SYMBOLIC COMPUTATION
OF
CONSERVED DENSITIES

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IMACS Conference
Applications of Computer Algebra
Tuesday, May 16, 1995
15:30

I. INTRODUCTION

Symbolic Software

- Solitons via Hirota's method (Macsyma & Mathematica)
- Painlevé test for ODEs or PDEs (Macsyma)
- Conservation laws of PDEs (Mathematica)
- Lie symmetries for ODEs and PDEs (Macsyma)

Purpose of the programs

- Study of integrability of nonlinear PDEs
- Exact solutions as bench mark for numerical algorithms
- Classification of nonlinear PDEs
- Lie symmetries $\longrightarrow$ solutions via reductions

Collaborators

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II. MATHEMATICA PROGRAM FOR CONSERVED DENSITIES

- **Purpose**

Compute polynomial-type conservation laws of single PDEs and systems of PDEs

Conservation law:

$$\rho_t + J_x = 0$$

both $\rho(u, u_x, u_{2x}, \ldots, u_{nx})$ and $J(u, u_x, u_{2x}, \ldots, u_{nx})$

Consequently

$$P = \int_{-\infty}^{+\infty} \rho dx = \text{constant}$$

provided J vanishes at infinity

Compare with constants of motions in classical mechanics

• **Example**

Consider the KdV equation

$$u_t + uu_x + u_{3x} = 0$$

Conserved densities:

$$\begin{aligned}
\rho_1 &= u \\
\rho_2 &= u^2 \\
\rho_3 &= u^3 - 3u_x^2 \\
&\vdots \\
\rho_6 &= u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \\
&+ \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2 \\
&\vdots
\end{aligned}$$

Integrable equations have ∞ conservation laws

- **Algorithm and Implementation**

Consider the scaling (weights) of the KdV

$$u \sim \frac{\partial^2}{\partial x^2}, \qquad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3}$$

Compute building blocks of ρ_3

(i) Start with building block u^3

Divide by u and differentiate twice $(u^2)_{2x}$

Produces the list of terms

$[u_x^2, uu_{2x}] \longrightarrow [u_x^2]$

Second list: remove terms that are total derivative
with respect to x or total derivative
up to terms earlier in the list

Divide by u^2 and differentiate twice $(u)_{4x}$

Produces the list: $[u_{4x}] \longrightarrow [\,]$

$[\,]$ is the empty list

Gather the terms:

$$\rho_3 = u^3 + c[1]u_x^2$$

where the constant c_1 must be determined

(ii) Compute $\dfrac{\partial \rho_3}{\partial t} = 3u^2 u_t + 2c_1 u_x u_{xt}$

Replace u_t by $-(uu_x + u_{xxx})$ and u_{xt} by $-(uu_x + u_{xxx})_x$

(iii) Integrate the result with respect to x

Carry out all integrations by parts

$$\frac{\partial \rho_3}{\partial t} = -[\frac{3}{4}u^4 + (c_1 - 3)uu_x^2 + 3u^2 u_{xx} - c_1 u_{xx}^2 + 2c_1 u_x u_{xxx}]_x$$
$$-(c_1 + 3)u_x^3$$

The last non-integrable term must vanish

Thus, $c_1 = -3$

Result:

$$\rho_3 = u^3 - 3u_x^2$$

(iv) Expression $[\ldots]$ yields

$$J_3 = \frac{3}{4}u^4 - 6uu_x^2 + 3u^2 u_{xx} + 3u_{xx}^2 - 6u_x u_{xxx}$$

Computer building blocks of ρ_6

(i) Start with u^6

Divide by u and differentiate twice

$(u^5)_{2x}$ produces the list of terms

$[u^3 u_x^2, u^4 u_{2x}] \longrightarrow [u^3 u_x^2]$

Next, divide u^6 by u^2, and compute $(u^4)_{4x}$

Corresponding list:

$[u_x^4, u u_x^2 u_{2x}, u^2 u_{2x}^2, u^2 u_x u_{3x}, u^3 u_{4x}] \longrightarrow [u_x^4, u^2 u_{2x}^2]$

