

Symbolic Computation of Lax Pairs of Systems of Nonlinear Partial Difference Equations and Their Gauge Equivalence

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This presentation is made in TeXpower

Outline

- What are nonlinear $P\Delta E$ s?
- Classification of integrable nonlinear $P\Delta E$ s in 2D
- Lax pairs of nonlinear PDEs & gauge equivalence
- Lax pair of nonlinear $P\Delta E$ s & gauge equivalence
- Examples of Lax pair of nonlinear $P\Delta E$ s
- Algorithm (Nijhoff 2001, Bobenko & Suris 2001)
- Software demonstration
- Lax pair for Hietarinta's generalized Boussinesq lattices & their gauge equivalence
- Conclusions and future work

What are nonlinear $P\Delta E$ s?

- Nonlinear maps with two (or more) lattice points

Some origins:

- ▶ full discretizations of PDEs
- ▶ discrete dynamical systems in 2 dimensions
- ▶ superposition principle (Bianchi permutability) for Bäcklund transformations between 3 solutions (2 parameters) of a completely integrable PDE
- **Example:** discrete potential Korteweg-de Vries (pKdV) equation

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0$$

- u is dependent variable or field (scalar case)
 n and m are lattice points
 p and q are parameters

- **Notation:**

$$(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (x, x_1, x_2, x_{12})$$

- Alternate notations (in the literature):

$$(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (u, \tilde{u}, \hat{u}, \hat{\tilde{u}})$$

$$(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (u_{00}, u_{10}, u_{01}, u_{11})$$

- **Example:** discrete pKdV equation

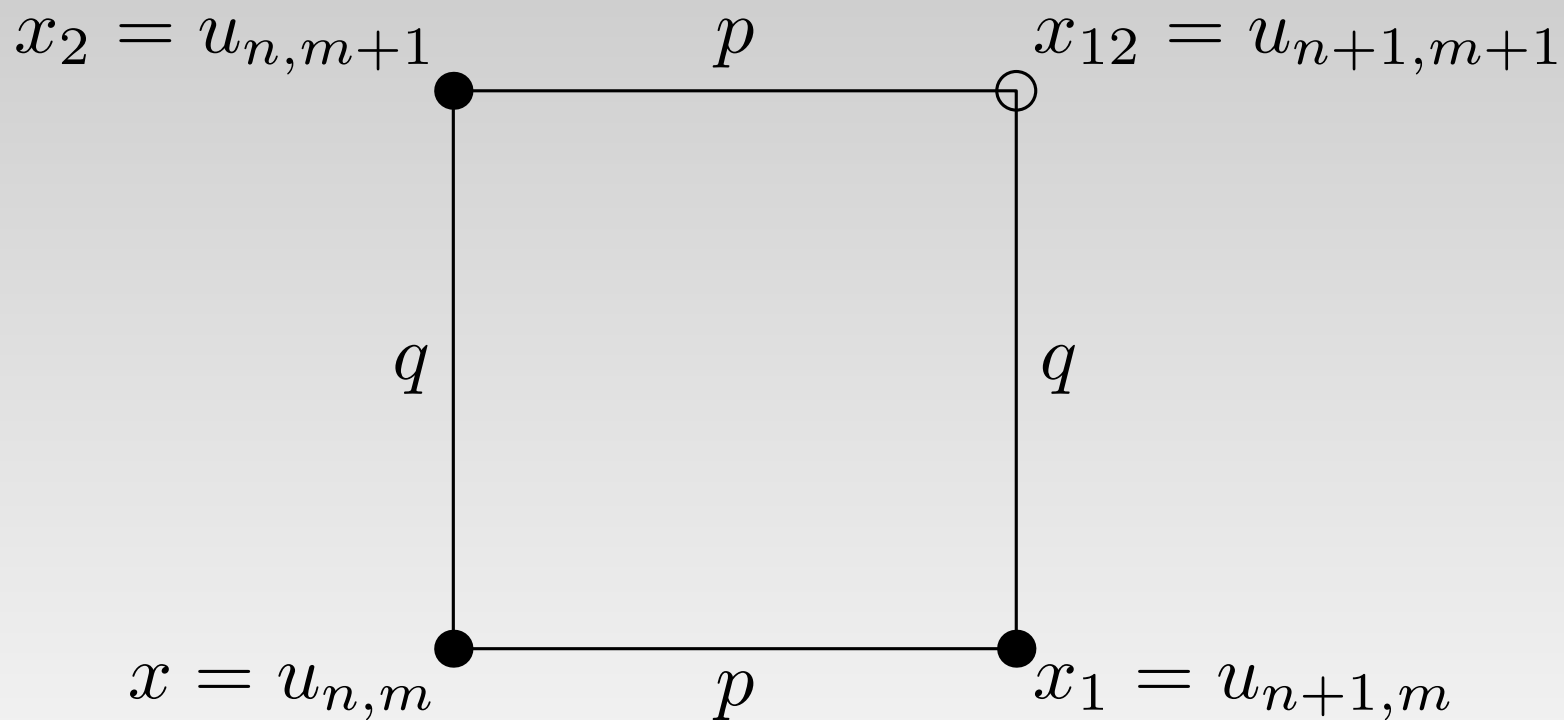
$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0$$

Short:

$$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0$$

Short: $(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$



Systems of $P\Delta E_s$

Example: Schwarzian-Boussinesq System

$$x_1 y - z_1 + z = 0$$

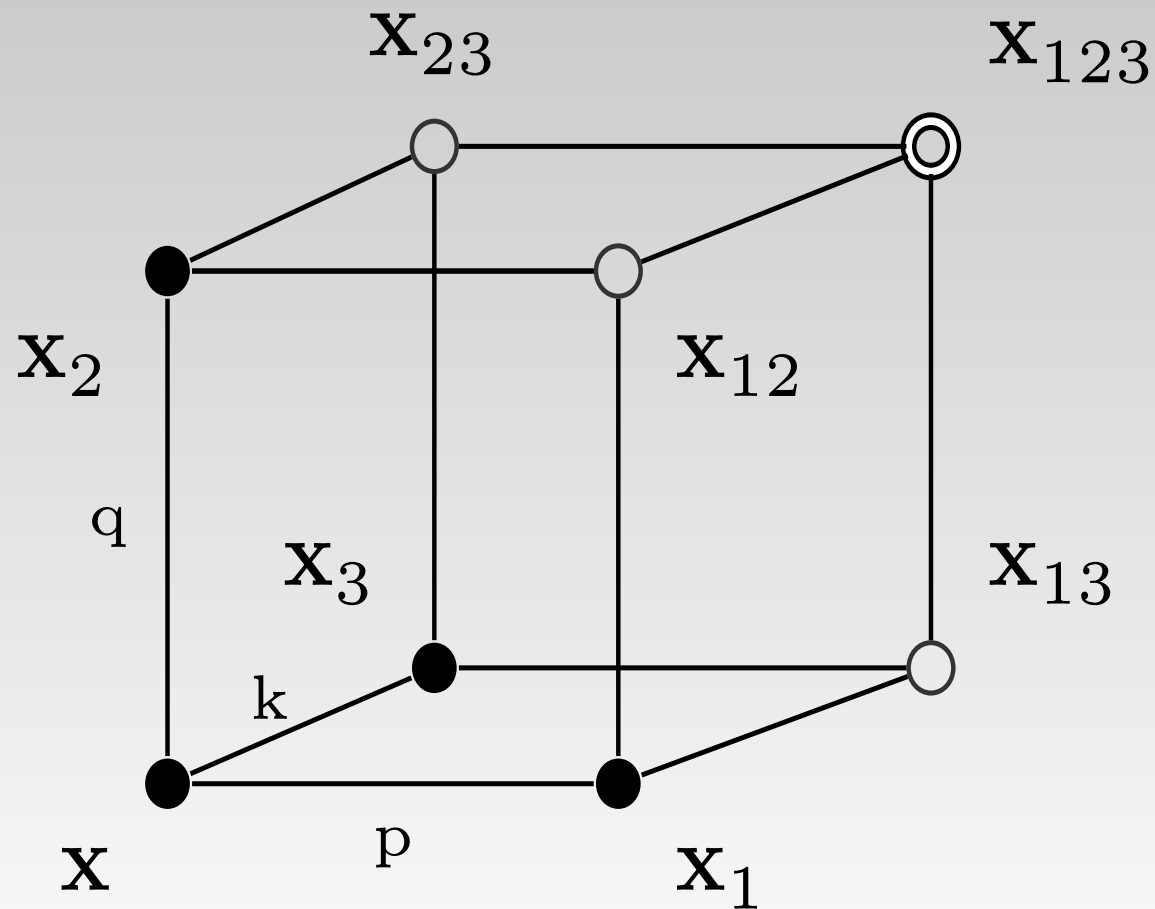
$$x_2 y - z_2 + z = 0$$

$$x y_{12}(y_1 - y_2) - y(p x_1 y_2 - q x_2 y_1) = 0$$

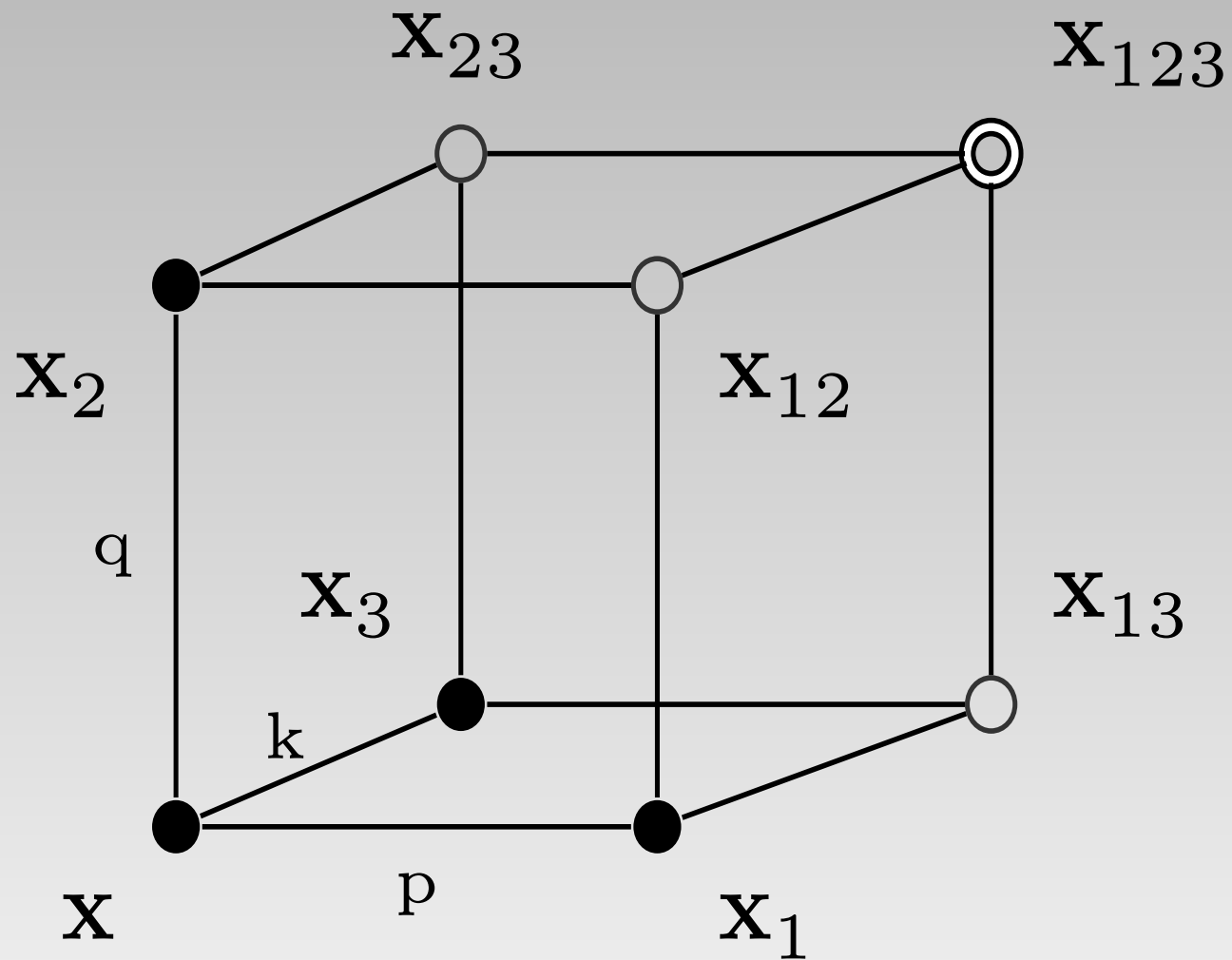
- Note: System has two **single-edge** equations and one **full-face** equation.

Concept of Consistency Around the Cube

Superposition of Bäcklund transformations between
4 solutions x, x_1, x_2, x_3 (3 parameters: p, q, k)



- Introduce a third lattice variable ℓ
- View u as dependent on three lattice points:
 n, m, ℓ . So, $x = u_{n,m} \longrightarrow x = u_{n,m,\ell}$
- Move in three directions:
 $n \rightarrow n + 1$ over distance p
 $m \rightarrow m + 1$ over distance q
 $\ell \rightarrow \ell + 1$ over distance k (spectral parameter)
- Require that the same lattice holds on the **front**, **bottom**, and **left** faces of the cube
- Require consistency for the computation of
 $x_{123} = u_{n+1,m+1,\ell+1}$ (3 ways $\rightarrow$ same answer)



Verification of Consistency Around the Cube

Example: discrete pKdV equation

★ Equation on front face of cube:

$$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

Solve for $x_{12} = x - \frac{p^2 - q^2}{x_1 - x_2}$

Compute x_{123} : $x_{12} \longrightarrow x_{123} = x_3 - \frac{p^2 - q^2}{x_{13} - x_{23}}$

★ Equation on floor of cube:

$$(x - x_{13})(x_1 - x_3) - p^2 + k^2 = 0$$

Solve for $x_{13} = x - \frac{p^2 - k^2}{x_1 - x_3}$

Compute x_{123} : $x_{13} \longrightarrow x_{123} = x_2 - \frac{p^2 - k^2}{x_{12} - x_{23}}$

★ Equation on **left face** of cube:

$$(x - x_{23})(x_3 - x_2) - k^2 + q^2 = 0$$

Solve for $x_{23} = x - \frac{q^2 - k^2}{x_2 - x_3}$

Compute x_{123} : $x_{23} \longrightarrow x_{123} = x_1 - \frac{q^2 - k^2}{x_{12} - x_{13}}$

★ Verify that all three coincide:

$$x_{123} = x_1 - \frac{q^2 - k^2}{x_{12} - x_{13}} = x_2 - \frac{p^2 - k^2}{x_{12} - x_{23}} = x_3 - \frac{p^2 - q^2}{x_{13} - x_{23}}$$

Upon substitution of x_{12} , x_{13} , and x_{23} :

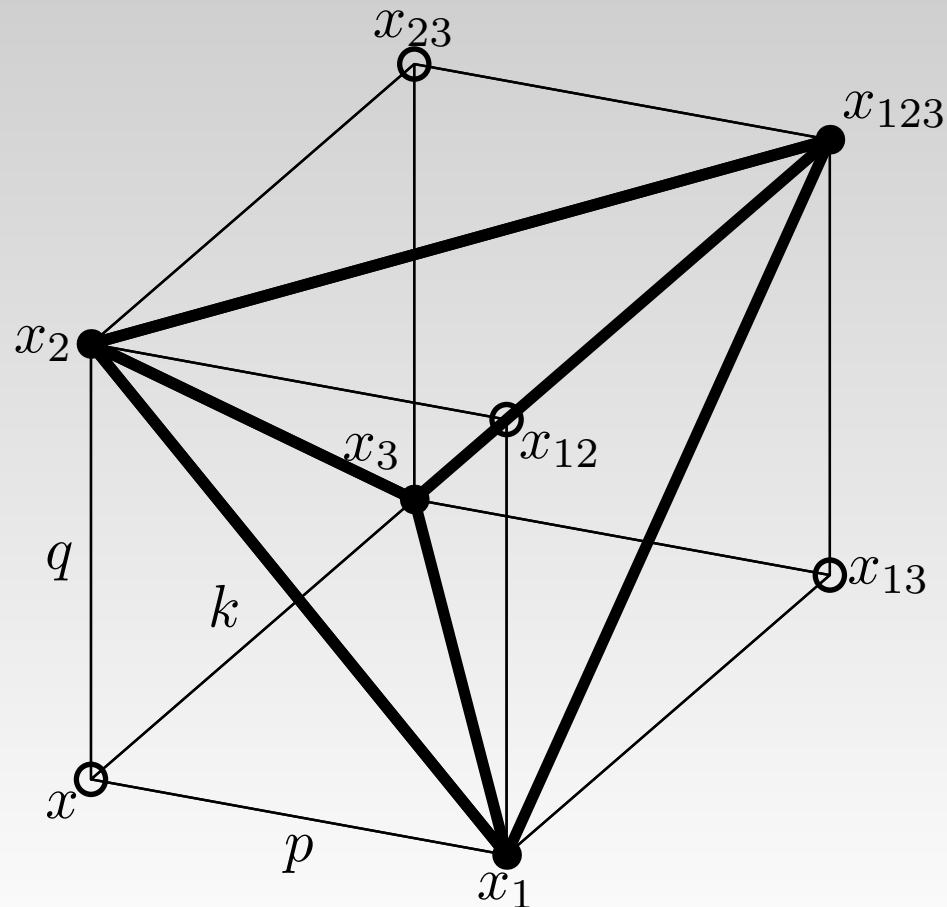
$$x_{123} = \frac{p^2 x_1 (x_2 - x_3) + q^2 x_2 (x_3 - x_1) + k^2 x_3 (x_1 - x_2)}{p^2 (x_2 - x_3) + q^2 (x_3 - x_1) + k^2 (x_1 - x_2)}$$

Consistency around the cube is satisfied!

Tetrahedron property

$$x_{123} = \frac{p^2 x_1 (x_2 - x_3) + q^2 x_2 (x_3 - x_1) + k^2 x_3 (x_1 - x_2)}{p^2 (x_2 - x_3) + q^2 (x_3 - x_1) + k^2 (x_1 - x_2)}$$

is independent of x . Connects x_{123} to x_1, x_2 and x_3 .



Classification of 2D scalar nonlinear PΔEs

Adler, Bobenko, Suris (ABS) 2003, 2007

- Consider a family PΔEs: $Q(x, x_1, x_2, x_{12}; p, q) = 0$

- Assumptions (ABS 2003):

- Affine linear

$$Q(x, x_1, x_2, x_{12}; p, q) = a_1 x x_1 x_2 x_{12} + a_2 x x_1 x_2 + \dots + a_{14} x_2 + a_{15} x_{12} + a_{16}$$

- Invariant under D_4 (symmetries of square)

$$\begin{aligned} Q(x, x_1, x_2, x_{12}; p, q) &= \epsilon Q(x, x_2, x_1, x_{12}; q, p) \\ &= \sigma Q(x_1, x, x_{12}, x_2; p, q) \end{aligned}$$

$$\epsilon, \sigma = \pm 1$$

- Consistency around the cube

Result of the ABS Classification

- List H

- ▶ H1

$$(x - x_{12})(x_1 - x_2) + q - p = 0$$

- ▶ H2

$$(x - x_{12})(x_1 - x_2) + (q - p)(x + x_1 + x_2 + x_{12}) + q^2 - p^2 = 0$$

- ▶ H3

$$p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) + \delta(p^2 - q^2) = 0$$

- List A

- ▶ A1

$$p(x+x_2)(x_1+x_{12})-q(x+x_1)(x_2+x_{12})-\delta^2 pq(p-q)=0$$

- ▶ A2

$$(q^2 - p^2)(xx_1x_2x_{12} + 1) + q(p^2 - 1)(xx_2 + x_1x_{12}) \\ - p(q^2 - 1)(xx_1 + x_2x_{12}) = 0$$

- List Q

- ▶ Q1

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta^2 pq(p-q) = 0$$

- ▶ Q2

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+pq(p-q) \\ (x+x_1+x_2+x_{12})-pq(p-q)(p^2-pq+q^2)=0$$

- ▶ Q3

$$(q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12}) \\ -p(q^2-1)(xx_2+x_1x_{12})-\frac{\delta^2}{4pq}(p^2-q^2)(p^2-1)(q^2-1)=0$$

► Q4 (mother) Hietarinta's Parametrization

$$\begin{aligned} & \operatorname{sn}(\alpha + \beta; k) (x_1 x_2 + x x_{12}) \\ & - \operatorname{sn}(\alpha; k) (x x_1 + x_2 x_{12}) - \operatorname{sn}(\beta; k) (x x_2 + x_1 x_{12}) \\ & + \operatorname{sn}(\alpha; k) \operatorname{sn}(\beta; k) \operatorname{sn}(\alpha + \beta; k) (1 + k^2 x x_1 x_2 x_{12}) = 0 \end{aligned}$$

where $\operatorname{sn}(\alpha; k)$ is the Jacobi elliptic sine function with modulus k .

- Other parameterizations (Adler, Nijhoff, Viallet) are given in the literature.



