial u_{5x}} + \cdots$$

$$\mathcal{L}_u^{(3)} = \frac{\partial}{\partial u_{3x}} - 4\mathrm{D}_x \frac{\partial}{\partial u_{4x}} + 10\mathrm{D}_x^2 \frac{\partial}{\partial u_{5x}} - 20\mathrm{D}_x^3 \frac{\partial}{\partial u_{6x}} + \cdots$$

The flux is

$$J(\mathbf{u}) = \int_0^1 \sum_{r=1}^{n} j_r(\mathbf{u})[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}.$$

where

$$j_r(\mathbf{u}) = \sum_{i=0}^{m-1} D_x^i (u_r \mathcal{L}_{u_r}^{(i+1)}(-E))$$

m is the order of E, and $j_r(\mathbf{u})[\lambda \mathbf{u}]$ means
$\mathbf{u} \rightarrow \lambda \mathbf{u}, \ \mathbf{u}_x \rightarrow \lambda \mathbf{u}_x, \ \mathbf{u}_{2x} \rightarrow \lambda \mathbf{u}_{2x}$, etc.

Demonstration (scalar case, $\mathbf{u} = u_1 = u, \ j_1(\mathbf{u}) = j(u)$):

Compute J via the homotopy operator!

$$-E = 3u^3 u_x + 3u^2 u_{3x} - 6u_x^3 - 6uu_x u_{2x} - 6u_x u_{4x}.$$

i	$\mathcal{L}_u^{(i+1)}(-E)$	$D_x^i(u\mathcal{L}_u^{(i+1)}(-E))$
0	$3u^3+24uu_{2x}+18u_{4x}+12u_x^2$	$3u^4+24u^2u_{2x}+18uu_{4x}+12uu_x^2$
1	$-24uu_x-36u_{3x}$	$-48uu_x^2-24u^2u_{2x}-36u_xu_{3x}-36uu_{4x}$
2	$3u^2+24u_{2x}$	$18uu_x^2+9u^2u_{2x}+24u_{2x}^2+48u_xu_{3x}+24uu_{4x}$
3	$-6u_x$	$-18u_{2x}^2-24u_xu_{3x}-6uu_{4x}$

Hence,

$$j(u) = 3u^4 - 18uu_x^2 - 12u_x u_{3x} + 9u^2 u_{2x} + 6u_{2x}^2.$$

Thus, the homotopy operator gives

$$\begin{aligned} J(u) &= \int_0^1 j(u)[\lambda u] \, \frac{d\lambda}{\lambda} \\ &= \int_0^1 (3\lambda^3 u^4 - 18\lambda^2 u u_x^2 - 12\lambda u_x u_{3x} + 9\lambda^2 u^2 u_{2x} + 6\lambda u_{2x}^2) \, d\lambda \\ &= \frac{3}{4} u^4 - 6u{u_x}^2 - 6u_x u_{3x} + 3u^2 u_{2x} + 3{u_{2x}}^2. \end{aligned}$$

Analogy PDEs and DDEs

Conservation laws for PDEs

$$\mathrm{D}_t \rho + \mathrm{D}_x J = 0$$

density ρ, flux J.

Compute $E = \mathrm{D}_t \rho$.

To guarantee the existence of J, apply the Euler operator

$$\begin{aligned}\mathcal{L}_{\mathbf{u}}^{(0)} &= \sum_{k=0}^{m} (-1)^k \mathrm{D}_x^k \frac{\partial}{\partial \mathbf{u}_{kx}} \\ &= \frac{\partial}{\partial \mathbf{u}} - \mathrm{D}_x(\frac{\partial}{\partial \mathbf{u}_x}) + \mathrm{D}_x^2(\frac{\partial}{\partial \mathbf{u}_{2x}}) + \cdots + (-1)^m \mathrm{D}_x^m(\frac{\partial}{\partial \mathbf{u}_{mx}}).\end{aligned}$$

to E of order m. D_x is the differential operator.

If $\mathcal{L}_{\mathbf{u}}^{(0)}(E) = 0$, then E is a total x-derivative $(-J_x)$.

If $\mathcal{L}_{\mathbf{u}}^{(0)}(E) \neq 0$, the nonzero terms must vanish identically.

E must be in the kernel of $\mathcal{L}_{\mathbf{u}}^{(0)}$ operator, or equivalently, E must be in the image of D_x operator.

Computation of flux J:

Apply the homotopy operator

$$J(u) = \int_0^1 \sum_{r=1}^{n} j_r(\mathbf{u})[\lambda \mathbf{u}] \, \frac{d\lambda}{\lambda}.$$

where $j_r(\mathbf{u})$ is computed with

$$j_r(\mathbf{u}) = \sum_{i=0}^{m-1} \mathrm{D}_x^i (u_r \mathcal{L}_{u_r}^{(i+1)}(-E))$$

with higher Euler operators (continuous):

$$\mathcal{L}_{\mathbf{u}}^{(i)} = \sum_{k=i}^{m} \binom{k}{i} (-\mathrm{D}_x)^{k-i} \frac{\partial}{\partial \mathbf{u}_{kx}}.$$

Conservation laws for DDEs

$$\dot{\rho}_n + J_{n+1} - J_n = 0$$

density ρ_n, flux J_n.

Compute $E = \dot{\rho}_n$.

To guarantee existence of J_n, apply the discrete Euler operator

$$\begin{aligned}
\mathcal{L}^{(0)}_{\mathbf{u}_n} &= \sum_{k=-q}^{p} \mathrm{D}^{-k} \frac{\partial}{\partial \mathbf{u}_{n+k}} \\
&= \frac{\partial}{\partial \mathbf{u}_n} + \mathrm{D}(\frac{\partial}{\partial \mathbf{u}_{n-1}}) + \mathrm{D}^2(\frac{\partial}{\partial \mathbf{u}_{n-2}}) + \cdots + \mathrm{D}^q(\frac{\partial}{\partial \mathbf{u}_{n-q}}) \\
&+ \mathrm{D}^{-1}(\frac{\partial}{\partial \mathbf{u}_{n+1}}) + \mathrm{D}^{-2}(\frac{\partial}{\partial \mathbf{u}_{n+2}}) + \cdots + \mathrm{D}^{-p}(\frac{\partial}{\partial \mathbf{u}_{n+p}})
\end{aligned}$$

to E with maximal negative and positive shifts on $\mathbf{u}$ are q and p.