Proceed with $(\frac{u^6}{u^3})_{6x} = (u^3)_{6x}, (\frac{u^6}{u^4})_{8x} = (u^2)_{8x}$

and $(\frac{u^6}{u^5})_{10x} = (u)_{10x}$

Obtain the lists:

$[u_{2x}^3, u_x u_{2x} u_{3x}, u u_{3x}^2, u_x^2 u_{4x}, u u_{2x} u_{4x}, u u_x u_{5x}, u^2 u_{6x}] \longrightarrow$
$[u_{2x}^3, u u_{3x}^2]$

$[u_{4x}^2, u_{3x} u_{5x}, u_{2x} u_{6x}, u_x u_{7x}, u u_{8x}] \longrightarrow [u_{4x}^2]$

and $[u_{10x}] \longrightarrow [\,]$

Gather the terms:

$\rho_6 = u^6 + c_1 u^3 u_x^2 + c_2 u_x^4 + c_3 u^2 u_{2x}^2 + c_4 u_{2x}^3 + c_5 u u_{3x}^2 + c_6 u_{4x}^2$

where the constants c_i must be determined

(ii) Compute $\frac{\partial}{\partial t}\rho_6$

Replace $u_t, u_{xt}, \ldots, u_{nx,t}$ by $-(uu_x + u_{xxx}), \ldots$

(iii) Integrate the result with respect to x

Carry out all integrations by parts

Require that non-integrabe part vanishes

Set to zero all the coefficients of the independent
combinations involving powers of u and
its derivatives with respect to x

Solve the linear system for unknowns $c_1, c_2, \ldots, c_6$

Result:

$$\begin{aligned}\rho_6 &= u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \\ &+ \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2\end{aligned}$$

(iv) Flux J_6 can be computed by substituting
the constants into the integrable part of ρ_6

- Further Examples

  * Conservation laws of generalized Schamel equation

$$n^2 u_t + (n+1)(n+2) u^{\frac{2}{n}} u_x + u_{xxx} = 0$$

n positive integer

$$\begin{aligned} \rho_1 &= u \\ \rho_2 &= u^2 \\ \rho_3 &= \frac{1}{2} u_x^2 - \frac{n^2}{2} u^{2+\frac{2}{n}} \end{aligned}$$

no further conservation laws

  * Conserved densities of modified vector derivative nonlinear Schrödinger equation

$$\frac{\partial \mathbf{B}_\perp}{\partial t} + \frac{\partial}{\partial x}(B_\perp^2 \mathbf{B}_\perp) + \alpha \mathbf{B}_{\perp 0} \mathbf{B}_{\perp 0} \cdot \frac{\partial \mathbf{B}_\perp}{\partial x} + \mathbf{e}_x \times \frac{\partial^2 \mathbf{B}_\perp}{\partial x^2} = 0$$

Replace vector equation by

$$\begin{aligned} & u_t + \left(u(u^2+v^2) + \beta u - v_x\right)_x = 0 \\ & v_t + \left(v(u^2+v^2) + u_x\right)_x = 0 \end{aligned}$$

u and v denote the components of $\mathbf{B}_\perp$ parallel and perpendicular to $\mathbf{B}_{\perp 0}$ and $\beta = \alpha B_{\perp 0}^2$

The first 5 conserved densities are:

$$\rho_1 = u^2 + v^2$$

$$\rho_2 = \frac{1}{2}(u^2 + v^2)^2 - uv_x + u_x v + \beta u^2$$

$$\rho_3 = \frac{1}{4}(u^2 + v^2)^3 + \frac{1}{2}(u_x^2 + v_x^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4)$$

$$\rho_4 = \frac{1}{4}(u^2 + v^2)^4 - \frac{2}{5}(u_x v_{xx} - u_{xx} v_x) + \frac{4}{5}(uu_x + vv_x)^2$$