Peter D. Lax (1926-)

Seminal paper: Integrals of nonlinear equations of evolution and solitary waves, Commun. Pure Appl. Math. **21** (1968) 467-490

Refresher: Lax Pairs of Nonlinear PDEs

- Historical example: Korteweg-de Vries equation

$$u_t + \alpha u u_x + u_{xxx} = 0 \quad \alpha \in \mathbb{R}$$

- Key idea: Replace the nonlinear PDE with a compatible linear system (Lax pair):

$$\psi_{xx} + \left(\frac{1}{6} \alpha u - \lambda \right) \psi = 0$$

$$\psi_t + 4\psi_{xxx} + \alpha u \psi_x + \frac{1}{2} \alpha u_x \psi = 0$$

ψ is eigenfunction; λ is constant eigenvalue
($\lambda_t = 0$) (isospectral)

Lax Pairs in Operator Form

- Replace a completely integrable nonlinear PDE by a pair of linear equations (called a **Lax pair**):

$$\mathcal{L}\psi = \lambda\psi \quad \text{and} \quad \mathbf{D}_t\psi = \mathcal{M}\psi$$

- Require compatibility of both equations

$$\mathcal{L}_t\psi + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})\psi \doteq 0$$

Lax equation:

$$\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \doteq \mathcal{O}$$

with commutator $[\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}$.

Furthermore, $\mathcal{L}_t\psi = [\mathbf{D}_t, \mathcal{L}]\psi = \mathbf{D}_t(\mathcal{L}\psi) - \mathcal{L}\mathbf{D}_t\psi$

and $\doteq$ means “evaluated on the PDE”

- **Example:** Lax operators for the KdV equation

$$\mathcal{L} = \mathbf{D}_x^2 + \frac{1}{6}\alpha u \mathbf{I}$$

$$\mathcal{M} = - \left(4 \mathbf{D}_x^3 + \alpha u \mathbf{D}_x + \frac{1}{2}\alpha u_x \mathbf{I} \right)$$

- **Note:** $\mathcal{L}_t \psi + [\mathcal{L}, \mathcal{M}] \psi = \frac{1}{6}\alpha (u_t + \alpha u u_x + u_{xxx}) \psi$

Alternate Operator Formulations

- Define $\tilde{\mathcal{L}} = \mathcal{L} - \lambda \mathbf{I}$ and $\tilde{\mathcal{M}} = \mathcal{M} - \mathbf{D}_t$
- Then, the Lax pair becomes

$$\tilde{\mathcal{L}}\psi = 0 \quad \text{and} \quad \tilde{\mathcal{M}}\psi = 0$$

and the Lax equation becomes $[\tilde{\mathcal{L}}, \tilde{\mathcal{M}}] \doteq \mathcal{O}$

Challenge: Find linear commuting operators modulo the (nonlinear) PDE

- If S is an arbitrary invertible operator, then

$$\hat{\mathcal{L}} = S\mathcal{L}S^{-1}, \quad \hat{\mathcal{M}} = S\mathcal{M}S^{-1}, \quad \hat{\mathbf{D}}_t = S\mathbf{D}_tS^{-1}$$

satisfy $\hat{\mathcal{L}}_t + [\hat{\mathcal{L}}, \hat{\mathcal{M}}] \doteq \mathcal{O}$

Lax Pairs in Matrix Form (AKNS Scheme)

- Express compatibility of

$$D_x \Psi = \mathbf{X} \Psi$$

$$D_t \Psi = \mathbf{T} \Psi$$

where $\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix}$, $\mathbf{X}$ and $\mathbf{T}$ are $N \times N$ matrices

- Lax equation (zero-curvature equation):

$$\boxed{D_t \mathbf{X} - D_x \mathbf{T} + [\mathbf{X}, \mathbf{T}] \doteq \mathbf{0}}$$

with commutator $[\mathbf{X}, \mathbf{T}] = \mathbf{X}\mathbf{T} - \mathbf{T}\mathbf{X}$

- **Example:** Lax pair for the KdV equation

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6}\alpha u & 0 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \frac{1}{6}\alpha u_x & -4\lambda - \frac{1}{3}\alpha u \\ -4\lambda^2 + \frac{1}{3}\alpha\lambda u + \frac{1}{18}\alpha^2 u^2 + \frac{1}{6}\alpha u_{2x} & -\frac{1}{6}\alpha u_x \end{bmatrix}$$

Substitution into the Lax equation yields

$$\mathbf{D}_t \mathbf{X} - \mathbf{D}_x \mathbf{T} + [\mathbf{X}, \mathbf{T}] = -\frac{1}{6}\alpha \begin{bmatrix} 0 & 0 \\ u_t + \alpha u u_x + u_{3x} & 0 \end{bmatrix}$$

Equivalence under Gauge Transformations

- Lax pairs are equivalent under a gauge transformation:

If $(\mathbf{X}, \mathbf{T})$ is a Lax pair then so is $(\mathcal{X}, \mathcal{T})$ with

$$\mathcal{X} = \mathbf{G}\mathbf{X}\mathbf{G}^{-1} + \mathbf{D}_x(\mathbf{G})\mathbf{G}^{-1}$$

$$\mathcal{T} = \mathbf{G}\mathbf{T}\mathbf{G}^{-1} + \mathbf{D}_t(\mathbf{G})\mathbf{G}^{-1}$$

$\mathbf{G}$ is arbitrary invertible matrix and $\Phi = \mathbf{G}\Psi$ where Φ goes with $(\mathcal{X}, \mathcal{T})$, i.e., $\mathbf{D}_x\Phi = \mathcal{X}\Phi$ and $\mathbf{D}_t\Phi = \mathcal{T}\Phi$.

Thus,

$$\mathcal{X}_t - \mathcal{T}_x + [\mathcal{X}, \mathcal{T}] \doteq \mathbf{0}$$

- **Example:** For the KdV equation

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6}\alpha u & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{X} = \begin{bmatrix} -ik & \frac{1}{6}\alpha u \\ -1 & ik \end{bmatrix}$$

Here,

$$\mathcal{X} = \mathbf{G}\mathbf{X}\mathbf{G}^{-1} \quad \text{and} \quad \mathcal{T} = \mathbf{G}\mathbf{T}\mathbf{G}^{-1}$$

with

$$\mathbf{G} = \begin{bmatrix} -i k & 1 \\ -1 & 0 \end{bmatrix}$$

where $\lambda = -k^2$

Reasons to Compute a Lax Pair

- Compatible **linear** system is the starting point for application of the IST and the Riemann-Hilbert method for boundary value problems
- Confirm the **complete integrability** of the PDE
- Zero-curvature representation of the PDE
- Compute conservation laws of the PDE
- Discover families of completely integrable PDEs

Question: How to find a Lax pair of a completely integrable PDE?

Answer: There is no completely systematic method

Lax Pair of Nonlinear PΔEs

- Replace the nonlinear PΔE by

$$\psi_1 = L \psi \quad (\text{recall } \psi_1 = \psi_{n+1,m})$$

$$\psi_2 = M \psi \quad (\text{recall } \psi_2 = \psi_{n,m+1})$$

For scalar PΔEs, L, M are 2×2 matrices; $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$

- Express compatibility:

$$\psi_{12} = L_2 \psi_2 = L_2 M \psi$$

$$\psi_{12} = M_1 \psi_1 = M_1 L \psi$$

$$\begin{array}{ccc}
 \psi_2 & \xrightarrow{L_2} & \psi_{12} \\
 \uparrow M & & \uparrow M_1 \\
 \psi & \xrightarrow{L} & \psi_1
 \end{array}$$

- Lax equation: $\boxed{L_2 M - M_1 L \doteq 0}$

Equivalence under Gauge Transformations

- Lax pairs of the same size are equivalent under a gauge transformation

If (L, M) is a Lax pair then so is $(\mathcal{L}, \mathcal{M})$ with

$$\begin{aligned}\mathcal{L} &= \mathcal{G}_1 L \mathcal{G}^{-1} \\ \mathcal{M} &= \mathcal{G}_2 M \mathcal{G}^{-1}\end{aligned}$$

where $\mathcal{G}$ is an invertible matrix and $\phi = \mathcal{G}\psi$
where ϕ goes with $(\mathcal{L}, \mathcal{M})$, i.e., $\phi_1 = \mathcal{L} \phi$, $\phi_2 = \mathcal{M} \phi$.

Proof: Trivial verification that

$$(\mathcal{L}_2 \mathcal{M} - \mathcal{M}_1 \mathcal{L}) \phi \doteq 0 \leftrightarrow (L_2 M - M_1 L) \psi \doteq 0$$

- Lax pairs of different sizes are equivalent under a gauge-like transformations

For example, if (L, M) is a 3×3 Lax pair and $(\mathcal{L}, \mathcal{M})$ is a 4×4 Lax pair then

$$\begin{aligned}\mathcal{L} &= \mathcal{H}_1 L \mathcal{H}_{\text{Left}}^{-1} \\ \mathcal{M} &= \mathcal{H}_2 M \mathcal{H}_{\text{Left}}^{-1}\end{aligned}$$

where $\mathcal{H}$ is 4×3 matrix with $\text{rank } \mathcal{H} = 3$ and $\phi = \mathcal{H}\psi$, where ϕ goes with $(\mathcal{L}, \mathcal{M})$.

Proof:

$$\phi_1 = \mathcal{L}\phi = \mathcal{H}_1 L \mathcal{H}_{\text{Left}}^{-1} \mathcal{H}\psi = \mathcal{H}_1 L\psi = \mathcal{H}_1\psi_1 = (\mathcal{H}\psi)_1$$

- Alternate Way:

For example, if (L, M) is a 3×3 Lax pair and $(\mathcal{L}, \mathcal{M})$ is a 4×4 Lax pair, then

$$\begin{aligned} L &= \tilde{\mathcal{H}}_1 \mathcal{L} \tilde{\mathcal{H}}_{\text{Right}}^{-1} \\ M &= \tilde{\mathcal{H}}_2 \mathcal{M} \tilde{\mathcal{H}}_{\text{Right}}^{-1} \end{aligned}$$

where $\tilde{\mathcal{H}}$ is 3×4 matrix with $\text{rank } \tilde{\mathcal{H}} = 3$ and $\psi = \tilde{\mathcal{H}}\phi$, where ϕ goes with $(\mathcal{L}, \mathcal{M})$.

Proof:

$$\psi_1 = L\psi = \tilde{\mathcal{H}}_1 \mathcal{L} \tilde{\mathcal{H}}_{\text{Right}}^{-1} \psi = \tilde{\mathcal{H}}_1 \mathcal{L} \phi = \tilde{\mathcal{H}}_1 \phi_1 = (\tilde{\mathcal{H}}\phi)_1$$

Which gauge equivalence should be used?

- Depends on the edge equations!
- Example: Lax pairs for generalized Boussinesq lattices due to Hietarinta.
- Lax pairs were computed by Zhao, Zhang, & Nijhoff (2012) and Bridgman & Hereman (2012). They are all valid but differ in size. Zhao *et al.* have 4×4 matrices. We obtained 3×3 matrices.
- Unresolved issue: When to use a left or right inverse.