D is the *up-shift* operator, D^{-1} the *down-shift* operator.

Applied to a monomial m

$$\mathrm{D}^{-1} m = m|_{n \to n-1} \quad \text{and} \quad Dm = m|_{n \to n+1}.$$

Note: D (up-shift operator) corresponds the differential operator D_x due to the forward difference

$$\frac{\partial J}{\partial x} \to \frac{J_{n+1} - J_n}{\Delta x} \quad (\Delta x = 1)$$

If $\mathcal{L}^{(0)}_{\mathbf{u}_n}(E) = 0$, then E matches $-(J_{n+1} - J_n)$.

If $\mathcal{L}^{(0)}_{\mathbf{u}_n}(E) \neq 0$, the nonzero terms must vanish identically.

In practice:

Compute $\tilde{E} = \mathrm{D}^q E$ (remove negative shifts) and apply

$$\mathcal{L}^{(0)}_{\mathbf{u}_n} = \frac{\partial}{\partial \mathbf{u}_n}(\sum_{k=0}^{p+q} \mathrm{D}^{-k}) = \frac{\partial}{\partial \mathbf{u}_n}(\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \cdots + \mathrm{D}^{-(p+q)})$$

Computation of flux $\tilde{J}_n$

Apply the homotopy operator

$$\tilde{J}_n = \int_0^1 \sum_{r=1}^{m} \tilde{j}_{r,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] \,\frac{d\lambda}{\lambda}.$$

where $\tilde{j}_{r,n}(\mathbf{u}_n)$ is computed with

$$\tilde{j}_{r,n}(\mathbf{u}_n) = \sum_{i=0}^{p+q-1} (\mathrm{D} - \mathrm{I})^i (u_{r,n} \mathcal{L}^{(i+1)}_{u_{r,n}}(-\tilde{E}))$$

with discrete higher Euler operators:

$$\mathcal{L}^{(i)}_{\mathbf{u}_n} = \frac{\partial}{\partial \mathbf{u}_n}(\sum_{k=i}^{p+q} \binom{k}{i} \mathrm{D}^{-k}).$$

Down-shift $\tilde{J}_n$ by q steps: $J_n = \mathrm{D}^{-q}\tilde{J}_n$.

Part II Algorithms for DDEs (lattices)

- **Tool: Up and Down Shift Operators**

D^{-1} and D are the *down-shift* and *up-shift* operators.

For a monomial m :

$$D^{-1}m = m|_{n\to n-1}, \quad \text{and} \quad Dm = m|_{n\to n+1}.$$

Example

$$D^{-1}u_{n+2}v_n = u_{n+1}v_{n-1}, \qquad\qquad Du_{n-2}v_{n-1} = u_{n-1}v_n.$$

Compositions of D^{-1} and D define an *equivalence relation*.
All shifted monomials are *equivalent*.

Example

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}.$$

- **Tool: Equivalence Criterion**

Two monomials m_1 and m_2 are equivalent, $m_1 \equiv m_2$, if

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial M_n.

Example: $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$u_{n-2}u_n = u_{n-1}u_{n+1}+[u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1}+[M_n - M_{n+1}]$.

Main representative of an equivalence class is the monomial with label n on u (or v).

Example: u_nu_{n+2} is main representative of class $\{u_{n-1}u_{n+1}, u_{n+1}u_{n+3}, \cdots\}$.

Use lexicographical ordering to resolve conflicts.

u_nv_{n+2} (not $u_{n-2}v_n$) is the main representative of class $\{u_{n-3}v_{n-1}, u_{n+2}v_{n+4}, \cdots\}$

- **Algorithm for Conserved Densities of DDEs.**

Three-step algorithm to find conserved densities:

(i) Determine the weights.
(ii) Construct the form of density.
(iii) Determine the coefficients.
(iv) Compute the flux with the discrete homotopy operator.

Example: Density of rank 3 of the Toda lattice,

$$\dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}).$$

Step 1: **Compute the weights.**

Require uniformity in rank for each equation:

$$w(u_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) = w(v_{n-1}) = w(v_n),$$
$$w(v_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) = w(v_n) + w(u_n) = w(v_n) + w(u_{n+1})$$

Weights are shift invariant. Set $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$ and solve the linear system: $w(u_n) = w(u_{n+1}) = 1$ and $w(v_n) = w(v_{n-1}) = 2$.

Step 2: **Construct the form of the density.**

List all monomials[1] in u_n and v_n of rank 3 or less:

$$\mathcal{G} = \{u_n^3, u_n^2, u_n v_n, u_n, v_n\}.$$

For each monomial in $\mathcal{G}$, introduce enough t-derivatives to obtain weight 3. Use the DDE to remove $\dot{u}_n$ and $\dot{v}_n$:

$$\frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n^3) = u_n^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,$$

[1]In general algorithm shifts are also needed: $u_n^3, u_n u_{n+1} u_{n-1}, u_n^2 u_{n+1}$, etc.

$$\frac{\mathrm{d}}{\mathrm{dt}}(u_n^2) = 2u_n v_{n-1} - 2u_n v_n,$$
$$\frac{\mathrm{d}}{\mathrm{dt}}(v_n) = u_n v_n - u_{n+1} v_n,$$
$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n.$$

Gather the resulting terms in a set

$$\mathcal{H} = \{u_n^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n\}.$$

Introduce main representatives.

Example: $u_n v_{n-1} \equiv u_{n+1} v_n$ are replaced by $u_n v_{n-1}$.

Linearly combine the monomials in

$$\mathcal{I} = \{u_n^3, u_n v_{n-1}, u_n v_n\}$$

to obtain

$$\rho_n = c_1\, u_n^3 + c_2\, u_n v_{n-1} + c_3\, u_n v_n.$$

Step 3: Determine the coefficients c_i.