$$+ \frac{6}{5}(u^2 + v^2)(u_x^2 + v_x^2) - (u^2 + v^2)^2(uv_x - u_x v)$$

$$+ \frac{\beta}{5}(2u_x^2 - 4u^3 v_x + 2u^6 + 3u^4 v^2 - v^6) + \frac{\beta^2}{5}u^4$$

$$
\begin{aligned}
\rho_5 \; &= \; \frac{7}{16}(u^2+v^2)^5 + \frac{1}{2}(u_{xx}^2+v_{xx}^2) \\
&- \; \frac{5}{2}(u^2+v^2)(u_x v_{xx} - u_{xx} v_x) + 5(u^2+v^2)(uu_x+vv_x)^2 \\
&+ \; \frac{15}{4}(u^2+v^2)^2(u_x^2+v_x^2)^2 - \frac{35}{16}(u^2+v^2)^3(uv_x - u_x v) \\
&+ \; \frac{\beta}{8}(5u^8+10u^6v^2-10u^2v^6-5v^8+20u^2u_x^2 \\
&- \; 12u^5v_x+60uv^4v_x-20v^2v_x^2) \\
&+ \; \frac{\beta^2}{4}(u^6+v^6)
\end{aligned}
$$

- Coupled Systems

  * Conserved densities for the Coupled KdV Equations (Hirota-Satsuma system)

$$
\begin{aligned}
u_t - a(u_{xxx} + 6uu_x) - 2bvv_x &= 0 \\
v_t + v_{xxx} + 3uv_x &= 0
\end{aligned}
$$

$$
\begin{aligned}
\rho_1 &= u \\
\rho_2 &= u^2 + \frac{2}{3}bv^2 \\
\rho_3 &= (1+a)(u^3 - \frac{1}{2}u_x^2) + b(uv^2 - v_x^2)
\end{aligned}
$$

and e.g.

$$
\begin{aligned}
\rho_4 = \; & u^4 - 2uu_x^2 + \frac{1}{5}u_{xx}^2 \\
& + \frac{4}{5}b(u^2v^2 + \frac{2}{3}uvv_{xx} + \frac{8}{3}uv_x^2 - \frac{2}{3}v_{xx}^2) + \frac{4}{15}b^2v^4
\end{aligned}
$$

provided $a = \frac{1}{2}$

There are infinitely many more conservation laws

* Conserved densities for the Ito system

$$\begin{aligned} u_t - u_{xxx} - 6uu_x - 2vv_x &= 0 \\ v_t - 2u_x v - 2uv_x &= 0 \end{aligned}$$

$$\begin{aligned} \rho_1 &= \frac{1}{2}u \\ \rho_2 &= \frac{1}{2}(u^2 + v^2) \\ \rho_3 &= u^3 - \frac{1}{2}u_x^2 + uv^2 \end{aligned}$$

and infinitely many more conservation laws

A Class of Fifth-order Evolution Equations

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

Special cases:

$\alpha = 30$	$\beta = 20$	$\gamma = 10$	Lax
$\alpha = 5$	$\beta = 5$	$\gamma = 5$	Sawada Kotera or Caudry−Dodd−Gibbon
$\alpha = 20$	$\beta = 25$	$\gamma = 10$	Kaup−Kuperschmidt
$\alpha = 2$	$\beta = 6$	$\gamma = 3$	Ito