- **Example 1:** Discrete pKdV Equation

$$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

- **Lax pair:**

$$L = tL_{\text{core}} = t \begin{bmatrix} x & p^2 - k^2 - xx_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = sM_{\text{core}} = s \begin{bmatrix} x & q^2 - k^2 - xx_2 \\ 1 & -x_2 \end{bmatrix}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{\text{Det}L_{\text{core}}}} = \frac{1}{\sqrt{k^2 - p^2}}$

and $s = \frac{1}{\sqrt{\text{Det}M_{\text{core}}}} = \frac{1}{\sqrt{k^2 - q^2}}$. Here, $\frac{t_2}{t} \frac{s}{s_1} = 1$.

- **Example 2:** Schwarzian-Boussinesq System

$$x_1 y - z_1 + z = 0$$

$$x_2 y - z_2 + z = 0$$

$$x y_{12}(y_1 - y_2) - y(p x_1 y_2 - q x_2 y_1) = 0$$

- **Lax pair:**

$$L = tL_{\text{core}} = t \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kzy_1}{x} & \frac{pyx_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix}$$

$$M = sM_{\text{core}} = s \begin{bmatrix} y_2 & -1 & 0 \\ \frac{kzy_2}{x} & \frac{qyx_2}{x} & -\frac{ky_2}{x} \\ 0 & -z_2 & y_2 \end{bmatrix}$$

with $t = s = \frac{1}{y}$, or $t = \frac{1}{y_1}$ and $s = \frac{1}{y_2}$,

or $t = \sqrt[3]{\frac{x}{y_1^2 y x_1}}$ and $s = \sqrt[3]{\frac{x}{y_2^2 y x_2}}$.

Here, $\frac{t_2}{t} \frac{s}{s_1} = \frac{y_1}{y_2}$.

Lax Pair Algorithm for Scalar $P\Delta E_s$

(Nijhoff 2001, Bobenko and Suris 2001)

Applies to $P\Delta E_s$ that are consistent around the cube

Example: Discrete pKdV Equation

- Step 1: Verify the consistency around the cube

Use the equation on floor

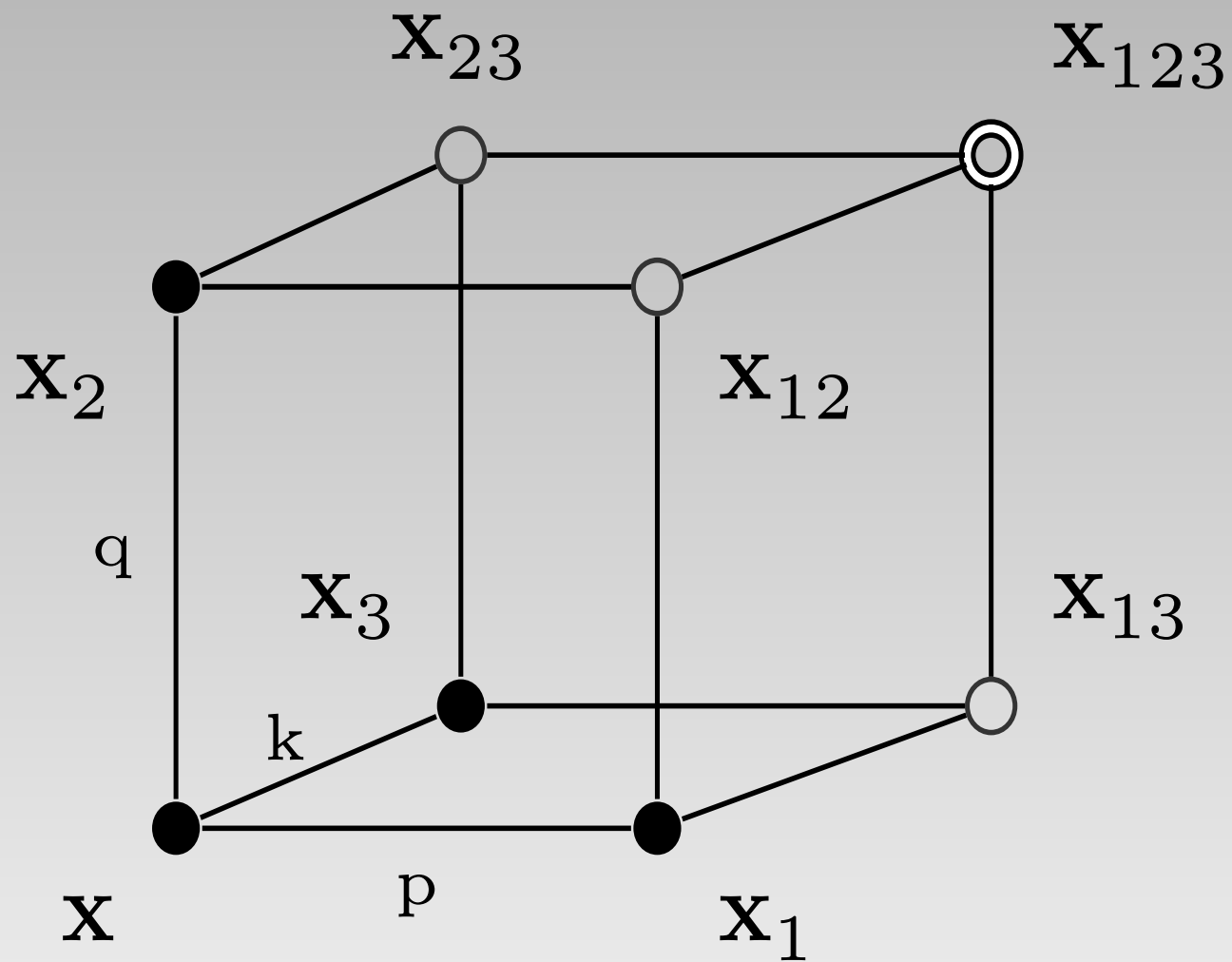
$$(x - x_{13})(x_1 - x_3) - p^2 + k^2 = 0$$

to compute $x_{13} = x - \frac{p^2 - k^2}{x_1 - x_3} = \frac{x_3 x - x x_1 + p^2 - k^2}{x_3 - x_1}$

Use the equation on left face

$$(x - x_{23})(x_3 - x_2) - k^2 + q^2 = 0$$

to compute $x_{23} = x - \frac{q^2 - k^2}{x_2 - x_3} = \frac{x_3 x - x x_2 + q^2 - k^2}{x_3 - x_2}$



- Step 2: Homogenization

★ Numerator and denominator of

$$x_{13} = \frac{x_3 x - x x_1 + p^2 - k^2}{x_3 - x_1} \quad \text{and} \quad x_{23} = \frac{x_3 x - x x_2 + q^2 - k^2}{x_3 - x_2}$$

are linear in x_3 .

★ Substitute $x_3 = \frac{f}{F}$ into x_{13} to get

$$x_{13} = \frac{x f + (p^2 - k^2 - x x_1) F}{f - x_1 F}$$

★ On the other hand, $x_3 = \frac{f}{F} \longrightarrow x_{13} = \frac{f_1}{F_1}$.

$$\text{Thus, } x_{13} = \frac{f_1}{F_1} = \frac{x f + (p^2 - k^2 - x x_1) F}{f - x_1 F}.$$

Hence, $f_1 = t (x f + (p^2 - k^2 - x x_1) F)$ and

$$F_1 = t (f - x_1 F)$$

★ In matrix form

$$\begin{bmatrix} f_1 \\ F_1 \end{bmatrix} = t \begin{bmatrix} x & p^2 - k^2 - xx_1 \\ 1 & -x_1 \end{bmatrix} \begin{bmatrix} f \\ F \end{bmatrix}$$

Matches $\psi_1 = \mathbf{L} \psi$ with $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$

★ Similarly, from $x_{23} = \frac{f_2}{F_2} = \frac{xf + (q^2 - k^2 - xx_2)F}{f - x_2 F}$

$$\psi_2 = \begin{bmatrix} f_2 \\ F_2 \end{bmatrix} = s \begin{bmatrix} x & q^2 - k^2 - xx_2 \\ 1 & -x_2 \end{bmatrix} \begin{bmatrix} f \\ F \end{bmatrix} = \mathbf{M} \psi.$$

Therefore,

$$L = t L_{\text{core}} = t \begin{bmatrix} x & p^2 - k^2 - xx_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = s M_{\text{core}} = s \begin{bmatrix} x & q^2 - k^2 - xx_2 \\ 1 & -x_2 \end{bmatrix}$$

- **Step 3:** Determine t and s

- ★ Substitute $L = t L_{\text{core}}, M = s M_{\text{core}}$ into

$$L_2 M - M_1 L = 0$$

$$\longrightarrow t_2 s (L_{\text{core}})_2 M_{\text{core}} - s_1 t (M_{\text{core}})_1 L_{\text{core}} = 0$$

- ★ Solve the equation from the (2-1)-element for

$$\frac{t_2}{t} \frac{s}{s_1} = f(x, x_1, x_2, p, q, \dots)$$

Find rational t and s .

- ★ Apply determinant to get

$$\frac{t_2}{t} \frac{s}{s_1} = \sqrt{\frac{\det L_{\text{core}}}{\det (L_{\text{core}})_2}} \sqrt{\frac{\det (M_{\text{core}})_1}{\det M_{\text{core}}}}$$

$$\text{Solution: } t = \frac{1}{\sqrt{\det L_{\text{core}}}}, \quad s = \frac{1}{\sqrt{\det M_{\text{core}}}}$$

→ Always works but introduces **roots!**

The ratio $\frac{t_2}{t} \frac{s}{s_1}$ is **invariant** under the change

$$t \rightarrow \frac{a_1}{a} t, \quad s \rightarrow \frac{a_2}{a} s,$$

where $a(x)$ is arbitrary.

Proper choice of $a(x) \implies$ **Rational** t and s .

No roots needed!

Lax Pair Algorithm – Systems of $P\Delta E_s$

(Nijhoff 2001, Bobenko and Suris 2001)

Applies to systems of $P\Delta E_s$ that are
consistent around the cube

Example: Schwarzian-Boussinesq System

$$x_1 y - z_1 + z = 0$$

$$x_2 y - z_2 + z = 0$$

$$x y_{12}(y_1 - y_2) - y(p x_1 y_2 - q x_2 y_1) = 0$$

- Note: System has two **single-edge** equations and one **full-face** equation

- Edge equations require **augmentation** of system with additional shifted, edge equations

$$x_{12} y_2 - z_{12} + z_2 = 0$$

$$x_{12} y_1 - z_{12} + z_1 = 0$$

- Edge equations will provide additional constraints during homogenization (Step 2).

The way you handle edge equations leads to gauge-equivalent Lax pairs!