Require that $\dot{\rho}_n + J_{n+1} - J_n = 0$ holds.

Compute $\dot{\rho}_n$. Use the DDE to remove $\dot{u}_n$ and $\dot{v}_n$. Thus,

$$\begin{aligned} E \;=\; \dot{\rho}_n &= (3c_1 - c_2) u_n^2 v_{n-1} + (c_3 - 3c_1) u_n^2 v_n + (c_3 - c_2) v_{n-1} v_n \\ &\quad + c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2. \end{aligned}$$

Shift E by $q = 1$ step up (remove negative shifts $n-1$). Apply

$$\mathcal{L}^{(0)}_{\mathbf{u}_n} = \frac{\partial}{\partial \mathbf{u}_n} \Big(\sum_{k=0}^{p+q} \mathrm{D}^{-k}\Big) = \frac{\partial}{\partial \mathbf{u}_n} (\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \cdots)$$

to $\tilde{E} = \mathrm{D}E$.

The maximal shift $p + q = 1 + 1 = 2$ on u_n. Hence,

$$
\begin{aligned}
\mathcal{L}^{(0)}_{u_n}(\tilde{E}) &= \frac{\partial}{\partial \mathbf{u}_n}(\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2})(\tilde{E}) \\
&= 2(3c_1 - c_2)u_n v_{n-1} + 2(c_3 - 3c_1)u_n v_n \\
&\quad + (c_2 - c_3)u_{n-1}v_{n-1} + (c_2 - c_3)u_{n+1}v_n \equiv 0
\end{aligned}
$$

The maximal shift $p + q = 0 + 1 = 1$ on v_n. Hence,

$$
\begin{aligned}
\mathcal{L}^{(0)}_{v_n}(\tilde{E}) &= = \frac{\partial}{\partial v_n}(\mathrm{I} + \mathrm{D}^{-1})(\tilde{E}) \\
&= (3c_1 - c_2)u_{n+1}^2 + (c_3 - c_2)v_{n+1} + (c_2 - c_3)u_n u_{n+1} \\
&\quad + 2(c_2 - c_3)v_n + (c_3 - 3c_1)u_n^2 + (c_3 - c_2)v_{n-1} \equiv 0.
\end{aligned}
$$

Solve the linear system

$$\mathcal{S} = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}.$$

The solution is $3c_1 = c_2 = c_3$. Choose $c_1 = \frac{1}{3}$, and $c_2 = c_3 = 1$:

Step 4: **Compute the flux** J_n.

- Method 1: Use equivalence criterion (simple cases)

  Start from

  $$E = \dot{\rho}_n = u_{n-1}u_n v_{n-1} + v_{n-1}^2 - u_n u_{n+1} v_n - v_n^2.$$

  Replace $u_{n-1}u_n v_{n-1}$ by $u_n u_{n+1} v_n + [u_{n-1}u_n v_{n-1} - u_n u_{n+1} v_n]$.
  Replace v_{n-1}^2 by $v_n^2 + [v_{n-1}^2 - v_n^2]$. Thus

  $$E = [u_{n-1}u_n v_{n-1} - u_n u_{n+1} v_n] + [v_{n-1}^2 - v_n^2]$$

  Group the first and second terms in the square brackets to match $[J_n - J_{n+1}]$.

  Hence

  $$E = [u_{n-1}u_n v_{n-1} + v_{n-1}^2] - [u_n u_{n+1} v_n + v_n^2].$$

  $$J_n = u_{n-1}u_n v_{n-1} + v_{n-1}^2.$$

– Method 2: Use the homotopy operator (most powerful)

Discrete higher Euler operators:

$$\mathcal{L}_{u_n}^{(i)} = \frac{\partial}{\partial u_n}(\sum_{k=i}^{p+q} \binom{k}{i} \mathrm{D}^{-k})$$

Examples (scalar case, $u_{1,n} = u_n$):

$$\begin{aligned}
\mathcal{L}_{u_n}^{(0)} &= \frac{\partial}{\partial u_n}(\mathrm{I} + \mathrm{D}^{-1} + \mathrm{D}^{-2} + \mathrm{D}^{-3} + \cdots) \\
\mathcal{L}_{u_n}^{(1)} &= \frac{\partial}{\partial u_n}(\mathrm{D}^{-1} + 2\mathrm{D}^{-2} + 3\mathrm{D}^{-3} + 4\mathrm{D}^{-4} + \cdots) \\
\mathcal{L}_{u_n}^{(2)} &= \frac{\partial}{\partial u_n}(\mathrm{D}^{-2} + 3\mathrm{D}^{-3} + 6\mathrm{D}^{-4} + 10\mathrm{D}^{-5} + \cdots) \\
\mathcal{L}_{u_n}^{(3)} &= \frac{\partial}{\partial u_n}(\mathrm{D}^{-3} + 4\mathrm{D}^{-4} + 10\mathrm{D}^{-5} + 20\mathrm{D}^{-6} + \cdots)
\end{aligned}$$

Similar formulas for $\mathcal{L}_{v_n}^{(i)}$.

The flux is

$$\tilde{J}_n = \int_0^1 (\tilde{j}_{1,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] + \tilde{j}_{2,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n])\frac{d\lambda}{\lambda}$$

where,

$$\begin{aligned}
\tilde{j}_{1,n}(\mathbf{u}_n) &= \sum_{i=0}^{p+q-1} (\mathrm{D} - \mathrm{I})^i (u_n \mathcal{L}_{u_n}^{(i+1)}(-\tilde{E})) \\
\tilde{j}_{2,n}(\mathbf{u}_n) &= \sum_{i=0}^{p+q-1} (\mathrm{D} - \mathrm{I})^i (v_n \mathcal{L}_{u_n}^{(i+1)}(-\tilde{E}))
\end{aligned}$$

Note that $p+q$ is the highest shift in $\tilde{E}$, and $\tilde{j}_{r,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n]$ means $\mathbf{u}_n \rightarrow \lambda \mathbf{u}_n,\ \mathbf{u}_{n+1} \rightarrow \lambda \mathbf{u}_{n+1},\ \mathbf{u}_{n+2} \rightarrow \lambda \mathbf{u}_{n+2},$ etc.