Table 1 Conserved Densities for Sawada-Kotera and Lax equations

Density	Sawada-Kotera equation	Lax equation
ρ_1	u	u
ρ_2	----	$\frac{1}{2}u^2$
ρ_3	$\frac{1}{3}u^3 - u_x^2$	$\frac{1}{3}u^3 - \frac{1}{6}u_x^2$
ρ_4	$\frac{1}{4}u^4 - \frac{9}{4}uu_x^2 + \frac{3}{4}u_{2x}^2$	$\frac{1}{4}u^4 - \frac{1}{2}uu_x^2 + \frac{1}{20}u_{2x}^2$
ρ_6	----	$\frac{1}{5}u^5 - u^2u_x^2 + \frac{1}{5}uu_{2x}^2 - \frac{1}{70}u_{3x}^2$
ρ_6	$\frac{1}{6}u^6 - \frac{25}{4}u^3u_x^2 - \frac{17}{8}u_x^4 + 6u^2u_{2x}^2$ $+2u_{2x}^3 - \frac{21}{8}uu_{3x}^2 + \frac{3}{8}u_{4x}^2$	$\frac{1}{6}u^6 - \frac{5}{3}u^3u_x^2 - \frac{5}{36}u_x^4 + \frac{1}{2}u^2u_{2x}^2$ $+\frac{5}{63}u_{2x}^3 - \frac{1}{14}uu_{3x}^2 + \frac{1}{252}u_{4x}^2$
ρ_7	$\frac{1}{7}u^7 - 9u^4u_x^2 - \frac{54}{5}uu_x^4 + \frac{57}{5}u^3u_{2x}^2$ $+\frac{648}{35}u_x^2u_{2x}^2 + \frac{489}{35}uu_{2x}^3 - \frac{261}{35}u^2u_{3x}^2$ $-\frac{288}{35}u_{2x}u_{3x}^2 + \frac{81}{35}uu_{4x}^2 - \frac{9}{35}u_{5x}^2$	$\frac{1}{7}u^7 - \frac{5}{2}u^4u_x^2 - \frac{5}{6}uu_x^4 + u^3u_{2x}^2$ $+\frac{1}{2}u_x^2u_{2x}^2 + \frac{10}{21}uu_{2x}^3 - \frac{3}{14}u^2u_{3x}^2$ $-\frac{5}{42}u_{2x}u_{3x}^2 + \frac{1}{42}uu_{4x}^2 - \frac{1}{924}u_{5x}^2$
ρ_8	----	$\frac{1}{8}u^8 - \frac{7}{2}u^5u_x^2 - \frac{35}{12}u^2u_x^4 + \frac{7}{4}u^4u_{2x}^2$ $+\frac{7}{2}uu_x^2u_{2x}^2 + \frac{5}{3}u^2u_{2x}^3 + \frac{7}{24}u_{2x}^4 + \frac{1}{2}u^3u_{3x}^2$ $-\frac{1}{4}u_x^2u_{3x}^2 - \frac{5}{6}uu_{2x}u_{3x}^2 + \frac{1}{12}u^2u_{4x}^2$ $+\frac{7}{132}u_{2x}u_{4x}^2 - \frac{1}{132}uu_{5x}^2 + \frac{1}{3432}u_{6x}^2$

Table 2 Conserved Densities for Kaup-Kuperschmidt and Ito equations		
Density	Kaup-Kuperschmidt equation	Ito equation
ρ_1	u	u
ρ_2	----	$\frac{u^2}{2}$
ρ_3	$\frac{u^3}{3} - \frac{1}{8}u_x^2$	----
ρ_4	$\frac{u^4}{4} - \frac{9}{16}uu_x^2 + \frac{3}{64}u_{2x}^2$	$\frac{u^4}{4} - \frac{9}{4}uu_x^2 + \frac{3}{4}u_{2x}^2$
ρ_5	----	----
ρ_6	$\frac{u^6}{6} - \frac{35}{16}u^3u_x^2 - \frac{31}{256}u_x^4 + \frac{51}{64}u^2u_{2x}^2$ $+\frac{37}{256}u_{2x}^3 - \frac{15}{128}uu_{3x}^2 + \frac{3}{512}u_{4x}^2$	----
ρ_7	$\frac{u^7}{7} - \frac{27}{8}u^4u_x^2 - \frac{369}{320}uu_x^4 + \frac{69}{40}u^3u_{2x}^2$ $+\frac{2619}{4480}u_x^2u_{2x}^2 + \frac{2211}{2240}uu_{2x}^3 - \frac{477}{1120}u^2u_{3x}^2$ $-\frac{171}{640}u_{2x}u_{3x}^2 + \frac{27}{560}uu_{4x}^2 - \frac{9}{4480}u_{5x}^2$	----
ρ_8	----	----