- **Step 1:** Verify the consistency around the cube
 - ★ System on the **front face**:

$$x_1 y - z_1 + z = 0$$

$$x_2 y - z_2 + z = 0$$

$$x y_{12} (y_1 - y_2) - y (p x_1 y_2 - q x_2 y_1) = 0$$

$$x_{12} y_2 - z_{12} + z_2 = 0$$

$$x_{12} y_1 - z_{12} + z_1 = 0$$

Solve for x_{12} , y_{12} , and z_{12} :

$$x_{12} = \frac{z_2 - z_1}{y_1 - y_2}$$

$$y_{12} = \frac{y (p x_1 y_2 - q x_2 y_1)}{x (y_1 - y_2)}$$

$$z_{12} = \frac{y_1 z_2 - y_2 z_1}{y_1 - y_2}$$

Compute x_{123} , y_{123} , and z_{123} :

$$x_{123} = \frac{z_{23} - z_{13}}{y_{13} - y_{23}}$$

$$y_{123} = \frac{y_3(px_{13}y_{23} - qx_{23}y_{13})}{x_3(y_{13} - y_{23})}$$

$$z_{123} = \frac{y_{13}z_{23} - y_{23}z_{13}}{y_{13} - y_{23}}$$

★ System on the bottom face:

$$x_1 y - z_1 + z = 0$$

$$x_3 y - z_3 + z = 0$$

$$x y_{13} (y_1 - y_3) - y (p x_1 y_3 - k x_3 y_1) = 0$$

$$x_{13} y_3 - z_{13} + z_3 = 0$$

$$x_{13} y_1 - z_{13} + z_1 = 0$$

Solve for x_{13} , y_{13} , and z_{13} :

$$x_{13} = \frac{z_3 - z_1}{y_1 - y_3}$$

$$y_{13} = \frac{y (p x_1 y_3 - k x_3 y_1)}{x (y_1 - y_3)}$$

$$z_{13} = \frac{y_1 z_3 - y_3 z_1}{y_1 - y_3}$$

Compute x_{123} , y_{123} , and z_{123} :

$$x_{123} = \frac{z_{23} - z_{12}}{y_{12} - y_{23}}$$

$$y_{123} = \frac{y_2(px_{12}y_{23} - kx_{23}y_{12})}{x_2(y_{12} - y_{23})}$$

$$z_{123} = \frac{y_{12}z_{23} - y_{23}z_{12}}{y_{12} - y_{23}}$$

★ System on the left face:

$$x_3 y - z_3 + z = 0$$

$$x_2 y - z_2 + z = 0$$

$$x y_{23} (y_3 - y_2) - y (p x_3 y_2 - q x_2 y_3) = 0$$

$$x_{23} y_2 - z_{23} + z_2 = 0$$

$$x_{23} y_1 - z_{23} + z_1 = 0$$

Solve for x_{23} , y_{23} , and z_{23} :

$$x_{23} = \frac{z_3 - z_2}{y_2 - y_3}$$

$$y_{23} = \frac{y(qx_2 y_3 - kx_3 y_2)}{x(y_2 - y_3)}$$

$$z_{23} = \frac{y_2 z_3 - y_3 z_2}{y_2 - y_3}$$

Compute x_{123} , y_{123} , and z_{123} :

$$x_{123} = \frac{z_{13} - z_{12}}{y_{12} - y_{13}}$$

$$y_{123} = \frac{y_1(qx_{12}y_{13} - kx_{13}y_{12})}{x_1(y_{12} - y_{13})}$$

$$z_{123} = \frac{y_{12}z_{13} - y_{13}z_{12}}{y_{12} - y_{13}}$$

Substitute x_{12} , y_{12} , y_{12} , x_{13} , y_{13} , z_{13} , x_{23} , y_{23} , z_{23} into the above to get

$$\begin{aligned}
 x_{123} &= \frac{x(x_1 - x_2)(y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2))}{(z_1 - z_2)(px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1))} \\
 y_{123} &= \frac{q(z_2 - z_1)(kx_3y_1 - px_1y_3) + k(z_3 - z_1)(px_1y_2 - qx_2y_1)}{x_1(px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1))} \\
 z_{123} &= \frac{px_1(y_3z_2 - y_2z_3) + qx_2(y_1z_3 - y_3z_1) + kx_3(y_2z_1 - y_1z_2)}{px_1(y_3 - y_2) + qx_2(y_1 - y_3) + kx_3(y_2 - y_1)}
 \end{aligned}$$

Answer is **unique** and independent of x and y .

Consistency around the cube is satisfied!

- Step 2: Homogenization

★ Observed that x_3 , y_3 and z_3 appear linearly in numerators and denominators of

$$x_{13} = \frac{z_3 - z_1}{y_1 - y_3}$$

$$y_{13} = \frac{y(px_1y_3 - kx_3y_1)}{x(y_1 - y_3)}$$

$$z_{13} = \frac{y_1z_3 - y_3z_1}{y_1 - y_3}$$

★ Substitute

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{G}, \quad \text{and} \quad z_3 = \frac{h}{H}.$$

★ Use constraints (from left face edges)

$$\begin{aligned} x_1 y - z_1 + z &= 0, \quad x_2 y - z_2 + z = 0 \\ \Rightarrow x_3 y - z_3 + z &= 0 \end{aligned}$$

Solve for $x_3 = \frac{z_3 - z}{y}$

Thus, $x_3 = \frac{f}{F} = \frac{h - zH}{yH}$, $y_3 = \frac{g}{G}$, and $z_3 = \frac{h}{H}$

★ Substitute x_3, y_3, z_3 into x_{13}, y_{13}, z_{13} :

$$x_{13} = \frac{G(h - z_1 H)}{H(y_1 G - g)}$$

$$y_{13} = \frac{y(px_1 g F - ky_1 f G)}{F x(y_1 G - g)}$$

$$z_{13} = \frac{y_1 h G - z_1 g H}{H(y_1 G - g)}$$

Require that numerators and denominators are

linear in f, g, h, F, G , and H . That forces $H = G = F$.

Hence, $x_3 = \frac{h - zF}{yF}$, $y_3 = \frac{g}{F}$, and $z_3 = \frac{h}{F}$.

★ Compute

$$x_3 = \frac{h - zF}{yF} \longrightarrow x_{13} = \frac{h_1 - z_1 F_1}{y_1 F_1}$$

$$y_3 = \frac{g}{F} \longrightarrow y_{13} = \frac{g_1}{F_1}$$

$$z_3 = \frac{h}{F} \longrightarrow z_{13} = \frac{h_1}{F_1}$$

Hence,

$$x_{13} = \frac{h - z_1 F}{y_1 F - g} = \frac{h_1 - z_1 F_1}{y_1 F_1}$$

$$y_{13} = \frac{y p x_1 g - k y_1 h + k z y_1 F}{x(y_1 F - g)} = \frac{g_1}{F_1}$$

$$z_{13} = \frac{y_1 h - z_1 g}{y_1 F - g} = \frac{h_1}{F_1}$$

Note that

$$x_{13} = \frac{h - z_1 F}{y_1 F - g} = \frac{h_1 - z_1 F_1}{y_1 F_1}$$

is automatically satisfied as a result of the relation $x_3 = \frac{z_3 - z}{y}$.

★ Write in matrix form:

$$\psi_1 = \begin{bmatrix} F_1 \\ g_1 \\ h_1 \end{bmatrix} = t \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kzy_1}{x} & \frac{pyx_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix} \begin{bmatrix} F \\ g \\ h \end{bmatrix} = L \psi$$

★ Repeat the same steps for x_{23}, y_{23}, z_{23} to obtain

$$\psi_2 = \begin{bmatrix} F_2 \\ g_2 \\ h_2 \end{bmatrix} = s \begin{bmatrix} y_2 & -1 & 0 \\ \frac{kzy_2}{x} & \frac{qyx_2}{x} & -\frac{ky_2}{x} \\ 0 & -z_2 & y_2 \end{bmatrix} \begin{bmatrix} F \\ g \\ h \end{bmatrix} = M \psi$$

★ Therefore,

$$L = tL_{\text{core}} = t \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kzy_1}{x} & \frac{pyx_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix}$$

$$M = sM_{\text{core}} = s \begin{bmatrix} y_2 & -1 & 0 \\ \frac{kzy_2}{x} & \frac{qyx_2}{x} & -\frac{ky_2}{x} \\ 0 & -z_2 & y_2 \end{bmatrix}$$

- Step 3: Determine t and s

★ Substitute $L = t L_{\text{core}}, M = s M_{\text{core}}$ into
 $L_2 M - M_1 L = 0$

$$\longrightarrow t_2 s (L_{\text{core}})_2 M_{\text{core}} - s_1 t (M_{\text{core}})_1 L_{\text{core}} = 0$$

★ Solve the equation from the (2-1)-element:

$$\frac{t_2}{t} \frac{s}{s_1} = \frac{y_1}{y_2}.$$

Thus, $t = s = \frac{1}{y}$, or $t = \frac{1}{y_1}$ and $s = \frac{1}{y_2}$,

or $t = \sqrt[3]{\frac{x}{y_1^2 y x_1}}$ and $s = \sqrt[3]{\frac{x}{y_2^2 y x_2}}$.

Summary: Lax Pair for Schwarzian-BSQ System

- Option a: Solving the edge equation for $x_3 = \frac{z_3 - z}{y}$ yields

$$x_3 = \frac{h - zF}{yF}, \quad y_3 = \frac{g}{F}, \quad \text{and} \quad z_3 = \frac{h}{F}, \quad \psi_a = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_a = \frac{1}{y} \begin{bmatrix} y_1 & -1 & 0 \\ \frac{kzy_1}{x} & \frac{pyx_1}{x} & -\frac{ky_1}{x} \\ 0 & -z_1 & y_1 \end{bmatrix}$$

- Option b: Solving the edge equation for $z_3 = x_3 y + z$ yields

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad \text{and} \quad z_3 = \frac{zF + yf}{F}, \quad \psi_b = \begin{bmatrix} F \\ f \\ g \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_b = \frac{1}{y} \begin{bmatrix} y_1 & 0 & -1 \\ z - z_1 & y & 0 \\ 0 & -\frac{ky y_1}{x_1} & \frac{py z_1}{x} \end{bmatrix}$$

- Gauge Equivalences between these Lax Matrices
- $L_b = \mathcal{G}_1 L_a \mathcal{G}^{-1}, \quad \psi_b = \mathcal{G} \psi_a$

with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{z}{y} & 0 & \frac{1}{y} \\ 0 & 1 & 0 \end{bmatrix}$$

Software Demonstration

New Results from Investigation of Gauge Equivalence of Generalized BSQ Lattices due to Hietarinta (2011)

- Four systems: A-2, B-2, C-3, and C-4.
- Only the B-2 system requires that $\mathbb{G}(a, c) + \mathbb{G}(c, b) = \mathbb{G}(a, b)$ is satisfied.
- Results for the A-2 and B-2 systems are similar.
- Results for the C-3 and C-4 systems are similar.