Demonstration (vector case, $\mathbf{u}_n = (u_n, v_n)$) :

Compute J_n via the homotopy operator!

Start from

$$-\tilde{E} = -\mathrm{D}E = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2.$$

To find: flux J_n such that $(\mathrm{D} - \mathrm{I}) J_n = -E$.

Homotopy operator inverts the operator $(\mathrm{D} - \mathrm{I})$.

i	$\mathcal{L}_{u_n}^{(i+1)}(-\tilde{E})$	$(\mathrm{D}-\mathrm{I})^i (u_n \mathcal{L}_{u_n}^{(i+1)}(-\tilde{E}))$
0	$u_{n-1} v_{n-1} + u_{n+1} v_n$	$u_n u_{n-1} v_{n-1} + u_n u_{n+1} v_n$
1	$u_{n-1} v_{n-1}$	$u_{n+1} u_n v_n - u_n u_{n-1} v_{n-1}$

i	$\mathcal{L}_{v_n}^{(i+1)}(-\tilde{E})$	$(\mathrm{D}-\mathrm{I})^i (v_n \mathcal{L}_{v_n}^{(i+1)}(-\tilde{E}))$
0	$u_n u_{n+1} + 2v_n$	$v_n u_n u_{n+1} + 2v_n^2$

Hence,

$$\tilde{j}_{1,n}(\mathbf{u}_n) = 2 u_n u_{n+1} v_n, \quad \tilde{j}_{2,n}(\mathbf{u}_n) = u_n u_{n+1} v_n + 2 v_n^2.$$

Thus, the homotopy operator gives

$$\begin{aligned} \tilde{J}_n &= \int_0^1 (\tilde{j}_{1,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n] + \tilde{j}_{2,n}(\mathbf{u}_n)[\lambda \mathbf{u}_n]) \, \frac{d\lambda}{\lambda} \\ &= \int_0^1 (3\lambda^2 u_n u_{n+1} v_n + 2\lambda v_n^2) \, d\lambda \\ &= u_n u_{n+1} v_n + v_n^2. \end{aligned}$$

Summary:

$\rho_n = \frac{1}{3} u_n^3 + u_n(v_{n-1} + v_n), \qquad J_n = \mathrm{D}^{-1} J_n = u_{n-1} u_n v_{n-1} + v_{n-1}^2.$

Analogously, conserved densities of rank ≤ 5:

$$\rho_n^{(1)} = u_n \qquad \rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n$$

$$\rho_n^{(3)} = \tfrac{1}{3}{u_n}^3 + u_n(v_{n-1} + v_n)$$

$$\rho_n^{(4)} = \tfrac{1}{4}{u_n}^4 + {u_n}^2(v_{n-1} + v_n) + u_n u_{n+1} v_n + \tfrac{1}{2}{v_n}^2 + v_n v_{n+1}$$

$$\begin{aligned} \rho_n^{(5)} &= \tfrac{1}{5}{u_n}^5 + {u_n}^3(v_{n-1} + v_n) + u_n u_{n+1} v_n(u_n + u_{n+1}) \\ &\quad + u_n v_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_n v_n(v_{n-1} + v_n + v_{n+1}). \end{aligned}$$

• Algorithm for Generalized Symmetries of DDEs.

Consider the Toda system

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}).$$

with

$$w(u_n) = 1 \quad \text{and} \quad w(v_n) = 2.$$

Compute the form of the symmetry of ranks $(3, 4)$, i.e. the first component of the symmetry has rank 3, the second rank 4.

Step 1: **Construct the form of the symmetry.**

List all monomials in u_n and v_n of rank 3 or less:

$$\mathcal{L}_1 = \{u_n^3, u_n^2, u_n v_n, u_n, v_n\},$$

and of rank 4 or less:

$$\mathcal{L}_2 = \{u_n^4, u_n^3, u_n^2 v_n, u_n^2, u_n v_n, u_n, v_n^2, v_n\}.$$

For each monomial in $\mathcal{L}_1$ and $\mathcal{L}_2$, introduce enough t-derivatives, so that each term exactly has rank 3 and 4, respectively.

Using the DDEs, for the monomials in $\mathcal{L}_1$:

$$\begin{aligned}
&\frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n^3) = u_n^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,\\
&\frac{\mathrm{d}}{\mathrm{dt}}(u_n^2) = 2u_n\dot{u}_n = 2u_n v_{n-1} - 2u_n v_n,\\
&\frac{\mathrm{d}}{\mathrm{dt}}(v_n) = \dot{v}_n = u_n v_n - u_{n+1}v_n,\\
&\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = \frac{\mathrm{d}}{\mathrm{dt}}(\dot{u}_n) = \frac{\mathrm{d}}{\mathrm{dt}}(v_{n-1} - v_n)\\
&\quad = u_{n-1}v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1}v_n.
\end{aligned}$$

Gather the resulting terms:

$$\mathcal{R}_1 = \{u_n^3, u_{n-1}v_{n-1}, u_nv_{n-1}, u_nv_n, u_{n+1}v_n\}.$$

$$\begin{aligned}\mathcal{R}_2 \;=\; &\{u_n^4, u_{n-1}^2v_{n-1}, u_{n-1}u_nv_{n-1}, u_n^2v_{n-1}, v_{n-2}v_{n-1}, v_{n-1}^2, u_n^2v_n,\\ &u_nu_{n+1}v_n, u_{n+1}^2v_n, v_{n-1}v_n, v_n^2, v_nv_{n+1}\}.\end{aligned}$$

Linearly combine the monomials in $\mathcal{R}_1$ and $\mathcal{R}_2$

$$\begin{aligned}G^{(1)} &= c_1\, u_n^3 + c_2\, u_{n-1}v_{n-1} + c_3\, u_nv_{n-1} + c_4\, u_nv_n + c_5\, u_{n+1}v_n,\\ G^{(2)} &= c_6\, u_n^4 + c_7\, u_{n-1}^2v_{n-1} + c_8\, u_{n-1}u_nv_{n-1} + c_9\, u_n^2v_{n-1}\\ &\quad + c_{10}\, v_{n-2}v_{n-1} + c_{11}\, v_{n-1}^2 + c_{12}\, u_n^2v_n + c_{13}\, u_nu_{n+1}v_n\\ &\quad + c_{14}\, u_{n+1}^2v_n + c_{15}\, v_{n-1}v_n + c_{16}\, v_n^2 + c_{17}\, v_nv_{n+1}.\end{aligned}$$

Step 2: Determine the unknown coefficients.