- Example 1: Hietarinta's A-2 System (2011)

$$zx_1 - y_1 - x = 0$$

$$zx_2 - y_2 - x = 0$$

$$y - xz_{12} + b_0x + \frac{\mathbb{G}(-p, -a)x_1 - \mathbb{G}(-q, -a)x_2}{z_2 - z_1} = 0$$

with $\mathbb{G}(\omega, \kappa) := \omega^3 - \kappa^3 + \alpha_2(\omega^2 - \kappa^2) + \alpha_1(\omega - \kappa) = 0$

System is CAC without any condition for $\mathbb{G}$.

- Option a: Solving for $x_3 = \frac{y_3+x}{z}$ yields

$$x_3 = \frac{xF+g}{zF}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and } \psi_a = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_a = \frac{1}{z} \begin{bmatrix} -zz_1 & 0 & z \\ xz_1 & z_1 & -zx_1 \\ \ell_{31} & -\frac{\mathbb{G}(-k, -a)}{x} & \frac{z(y+b_0x)}{x} \end{bmatrix}$$

$$\ell_{31} = \frac{1}{x} \left(\mathbb{G}(-p, -a)zx_1 - \mathbb{G}(-k, -a)x - zz_1(y + b_0x) \right)$$

- Option b: Solving for $y_3 = zx_3 - x$ yields

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{-xF + zf}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and } \psi_b = \begin{bmatrix} F \\ f \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_b = \begin{bmatrix} -z_1 & 0 & 1 \\ -x_1 & 1 & 0 \\ \ell_{31} & -\frac{\mathbb{G}(-k, -a)}{x} & \frac{y + b_0 x}{x} \end{bmatrix}$$

$$\ell_{31} = \frac{1}{x} \left(\mathbb{G}(-p, -a)x_1 - z_1(y + b_0 x) \right)$$

- Option c: Substituting

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and setting } \psi_c = \begin{bmatrix} F \\ f \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_c = \begin{bmatrix} -z_1 & 0 & 0 & 1 \\ -x_1 & 1 & 0 & 0 \\ 0 & z_1 & 0 & -x_1 \\ \ell_{41} & -\frac{\mathbb{G}(-k, -a)}{x} & 0 & \frac{y+b_0x}{x} \end{bmatrix}$$

$$\ell_{41} = \frac{1}{x} \left(\mathbb{G}(-p, -a)x_1 - z_1(y + b_0x) \right)$$

- Gauge Equivalences between these Lax Matrices
- $L_b = \mathcal{G}_1 L_a \mathcal{G}^{-1}, \quad \psi_b = \mathcal{G} \psi_a$

with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{x}{z} & \frac{1}{z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_c = \mathcal{H}_1 L_a \mathcal{H}_{\text{Left}}^{-1}, \quad \psi_c = \mathcal{H} \psi_a$

$$\mathcal{H} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{x}{z} & \frac{1}{z} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathcal{H}_{\text{Left}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -x & z & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $L_c = \mathcal{J}_1 L_b \mathcal{J}_{\text{Left}}^{-1}, \quad \psi_c = \mathcal{J} \psi_b$

$$\mathcal{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x & z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathcal{J}_{\text{Left}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Example 2:** Hietarinta's B-2 System (2011)

$$xx_1 - z_1 - y = 0$$

$$xx_2 - z_2 - y = 0$$

$$y_{12} + \alpha_1 + z + \alpha_2(x_{12} - x) - xx_{12} + \frac{\mathbb{G}(-p, -q)}{x_2 - x_1} = 0$$

with $\mathbb{G}(\omega, \kappa) := \omega^3 - \kappa^3 + \alpha_2(\omega^2 - \kappa^2) + \alpha_1(\omega - \kappa) = 0$

System is 3D consistent if $\mathbb{G}(a, c) + \mathbb{G}(c, b) = \mathbb{G}(a, b)$ holds.

- Option a: Solving for $x_3 = \frac{z_3+y}{x}$ yields

$$x_3 = \frac{yF+h}{xF}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and } \psi_a = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_a = \frac{1}{x} \begin{bmatrix} y - xx_1 & 0 & 1 \\ \ell_{21} & x(x - \alpha_2) & \alpha_2 x - \alpha_1 - z \\ -yy_1 & xx_1 & -y_1 \end{bmatrix}$$

$$\ell_{21} = (\alpha_2 x - \alpha_1 - z)(y - xx_1) - \mathbb{G}(-p, -k)x - xy_1(x - \alpha_2)$$

- Option b: Solving for $z_3 = xx_3 - y$ yields

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{-yF + xf}{F}, \quad \text{and } \psi_b = \begin{bmatrix} F \\ f \\ g \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_b = \begin{bmatrix} -x_1 & 1 & 0 \\ -y_1 & 0 & 1 \\ \ell_{31} & \alpha_2 x - \alpha_1 - z & x - \alpha_2 \end{bmatrix}$$

$$\ell_{31} = -\left(x_1(\alpha_2 x - \alpha_1 - z) + y_1(x - \alpha_2) + \mathbb{G}(-p, -k)\right)$$

- Option c: Substituting

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and setting } \psi_c = \begin{bmatrix} F \\ f \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_c = \begin{bmatrix} -x_1 & 1 & 0 & 0 \\ -y_1 & 0 & 1 & 0 \\ \ell_{31} & \alpha_2 x - \alpha_1 - z & x - \alpha_2 & 0 \\ 0 & -y_1 & x_1 & 0 \end{bmatrix}$$

$$\ell_{31} = -\left(x_1(\alpha_2 x - \alpha_1 - z) + y_1(x - \alpha_2) + \mathbb{G}(-p, -k)\right)$$

- Gauge Equivalences between these Lax Matrices
- $L_b = \mathcal{G}_1 L_a \mathcal{G}^{-1}, \quad \psi_b = \mathcal{G} \psi_a$

with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{y}{x} & 0 & \frac{1}{x} \\ 0 & 1 & 0 \end{bmatrix}$$

- $L_c = \mathcal{H}_1 L_a \mathcal{H}_{\text{Left}}^{-1}, \quad \psi_c = \mathcal{H} \psi_a$

$$\mathcal{H} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{y}{x} & 0 & \frac{1}{x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathcal{H}_{\text{Left}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -y & x & 0 & 0 \end{bmatrix}$$

- $L_c = \mathcal{J}_1 L_b \mathcal{J}_{\text{Left}}^{-1}, \quad \psi_c = \mathcal{J} \psi_b$

$$\mathcal{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -y & x & 0 \end{bmatrix}$$

and

$$\mathcal{J}_{\text{Left}}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- **Example 3:** Hietarinta's C-3 System (2011)

$$y_1 z + x_1 - x = 0 \quad = \quad 0$$

$$y_2 z + x_2 - x \quad = \quad 0$$

$$\begin{aligned} & \mathbb{G}(-a, -b)x_{12} - yz_{12} \\ & + z \left(\frac{\mathbb{G}(-q, -b)z_1 y_2 - \mathbb{G}(-p, -b)z_2 y_1}{z_1 - z_2} \right) = 0 \end{aligned}$$

with $\mathbb{G}(\omega, \kappa) := \omega^3 - \kappa^3 + \alpha_2(\omega^2 - \kappa^2) + \alpha_1(\omega - \kappa) = 0$

System is CAC without any condition on $\mathbb{G}$.

- Solving for $x_3 = x - y_3 z$ yields

$$x_3 = \frac{xF - zg}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and } \psi_a = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_a = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 1 \\ zy_1 & -z & 0 \\ -\mathbb{G}(-a, -b) \frac{xz_1}{y} & \ell_{32} & \ell_{33} \end{bmatrix}$$

$$\ell_{32} = \frac{zz_1}{y} \left(\mathbb{G}(-a, -b) - \mathbb{G}(-k, -b) \right)$$

$$\ell_{33} = \frac{1}{y} \left(\mathbb{G}(-a, -b)(x - y_1 z) + \mathbb{G}(-p, -b)zy_1 \right)$$

- Solving for $y_3 = \frac{x-x_3}{z}$ yields

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{x_3 F - f}{z F}, \quad z_3 = \frac{h}{F}, \quad \text{and} \quad \psi_b = \begin{bmatrix} F \\ f \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_b = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 1 \\ 0 & -z_1 & x_1 \\ -\mathbb{G}(-k, -b) \frac{x z_1}{y} & \ell_{32} & \ell_{33} \end{bmatrix}$$

$$\ell_{32} = \frac{z_1}{y} \left(\mathbb{G}(-k, -b) - \mathbb{G}(-a, -b) \right)$$

$$\ell_{33} = \frac{1}{y} \left(x_1 \mathbb{G}(-a, -b) + y_1 z \mathbb{G}(-p, -b) \right)$$

- Substituting

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and setting } \psi_c =$$

$$\begin{bmatrix} F \\ f \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_c = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 0 & 1 \\ 0 & -z_1 & 0 & x_1 \\ zy_1 & 0 & -z & 0 \\ 0 & -\mathbb{G}(-a, -b) \frac{z_1}{y} & \ell_{34} & \ell_{44} \end{bmatrix}$$

$$\ell_{34} = -\mathbb{G}(-k, -b) \frac{zz_1}{y}$$

$$\ell_{44} = \frac{1}{y} \left(\mathbb{G}(-a, -b)x_1 + \mathbb{G}(-p, -b)zy_1 \right)$$

- Gauge Equivalences between these Lax Matrices
- $L_b = \mathcal{G}_1 L_a \mathcal{G}^{-1}, \quad \psi_b = \mathcal{G} \psi_a$

with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ x & -z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_a = \tilde{\mathcal{H}}_1 L_c \tilde{\mathcal{H}}_{\text{Right}}^{-1}, \quad \psi_a = \tilde{\mathcal{H}} \psi_c$

$$\tilde{\mathcal{H}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{z} & -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\tilde{\mathcal{H}}_{\text{Right}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ x & -z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_b = \tilde{\mathcal{J}}_1 L_c \tilde{\mathcal{J}}_{\text{Right}}^{-1}, \quad \psi_b = \tilde{\mathcal{J}} \psi_c$

$$\tilde{\mathcal{J}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 0 & -z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\tilde{\mathcal{J}}_{\text{Right}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{x}{z} & -\frac{1}{z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Example 4:** Hietarinta's C-4 System (2011)

$$y_1 z + x_1 - x = 0 = 0$$

$$y_2 z + x_2 - x = 0$$

$$y z_{12} - z \left(\frac{z_2 y_1 \mathcal{G}(p) - z_1 y_2 \mathcal{G}(q)}{z_1 - z_2} \right) - x_{12} x + \frac{1}{4} \mathbb{G}(-a, -b)^2 = 0$$

with $\mathbb{G}(\omega, \kappa) := \omega^3 - \kappa^3 + \alpha_2(\omega^2 - \kappa^2) + \alpha_1(\omega - \kappa) = 0$

System is CAC without any condition on $\mathbb{G}$.