Require that the symmetry condition $\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}]$ holds.

Solution:

$$\begin{aligned}c_1 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0,\\ -c_2 = -c_3 = c_4 = c_5 = -c_{12} = c_{14} = -c_{15} = c_{17}.\end{aligned}$$

Therefore, with $c_{17} = 1$, the symmetry of rank $(3,4)$ is:

$$\begin{aligned}G^{(1)} &= u_nv_n - u_{n-1}v_{n-1} + u_{n+1}v_n - u_nv_{n-1},\\ G^{(2)} &= u_{n+1}^2v_n - u_n^2v_n + v_nv_{n+1} - v_{n-1}v_n.\end{aligned}$$

Analogously, the symmetry of rank $(4,5)$ reads

$$\begin{aligned}G^{(1)} &= u_n^2v_n + u_nu_{n+1}v_n + u_{n+1}^2v_n + v_n^2 + v_nv_{n+1} - u_{n-1}^2v_{n-1}\\ &\quad - u_{n-1}u_nv_{n-1} - u_n^2v_{n-1} - v_{n-2}v_{n-1} - v_{n-1}^2,\\ G^{(2)} &= u_{n+1}v_n^2 + 2u_{n+1}v_nv_{n+1} + u_{n+2}v_nv_{n+1} - u_n^3v_n + u_{n+1}^3v_n\\ &\quad - u_{n-1}v_{n-1}v_n - 2u_nv_{n-1}v_n - u_nv_n^2.\end{aligned}$$

• Recursion Operators of DDEs.

Key Observation

$\star$ Recursion operator for the Kac-van Moerbeke lattice

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}),$$

is

$$\begin{aligned}\mathcal{R} &= u_n\mathrm{D} + u_n\mathrm{D}^{-1} + (u_n + u_{n+1})\mathrm{I} + u_n(u_{n+1} - u_{n-1})(\mathrm{D} - \mathrm{I})^{-1}\frac{1}{u_n}\mathrm{I} \\ &= u_n(\mathrm{I} + \mathrm{D})(u_n\mathrm{D} - \mathrm{D}^{-1}u_n)(\mathrm{D} - \mathrm{I})^{-1}\frac{1}{u_n}\mathrm{I}\end{aligned}$$

D^{-1} and D are down and up-shift operators.

I is the identity operator.

$\mathrm{D} - \mathrm{I}$ is the discretized version of D_x (PDE case).

$(\mathrm{D} - \mathrm{I})^{-1}$ corresponding to the integral operator D_x^{-1} (PDE case).

The recursion operator has rank 1. Indeed, compare the ranks of successive symmetries (ranks 2 and 3):

$$\begin{aligned}G^{(1)} &= u_n(u_{n+1} - u_{n-1}), \\ G^{(2)} &= u_n u_{n+1}(u_n + u_{n+1} + u_{n+2}) - u_{n-1}u_n(u_{n-2} + u_{n-1} + u_n),\end{aligned}$$

which are linked via $\mathcal{R}G^{(1)} = G^{(2)}$.

Recursion operator splits into $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$.

$\mathcal{R}_0$ has linear combinations of $\mathrm{D}^{-1}, \mathrm{D}, \mathrm{I}$ and $u_{n\pm p}$.

$\mathcal{R}_1$ is of the form

$$\mathcal{R}_1 = \sum_j \sum_k G^{(j)}(\mathrm{D} - \mathrm{I})^{-1}\rho'_{(k)}$$

- **Algorithm for Recursion Operators of DDEs.**

Scalar Case

Step 1: **Determine the rank of the recursion operator.**

Recall: first two higher symmetries of Kac Van Moerbeke equation are

$$
\begin{aligned}
G^{(1)} &= u_n(u_{n+1} - u_{n-1}), \\
G^{(2)} &= u_n u_{n+1}(u_n + u_{n+1} + u_{n+2}) - u_{n-1}u_n(u_{n-2} + u_{n-1} + u_n), .
\end{aligned}
$$

Hence,

$$R = \text{rank } \mathcal{R} = \text{rank } G^{(2)} - \text{rank } G^{(1)} = 3 - 2 = 1.$$

Step 2: **Construct the form of the recursion operator.**

(i) Determine the pieces of operator $\mathcal{R}_0$

Compute the required shift (p) and linearly combine terms with $\mathrm{D}^{-1}, \mathrm{D}, \mathrm{I}$ and $u_{n\pm p}$.

Example: For the Kac-van Moerbeke lattice:

$$
\begin{aligned}
\mathcal{R}_0 &= (c_1 u_{n-1} + c_2 u_n + c_3 u_{n+1})\mathrm{D}^{-1} + (c_4 u_{n-1} + c_5 u_n + c_6 u_{n+1})\mathrm{I} \\
&\quad + (c_7 u_{n-1} + c_8 u_n + c_9 u_{n+1})\mathrm{D}^{+1},
\end{aligned}
$$

where the c_i's are constant coefficients.

(ii) Determine the pieces of operator $\mathcal{R}_1$

Combine the symmetries $G^{(j)}$ with $(\mathrm{D} - \mathrm{I})^{-1}$ and ${\rho_{(k)}}'(u)$, so that every term in

$$\mathcal{R}_1 = \sum_j \sum_k G^{(j)}(\mathrm{D} - \mathrm{I})^{-1}\rho'_{(k)}$$

has rank R.
The indices j and k are taken so that

$$\text{rank }(G^{(j)}) + \text{rank }({\rho_{(k)}}'(u)) - 1 = R.$$

Example: For the Kac-van Moerbeke lattice:

$$\mathcal{R}_1 = c_{10} u_n (u_{n+1} - u_{n-1})(\mathrm{D}^{+1} - \mathrm{I})^{-1}(\frac{1}{u_n}),$$

with c_{10} a constant coefficient.