- Solving for $x_3 = x - y_3 z$ yields

$$x_3 = \frac{xF - zg}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and } \psi_a = \begin{bmatrix} F \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_a = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 1 \\ x - x_1 & -z & 0 \\ \ell_{31} & \frac{zz_1}{y} \left(x + \mathcal{G}(k) \right) & \ell_{33} \end{bmatrix}$$

$$\ell_{31} = - \frac{z_1}{y} \left(x^2 - \frac{1}{4} \mathbb{G}(-a, -b)^2 \right)$$

$$\ell_{33} = \frac{1}{y} \left(xx_1 - \frac{1}{4} \mathbb{G}(-a, -b)^2 - y_1 z \mathcal{G}(p) \right)$$

- Solving for $y_3 = \frac{x-x_3}{z}$ yields

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{xF-f}{zF}, \quad z_3 = \frac{h}{F}, \quad \text{and} \quad \psi_b = \begin{bmatrix} F \\ f \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_b = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 1 \\ 0 & -z_1 & x_1 \\ \ell_{31} & -\frac{z_1}{y} \left(x + \mathcal{G}(k) \right) & \ell_{33} \end{bmatrix}$$

$$\ell_{31} = \frac{z_1}{y} \left(\frac{1}{4} \mathbb{G}(-a, -b)^2 + x \mathcal{G}(k) \right)$$

$$\ell_{33} = \frac{1}{y} \left(xx_1 - \frac{1}{4} \mathbb{G}(-a, -b)^2 - y_1 z \mathcal{G}(p) \right)$$

- Substituting

$$x_3 = \frac{f}{F}, \quad y_3 = \frac{g}{F}, \quad z_3 = \frac{h}{F}, \quad \text{and setting } \psi_c = \begin{bmatrix} F \\ f \\ g \\ h \end{bmatrix}$$

- Corresponding Lax matrix:

$$L_c = \frac{1}{z} \begin{bmatrix} -z_1 & 0 & 0 & 1 \\ 0 & -z_1 & 0 & x_1 \\ zy_1 & 0 & -z & 0 \\ \ell_{11} & -\frac{xz_1}{y} & \frac{zz_1}{y} \mathcal{G}(k) & \ell_{44} \end{bmatrix}$$

$$\ell_{11} = \frac{1}{4} \mathbb{G}(-a, -b)^2 \frac{z_1}{y}$$

$$\ell_{44} = \frac{1}{y} \left(xx_1 - \frac{1}{4} \mathbb{G}(-a, -b)^2 - y_1 z \mathcal{G}(p) \right)$$

- Gauge Equivalences between these Lax Matrices
- $L_b = \mathcal{G}_1 L_a \mathcal{G}^{-1}, \quad \psi_b = \mathcal{G} \psi_a$

with

$$\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 \\ x & -z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_a = \tilde{\mathcal{H}}_1 L_c \tilde{\mathcal{H}}_{\text{Right}}^{-1}, \quad \psi_a = \tilde{\mathcal{H}} \psi_c$

$$\tilde{\mathcal{H}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{x}{z} & -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\tilde{\mathcal{H}}_{\text{Right}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ x & -z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $L_b = \tilde{\mathcal{J}}_1 L_c \tilde{\mathcal{J}}_{\text{Right}}^{-1}, \quad \psi_b = \tilde{\mathcal{J}} \psi_c$

$$\tilde{\mathcal{J}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 0 & -z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\tilde{\mathcal{J}}_{\text{Right}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{x}{z} & -\frac{1}{z} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Example 5:** Lattice due to Hietarinta (2011)

$$x_1 z - y_1 - x = 0$$

$$x_2 z - y_2 - x = 0$$

$$z_{12} - \frac{y}{x} - \frac{1}{x} \left(\frac{px_1 - qx_2}{z_1 - z_2} \right) = 0$$

Note: System has two **single-edge** equations and one **full-face** equation.

- **Lax pair:**

$$L = t \begin{bmatrix} \frac{yz}{x} & \frac{k}{x} & \frac{kx - px_1 z - yzz_1}{x} \\ -x_1 z & z_1 & xz_1 \\ z & 0 & -zz_1 \end{bmatrix}$$

and

$$M = s \begin{bmatrix} \frac{yz}{x} & \frac{k}{x} & \frac{kx - qx_2z - yzz_2}{x} \\ -x_2z & z_2 & xz_2 \\ z & 0 & -zz_2 \end{bmatrix},$$

where $t = s = \frac{1}{z}$, or $t = \frac{1}{z_1}$, $s = \frac{1}{z_2}$,

$$\text{or } t = \sqrt[3]{\frac{x}{x_1 z^2 z_1}}, \quad s = \sqrt[3]{\frac{x}{x_2 z^2 z_2}}.$$

Here, $\frac{t_2}{t} \frac{s}{s_1} = \frac{z_1}{z_2}$.

- **Example 6:** Discrete Boussinesq System
(Tongas and Nijhoff 2005)

$$z_1 - xx_1 + y = 0$$

$$z_2 - xx_2 + y = 0$$

$$(x_2 - x_1)(z - xx_{12} + y_{12}) - p + q = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} & -x_1 & 1 & 0 \\ & -y_1 & 0 & 1 \\ p - k - xy_1 + x_1z & -z & x \end{bmatrix}$$

$$M = s \begin{bmatrix} -x_2 & 1 & 0 \\ -y_2 & 0 & 1 \\ q - k - xy_2 + x_2z & -z & x \end{bmatrix}$$

with $t = s = 1$, or $t = \frac{1}{\sqrt[3]{p-k}}$ and $s = \frac{1}{\sqrt[3]{q-k}}$.

Note: $x_3 = \frac{f}{F}$, $y_3 = \frac{g}{F}$, and $\psi = \begin{bmatrix} f \\ F \\ g \end{bmatrix}$, and $\frac{t_2}{t} \frac{s}{s_1} = 1$.

- **Example 7:** System of pKdV Lattices
(Xenitidis and Mikhailov 2009)

$$(x - x_{12})(y_1 - y_2) - p^2 + q^2 = 0$$

$$(y - y_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

- **Lax pair:**

$$L = \begin{bmatrix} 0 & 0 & tx & t(p^2 - k^2 - xy_1) \\ 0 & 0 & t & -ty_1 \\ Ty & T(p^2 - k^2 - x_1y) & 0 & 0 \\ T & -Tx_1 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & sx & s(q^2 - k^2 - xy_2) \\ 0 & 0 & s & -sy_2 \\ Sy & S(q^2 - k^2 - x_2y) & 0 & 0 \\ S & -Sx_2 & 0 & 0 \end{bmatrix}$$

with $t = s = T = S = 1$,

or $tT = \frac{1}{\sqrt{\text{Det}L_c}} = \frac{1}{p-k}$ and $sS = \frac{1}{\sqrt{\text{Det}M_c}} = \frac{1}{q-k}$.

Note: $\frac{t_2}{T} \frac{S}{s_1} = 1$ and $\frac{T_2}{t} \frac{s}{S_1} = 1$,

or $\frac{T_2}{\mathcal{T}} \frac{\mathcal{S}}{S_1} = 1$, with $\mathcal{T} = tT$, $\mathcal{S} = sS$.

Here, $x_3 = \frac{f}{F}$, $y_3 = \frac{g}{G}$, and $\psi = [f \ F \ g \ G]^T$.

- **Example 8:** Discrete NLS System
(Xenitidis and Mikhailov 2009)

$$y_1 - y_2 - y \left((x_1 - x_2)y + p - q \right) = 0$$

$$x_1 - x_2 + x_{12} \left((x_1 - x_2)y + p - q \right) = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} -1 & x_1 \\ y & k - p - yx_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} -1 & x_2 \\ y & k - q - yx_2 \end{bmatrix}$$

with $t = s = 1$, or $t = \frac{1}{\sqrt{\text{Det} L_c}} = \frac{1}{\sqrt{\alpha - k}}$ and $s = \frac{1}{\sqrt{\beta - k}}$.

Note: $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and $\frac{t_2}{t} \frac{s}{s_1} = 1$.

- **Example 9:** Schwarzian-Boussinesq Lattice
(Nijhoff 1999)

$$z_1 y - x_1 + x = 0$$

$$z_2 y - x_2 + x = 0$$

$$z y_{12} (y_1 - y_2) - y(p y_2 z_1 - q y_1 z_2) = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} y & 0 & -yz_1 \\ -\frac{ky y_1}{z} & \frac{py z_1}{z} & 0 \\ 0 & -1 & y_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} y & 0 & -yz_2 \\ -\frac{ky y_2}{z} & \frac{py z_2}{z} & 0 \\ 0 & -1 & y_2 \end{bmatrix}$$

with $t = s = \frac{1}{y}$, or $t = \frac{1}{y_1}$ and $s = \frac{1}{y_2}$,

or $t = \sqrt[3]{\frac{z}{y^2 y_1 z_1}}$ and $s = \sqrt[3]{\frac{z}{y^2 y_2 z_2}}$.

Note: $x_3 = \frac{fy + Fx}{F}$, $y_3 = \frac{g}{F}$, $z_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ g \\ F \end{bmatrix}$,

and $\frac{t_2}{t} \frac{s}{s_1} = \frac{y_1}{y_2}$.

- **Example 10:** Toda modified Boussinesq System (Nijhoff 1992)

$$y_{12}(p - q + x_2 - x_1) - (p - 1)y_2 + (q - 1)y_1 = 0$$

$$y_1 y_2 (p - q - z_2 + z_1) - (p - 1)y y_2 + (q - 1)y y_1 = 0$$

$$y(p + q - z - x_{12})(p - q + x_2 - x_1) - (p^2 + p + 1)y_1 + (q^2 + q + 1)y_2 = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} k + p - z & \frac{1+k+k^2}{y} & \frac{-k^2 y - y_1 - p^2(y_1 - y) - k y(x_1 - z) + y z x_1}{y} \\ 0 & p - 1 & (1 - k)y_1 \\ 1 & 0 & p - k - x_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} k + q - z & \frac{1+k+k^2}{y} & \frac{-k^2y-y_2-q^2(y_2-y)-ky(x_2-z)+yzx_2}{y} \\ 0 & q-1 & (1-k)y_2 \\ 1 & 0 & q-k-x_2 \end{bmatrix}$$

with $t = s = 1$, or $t = \sqrt[3]{\frac{y_1}{y}}$ and $s = \sqrt[3]{\frac{y_2}{y}}$.