(iii) Build the operator $\mathcal{R}$

Build $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$.

Example: For the Kac-van Moerbeke lattice:

$$\begin{aligned}\mathcal{R} \;=\; & (c_1 u_{n-1} + c_2 u_n + c_3 u_{n+1})\mathrm{D}^{-1} + (c_4 u_{n-1} + c_5 u_n + c_6 u_{n+1})\mathrm{I} \\ & + (c_7 u_{n-1} + c_8 u_n + c_9 u_{n+1})\mathrm{D}^{+1} \\ & + c_{10} u_n (u_{n+1} - u_{n-1})(\mathrm{D}^{+1} - \mathrm{I})^{-1}(\frac{1}{u_n}).\end{aligned}$$

Step 3: Determine the unknown coefficients.

Substitute in the determining equation, alternatively, require that

$$\mathcal{R} G^{(k)} = G^{(k+1)}, \quad k = 1, 2, 3, ...$$

Solution of the linear system:

$$c_1 = c_3 = c_4 = c_7 = c_9 = 0, c_2 = c_5 = c_6 = c_8 = c_{10} = 1.$$

Final result:

Recursion operator for Kac-van Moerbeke lattice:

$$\begin{aligned}\mathcal{R} &= u_n \mathrm{D} + u_n \mathrm{D}^{-1} + (u_n + u_{n+1})\mathrm{I} + u_n (u_{n+1} - u_{n-1})(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{u_n} \mathrm{I} \\ &= u_n (\mathrm{I} + \mathrm{D})(u_n \mathrm{D} - \mathrm{D}^{-1} u_n)(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{u_n} \mathrm{I}\end{aligned}$$

Matrix Case

Recursion operator (matrix) splits naturally in $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$.

Entries of matrix $\mathcal{R}_0$ are linear combinations of $(\mathbf{u}_n, \mathbf{u}_{n\pm1}, \mathbf{u}_{n\pm2}, ...)$ and $(\mathrm{I}, \mathrm{D}, \mathrm{D}^{-1}, ...)$ of rank R.

Matrix $\mathcal{R}_1$ is of the form

$$\sum_j \sum_k \mathbf{G}^{(j)} (\mathrm{D} - \mathrm{I})^{-1} \otimes \rho'_{(k)}$$

where $\otimes$ denotes the matrix outer product, and

$\rho'_{(k)}$ is the Fréchet derivative of $\rho_{(k)}$.

Example.

The Toda lattice

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}).$$

Recursion operator:

$$\mathcal{R} = \begin{pmatrix} -u_n \mathrm{I} & -\mathrm{D}^{-1} - \mathrm{I} + (v_{n-1} - v_n)(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \\ -v\mathrm{I} - v\mathrm{D} & u_{n+1}\mathrm{I} + v_n(u_n - u_{n+1})(\mathrm{D} - \mathrm{I})^{-1} \frac{1}{v_n} \mathrm{I} \end{pmatrix}$$

- **Example: The Ablowitz-Ladik Lattice.**

Consider the Ablowitz and Ladik discretization,

$$i\,\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \kappa u_n^* u_n (u_{n+1} + u_{n-1}),$$

of the NLS equation,

$$iu_t + u_{xx} + \kappa u^2 u^* = 0$$

u_n^* is the complex conjugate of u_n. Treat u_n and $v_n = u_n^*$ as independent variables and add the complex conjugate equation. Set $\kappa = 1$ (scaling) and absorb i in the scale on t :

$$\begin{aligned} \dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}), \\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

Since $v_n = u_n^*$, $w(v_n) = w(u_n)$.

No uniformity in rank! Introduce an auxiliary parameter α with weight.

$$\begin{aligned} \dot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}), \\ \dot{v}_n &= -\alpha(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

Uniformity in rank leads to

$$\begin{aligned} w(u_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) &= w(\alpha) + w(u_n) = 2w(u_n) + w(v_n), \\ w(v_n) + w(\frac{\mathrm{d}}{\mathrm{dt}}) &= w(\alpha) + w(v_n) = 2w(v_n) + w(u_n). \end{aligned}$$

For $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$,

$$w(u_n) + w(v_n) = w(\alpha) = 1.$$

So, one solution is

$$w(u_n) = w(v_n) = \frac{1}{2}, \quad w(\alpha) = 1.$$

Alternatively, for $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 0$,

$$w(u_n) + w(v_n) = 0, \quad w(\alpha) = 0.$$

The second scale helps eliminate terms in candidate density ρ.

Conserved densities (for $\alpha = 1$, in original variables):

$$\rho_n^{(1)} = u_n u_{n-1}^*$$

$$\rho_n^{(2)} = u_n u_{n+1}^*$$

$$\rho_n^{(3)} = \tfrac{1}{2} u_n^2 u_{n-1}^{*2} + u_n u_{n+1} u_{n-1}^* v_n + u_n u_{n-2}^*$$

$$\rho_n^{(4)} = \tfrac{1}{2} u_n^2 u_{n+1}^{*2} + u_n u_{n+1} u_{n+1}^* u_{n+2}^* + u_n u_{n+2}^*$$

$$\begin{aligned} \rho_n^{(5)} = {} & \tfrac{1}{3} u_n^3 u_{n-1}^{*3} + u_n u_{n+1} u_{n-1}^* u_n^* (u_n u_{n-1}^* + u_{n+1} u_n^* + u_{n+2} u_{n+1}^*) \\ & + u_n u_{n-1}^* (u_n u_{n-2}^* + u_{n+1} u_{n-1}^*) + u_n u_n^* (u_{n+1} u_{n-2}^* + u_{n+2} u_{n-1}^*) + u_n u_{n-3}^* \end{aligned}$$

$$\begin{aligned} \rho_n^{(6)} = {} & \tfrac{1}{3} u_n^3 u_{n+1}^{*3} + u_n u_{n+1} u_{n+1}^* u_{n+2}^* (u_n u_{n+1}^* + u_{n+1} u_{n+2}^* + u_{n+2} u_{n+3}^*) \\ & + u_n u_{n+2}^* (u_n u_{n+1}^* + u_{n+1} u_{n+2}^*) + u_n u_{n+3}^* (u_{n+1} u_{n+1}^* + u_{n+2} u_{n+2}^*) + u_n u_{n+3}^* \end{aligned}$$

The Ablowitz-Ladik lattice has infinitely many conserved densities.

Density we missed

$$\rho_n^{(0)} = \ln(1 + u_n u_n^*).$$

We cannot find the Hamiltonian (constant of motion):

$$H = -i \sum [u_n^* (u_{n-1} + u_{n+1}) - 2 \ln(1 + u_n u_n^*)],$$

since it has a logarithmic term.

- **Application: Discretization of combined KdV-mKdV equation.**

  Consider the integrable discretization

$$\begin{aligned}\dot{u}_n \;=\; & -(1+\alpha h^2 u_n + \beta h^2 u_n^2)\left\{ \tfrac{1}{h^3}\left(\tfrac{1}{2}u_{n+2} - u_{n+1} + u_{n-1} - \tfrac{1}{2}u_{n-2}\right)\right. \\ & + \tfrac{\alpha}{2h}[u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1}-u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2}] \\ & \left. + \tfrac{\beta}{2h}[u_{n+1}^2(u_{n+2}+u_n) - u_{n-1}^2(u_{n-2}+u_n)] \right\}\end{aligned}$$

  of a combined KdV-mKdV equation

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx} = 0.$$

  Discretizations the KdV and mKdV equations are special cases.

  Set $h = 1$ (scaling). No uniformity in rank!

  Introduce auxiliary parameters γ and δ with weights.

$$\begin{aligned}\dot{u}_n \;=\; & -(\gamma+\alpha u_n + \beta u_n^2)\left\{ \delta\left(\tfrac{1}{2}u_{n+2} - u_{n+1} + u_{n-1} - \tfrac{1}{2}u_{n-2}\right)\right. \\ & + \tfrac{\alpha}{2}[u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1}-u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2}] \\ & \left. + \tfrac{\beta}{2}[u_{n+1}^2(u_{n+2}+u_n) - u_{n-1}^2(u_{n-2}+u_n)] \right\},\end{aligned}$$

  Uniformity in rank requires

$$w(\gamma) = w(\delta) = 2w(u_n), \quad w(\alpha) = w(u_n), \quad w(\beta) = 0.$$

  Then,

$$w(u_n) + 1 = 5w(u_n),$$

  Hence,

$$w(u_n) = w(\alpha) = \tfrac{1}{4}, \quad w(\gamma) = w(\delta) = \tfrac{1}{2}, \quad w(\beta) = 0,$$

Conserved densities:

For the combined KdV-mKdV case ($\alpha \neq 0, \beta \neq 0$) :

Rank $\frac{1}{2}$ and 1 (after splitting):

$$\rho_n^{(1)} = \alpha u_n + \beta u_n u_{n+1}$$

$$\begin{aligned}\rho_n^{(2)} &= \frac{\alpha^2}{2\beta} {u_n}^2 + \frac{\alpha^2}{\beta} u_n u_{n+1} - u_n u_{n+1} + \alpha {u_n}^2 u_{n+1} + \alpha u_n {u_{n+1}}^2 \\ &+ \frac{1}{2} \beta {u_n}^2 {u_{n+1}}^2 + u_n u_{n+2} + \alpha u_n u_{n+1} u_{n+2} + \beta u_n {u_{n+1}}^2 u_{n+2}.\end{aligned}$$

For the KdV case ($\beta = 0$) :

$$\begin{aligned}\dot{u}_n &= -(\gamma + \alpha h^2 u_n) \Big\{ \tfrac{\delta}{h^3} (\tfrac{1}{2} u_{n+2} - u_{n+1} + u_{n-1} - \tfrac{1}{2} u_{n-2}) \\ &\quad + \tfrac{\alpha}{2h} [u_{n+1}^2 - u_{n-1}^2 + u_n (u_{n+1} - u_{n-1}) + u_{n+1} u_{n+2} - u_{n-1} u_{n-2}] \Big\}\end{aligned}$$

with $\gamma = \delta = 1$ is a completely integrable discretization of the KdV equation

$$u_t + 6\alpha u u_x + u_{xxx} = 0.$$

Now,

$$w(\gamma) = w(\delta) = w(u_n), \quad w(\alpha) = 0.$$

Then,

$$w(u_n) + 1 = 3 w(u_n).$$

So,

$$w(u_n) = w(\gamma) = w(\delta) = \tfrac{1}{2}, \quad w(\alpha) = 0.$$

From rank $\frac{3}{2}$ and $\frac{5}{2}$ (after splitting):

$$\rho_n^{(1)} = u_n,$$

$$\rho_n^{(2)} = u_n(\tfrac{1}{2}u_n + u_{n+1}),$$

$$\rho_n^{(3)} = u_n(\tfrac{1}{3}u_n^2 + u_n u_{n+1} + u_{n+1}^2 + \tfrac{1}{\alpha}u_{n+2} + u_{n+1}u_{n+2})$$

$$\rho_n^{(4)} = u_n(\tfrac{1}{4}u_n^3 + u_n^2 u_{n+1} + \tfrac{3}{2}u_n u_{n+1}^2 + u_{n+1}^3 + \cdots + u_{n+1}u_{n+2}u_{n+3})$$

$$\rho_n^{(5)} = u_n(\tfrac{1}{5}\alpha u_n^4 - \tfrac{1}{2}u_n^3 - 2u_n^2 u_{n+1} + \cdots +$$

$$\alpha u_{n+1}u_{n+2}u_{n+3}u_{n+4})$$

<u>For the mKdV case</u> $(\alpha = 0)$:

$$\begin{aligned} \dot{u}_n &= -(\gamma + \beta h^2 u_n^2) \left\{ \tfrac{\delta}{h^3}(\tfrac{1}{2}u_{n+2} - u_{n+1} + u_{n-1} - \tfrac{1}{2}u_{n-2}) \right. \\ &\quad \left. + \tfrac{\beta}{2h}[u_{n+1}^2(u_{n+2} + u_n) - u_{n-1}^2(u_{n-2} + u_n)] \right\} \end{aligned}$$

with $\gamma = \delta = 1$ is a completely integrable discretization of the modified KdV equation

$$u_t + 6\beta u^2 u_x + u_{xxx} = 0.$$

Now,

$$w(\gamma) = w(\delta) = 2w(u_n), \quad w(\beta) = 0.$$

Then,

$$w(u_n) + 1 = 5w(u_n).$$

So,

$$w(u_n) = \tfrac{1}{4}, \quad w(\gamma) = w(\delta) = \tfrac{1}{2}, \quad w(\beta) = 0.$$

From rank $\frac{3}{2}$ and $\frac{5}{2}$ (after splitting):

$$
\begin{aligned}
\rho_n^{(1)} &= u_n u_{n+1}, \\
\rho_n^{(2)} &= u_n(\tfrac{1}{2}u_n u_{n+1}^2 + \tfrac{1}{\beta}u_{n+2} + u_{n+1}^2 u_{n+2}) \\
\rho_n^{(3)} &= u_n(\tfrac{1}{3}u_n^2 u_{n+1}^3 + \tfrac{1}{\beta}u_n u_{n+1} u_{n+2} + + \cdots + u_{n+1}^2 u_{n+2}^2 u_{n+3}) \\
\rho_n^{(4)} &= u_n(\tfrac{1}{4}\beta u_n^3 u_{n+1}^4 + u_n^2 u_{n+1}^2 u_{n+2} + \cdots + \beta u_{n+1}^2 u_{n+2}^2 u_{n+3}^2 u_{n+4})
\end{aligned}
$$

Part III Software, Future Work, Publications

- **Scope and Limitations of Algorithms.**
  - Systems of DDEs must be polynomial in dependent variables.
  - One discretized space variable (lattice point n)
  - Program only computes polynomial conservation laws and generalized symmetries (no recursion operators yet). (Non-polynomial densities in progress).
  - Program does not compute conservation laws and symmetries that explicitly depend on n.
  - No limit on the number of equations in the system. In practice: time and memory constraints.
  - Input systems may have (nonzero) parameters. Program computes the compatibility conditions for parameters such that conservation laws and symmetries (of a given rank) exist.
  - Systems can also have parameters with (unknown) weight. This allows one to test lattice equations of non-uniform rank.
  - For systems where one or more of the weights is free, the program prompts the user for info.
  - Fractional weights and ranks are permitted.
  - Lattice equations must be of first-order in t.

- **Conclusions and Future Research**
  - Compute simple logarithmic and rational densities.
  - Implement the recursion operator algorithm for DDEs.
  - Improve software, compare with other strategies & packages.
  - Add tools for parameter analysis (Gröbner basis, Ritt-Wu or characteristic sets algorithms).
  - Introduce multiple sets of weights based on $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 0$ and $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$.
  - Application: test model DDEs for integrability.

    (study the integrable discretization of KdV-mKdV equation).

- **Implementation in Mathematica – Software**

  * P.J. Adams and W. Hereman
    TransPDEDensityFlux.m: Symbolic computation of conserved densities and fluxes for systems of partial differential equations with transcendental nonlinearities (2002).
  * H. Eklund and W. Hereman
    DDEDensityFlux.m: Symbolic computation of conserved densities and fluxes for nonlinear systems of differential-difference equations (2002).
  * Ü. Göktaş and W. Hereman
    InvariantsSymmetries.m: A Mathematica integrability package for the computation of invariants and symmetries (1997).
    Available from MathSource
    (Item: 0208-932, Applications/Mathematics) via FTP:
    mathsource.wolfram.com or URL
    http://www.mathsource.com/cgi-bin/MathSource/Applications/
  * Ü. Göktaş and W. Hereman
    CONDENS.M: A Mathematica program for the symbolic computation of conserved densities for systems of nonlinear evolution equations (1996).
  * Ü. Göktaş and W. Hereman
    DIFFDENS.M: A Mathematica program for the symbolic computation of conserved densities for systems of nonlinear differential-difference equations (1997).

    All codes are available via the Internet
    URL: http://www.mines.edu/fs_home/whereman/
    and via anonymous FTP from mines.edu in directory
    pub/papers/math_cs_dept/software/

- **Publications**

1). P. J. Adams, *Symbolic Computation of Conserved Densities and Fluxes for Systems of Partial Differential Equations with Transcendental Nonlinearities*, MS Thesis, CSMines, Dec. 2002.

2). H. Eklund, *Symbolic Computation of Conserved Densities and Fluxes for Nonlinear Systems of Differential-Difference Equations*, MS Thesis, Colorado School of Mines, Dec. 2002.

3). Ü. Göktaş and W. Hereman, Symbolic computation of conserved densities for systems of nonlinear evolution equations, J. Symb. Comput., 24 (1997) 591–621.

4). Ü. Göktaş, W. Hereman, and G. Erdmann, Computation of conserved densities for systems of nonlinear differential-difference equations, Phys. Lett. A, 236 (1997) 30–38.

5). Ü. Göktaş and W. Hereman, Computation of conserved densities for nonlinear lattices, Physica D, 123 (1998) 425–436.

6). Ü. Göktaş and W. Hereman, Algorithmic computation of higher-order symmetries for nonlinear evolution and lattice equations, Adv. in Comput. Math. 11 (1999), 55-80.

7). W. Hereman and Ü. Göktaş, Integrability Tests for Nonlinear Evolution Equations. In: *Computer Algebra Systems: A Practical Guide*, Ed.: M. Wester, Wiley & Sons, New York (1999) Chap. 12, pp. 211-232.

8). W. Hereman, Ü. Göktaş, M. Colagrosso, and A. Miller, Algorithmic integrability tests for nonlinear differential and lattice equations, Comp. Phys. Comm. 115 (1998) 428–446.

9). M. Hickman and W. Hereman, Computation of Densities and Fluxes of Nonlinear Differential-Difference Equations, *Proc. Roy. Soc. Lon. A* (2003) in press.