Note: $x_3 = \frac{f}{F}$, $y_3 = \frac{g}{F}$, $\psi = \begin{bmatrix} f \\ g \\ F \end{bmatrix}$

Here, $\frac{t_2}{t} \frac{s}{s_1} = 1$.

Conclusions and Future Work

- *Mathematica* code works for scalar $P\Delta E_s$ in 2D defined on quad-graphs (quadrilateral faces).
- *Mathematica* code has been extended to systems of $P\Delta E_s$ in 2D defined on quad-graphs.
- Code can be used to test (i) consistency around the cube and compute or test (ii) Lax pairs.
- Consistency around cube $\implies P\Delta E$ has Lax pair.
- $P\Delta E$ has Lax pair $\nRightarrow$ consistency around cube.
Indeed, there are $P\Delta E_s$ with a Lax pair that are not consistent around the cube.
Example: discrete sine-Gordon equation.

- Avoid the determinant method to avoid square roots! Factorization plays an essential role!
- Hard cases: $Q4$ equation (elliptic curves, Weierstraß functions) (Nijhoff 2001) and $Q5$
- Future Work: Extension to more complicated systems of $P\Delta E_s$.
- $P\Delta E_s$ in 3D: Lax pair will be expressed in terms of tensors. Consistency around a “hypercube”.
Examples: discrete Kadomtsev-Petviashvili (KP) equations.

Thank You

Additional Examples

- **Example 9:** (H1) Equation (ABS classification)

$$(x - x_{12})(x_1 - x_2) + q - p = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} x & p - k - xx_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} x & q - k - xx_2 \\ 1 & -x_2 \end{bmatrix}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{k-p}}$ and $s = \frac{1}{\sqrt{k-q}}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = 1$.

- **Example 10:** (H2) Equation (ABS 2003)

$$(x - x_{12})(x_1 - x_2) + (q - p)(x + x_1 + x_2 + x_{12}) + q^2 - p^2 = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} p - k + x & p^2 - k^2 + (p - k)(x + x_1) - xx_1 \\ 1 & -(p - k + x_1) \end{bmatrix}$$

$$M = s \begin{bmatrix} q - k + x & q^2 - k^2 + (q - k)(x + x_2) - xx_2 \\ 1 & -(q - k + x_2) \end{bmatrix}$$

with $t = \frac{1}{\sqrt{2(k-p)(p+x+x_1)}}$ and $s = \frac{1}{\sqrt{2(k-q)(q+x+x_2)}}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{p+x+x_1}{q+x+x_2}.$

- **Example 11:** (H3) Equation (ABS 2003)

$$p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) + \delta(p^2 - q^2) = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} kx & -(\delta(p^2 - k^2) + pxx_1) \\ p & -kx_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} kx & -(\delta(q^2 - k^2) + qxx_2) \\ q & -kx_2 \end{bmatrix}$$

with $t = \frac{1}{\sqrt{(p^2 - k^2)(\delta p + xx_1)}}$ and $s = \frac{1}{\sqrt{(q^2 - k^2)(\delta q + xx_2)}}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{\delta p + xx_1}{\delta q + xx_2}$.

- Example 12: (H3) Equation ($\delta = 0$) (ABS 2003)

$$p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) = 0$$

- Lax pair:

$$L = t \begin{bmatrix} kx & -pxx_1 \\ p & -kx_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} kx & -qxx_2 \\ q & -kx_2 \end{bmatrix}$$

with $t = s = \frac{1}{x}$ or $t = \frac{1}{x_1}$ and $s = \frac{1}{x_2}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{x x_1}{x x_2} = \frac{x_1}{x_2}$.

- **Example 13:** (Q1) Equation (ABS 2003)

$$p(x - x_2)(x_1 - x_{12}) - q(x - x_1)(x_2 - x_{12}) + \delta^2 pq(p - q) = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} (p - k)x_1 + kx & -p(\delta^2 k(p - k) + xx_1) \\ p & -((p - k)x + kx_1) \end{bmatrix}$$

$$M = s \begin{bmatrix} (q - k)x_2 + kx & -q(\delta^2 k(q - k) + xx_2) \\ q & -((q - k)x + kx_2) \end{bmatrix}$$

with $t = \frac{1}{\delta p \pm (x - x_1)}$ and $s = \frac{1}{\delta q \pm (x - x_2)}$,
 or $t = \frac{1}{\sqrt{k(p - k)((\delta p + x - x_1)(\delta p - x + x_1))}}$ and

$$s = \frac{1}{\sqrt{k(q - k)((\delta q + x - x_2)(\delta q - x + x_2))}}$$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{q(\delta p + (x - x_1))(\delta p - (x - x_1))}{p(\delta q + (x - x_2))(\delta q - (x - x_2))}.$

- **Example 14:** (Q1) Equation ($\delta = 0$) (ABS 2003)

$$p(x - x_2)(x_1 - x_{12}) - q(x - x_1)(x_2 - x_{12}) = 0$$

which is the cross-ratio equation

$$\frac{(x - x_1)(x_{12} - x_2)}{(x_1 - x_{12})(x_2 - x)} = \frac{p}{q}$$

- **Lax pair:**

$$L = t \begin{bmatrix} (p - k)x_1 + kx & -p x x_1 \\ p & -((p - k)x + kx_1) \end{bmatrix}$$

$$M = s \begin{bmatrix} (q - k)x_2 + kx & -q x x_2 \\ q & -((q - k)x + kx_2) \end{bmatrix}$$

Here, $\frac{t_2}{t} \frac{s}{s_1} = \frac{q(x-x_1)^2}{p(x-x_2)^2}$. So, $t = \frac{1}{x-x_1}$ and $s = \frac{1}{x-x_2}$

or $t = \frac{1}{\sqrt{k(k-p)}(x-x_1)}$ and $s = \frac{1}{\sqrt{k(k-q)}(x-x_2)}$.

- **Example 15:** (Q2) Equation (ABS 2003)

$$p(x - x_2)(x_1 - x_{12}) - q(x - x_1)(x_2 - x_{12}) + pq(p - q)(x + x_1 + x_2 + x_{12}) - pq(p - q)(p^2 - pq + q^2) = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} (k - p)(kp - x_1) + kx & \\ -p(k(k - p)(k^2 - kp + p^2 - x - x_1) + xx_1) & \\ p & -((k - p)(kp - x) + kx_1) \end{bmatrix}$$

$$M = s \begin{bmatrix} (k - q)(kq - x_2) + kx & \\ -q(k(k - q)(k^2 - kq + q^2 - x - x_2) + xx_2) & \\ q & -((k - q)(kq - x) + kx_2) \end{bmatrix}$$

- with

$$t = \frac{1}{\sqrt{k(k-p)\left((x-x_1)^2 - 2p^2(x+x_1) + p^4\right)}}$$

and

$$s = \frac{1}{\sqrt{k(k-q)\left((x-x_2)^2 - 2q^2(x+x_2) + q^4\right)}}$$

Note:

$$\begin{aligned} \frac{t_2}{t} \frac{s}{s_1} &= \frac{q \left((x - x_1)^2 - 2p^2(x + x_1) + p^4 \right)}{p \left((x - x_2)^2 - 2q^2(x + x_2) + q^4 \right)} \\ &= \frac{p \left((X + X_1)^2 - p^2 \right) \left((X - X_1)^2 - p^2 \right)}{q \left((X + X_2)^2 - q^2 \right) \left((X - X_2)^2 - q^2 \right)} \end{aligned}$$

with $x = X^2$, and, consequently, $x_1 = X_1^2$, $x_2 = X_2^2$.

- **Example 16:** (Q3) Equation (ABS 2003)

$$(q^2 - p^2)(xx_{12} + x_1x_2) + q(p^2 - 1)(xx_1 + x_2x_{12}) - p(q^2 - 1)(xx_2 + x_1x_{12}) - \frac{\delta^2}{4pq}(p^2 - q^2)(p^2 - 1)(q^2 - 1) = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} -4kp \left(p(k^2 - 1)x + (p^2 - k^2)x_1 \right) & -(p^2 - 1)(\delta^2 k^2 - \delta^2 k^4 - \delta^2 p^2 + \delta^2 k^2 p^2 - 4k^2 pxx_1) \\ -4k^2 p(p^2 - 1) & 4kp \left(p(k^2 - 1)x_1 + (p^2 - k^2)x \right) \end{bmatrix}$$

$$M = s \begin{bmatrix} -4kq \left(q(k^2 - 1)x + (q^2 - k^2)x_2 \right) & -(q^2 - 1)(\delta^2 k^2 - \delta^2 k^4 - \delta^2 q^2 + \delta^2 k^2 q^2 - 4k^2 qxx_2) \\ -4k^2 q(q^2 - 1) & 4kq \left(q(k^2 - 1)x_2 + (q^2 - k^2)x \right) \end{bmatrix}$$

- with

$$t = \frac{1}{2k \sqrt{p(k^2-1)(k^2-p^2) \left(4p^2(x^2+x_1^2) - 4p(1+p^2)xx_1 + \delta^2(1-p^2)^2 \right)}}$$

and

$$s = \frac{1}{2k \sqrt{q(k^2-1)(k^2-q^2) \left(4q^2(x^2+x_2^2) - 4q(1+q^2)xx_2 + \delta^2(1-q^2)^2 \right)}} .$$

Note:

$$\begin{aligned} & \frac{t_2}{t} \frac{s}{s_1} \\ &= \frac{q(q^2-1) (4p^2(x^2+x_1^2) - 4p(1+p^2)xx_1 + \delta^2(1-p^2)^2)}{p(p^2-1) (4q^2(x^2+x_2^2) - 4q(1+q^2)xx_2 + \delta^2(1-q^2)^2)} \\ &= \frac{q(q^2-1) (4p^2(x-x_1)^2 - 4p(p-1)^2xx_1 + \delta^2(1-p^2)^2)}{p(p^2-1) (4q^2(x-x_2)^2 - 4q(q-1)^2xx_2 + \delta^2(1-q^2)^2)} \\ &= \frac{q(q^2-1) (4p^2(x+x_1)^2 - 4p(p+1)^2xx_1 + \delta^2(1-p^2)^2)}{p(p^2-1) (4q^2(x+x_2)^2 - 4q(q+1)^2xx_2 + \delta^2(1-q^2)^2)} \end{aligned}$$

where

$$\begin{aligned} & 4p^2(x^2 + x_1^2) - 4p(1 + p^2)xx_1 + \delta^2(1 - p^2)^2 \\ &= \delta^2(p - e^{X+X_1})(p - e^{-(X+X_1)})(p - e^{X-X_1})(p - e^{-(X-X_1)}) \\ &= \delta^2(p - \cosh(X + X_1) + \sinh(X + X_1)) \\ &\quad (p - \cosh(X + X_1) - \sinh(X + X_1)) \\ &\quad (p - \cosh(X - X_1) + \sinh(X - X_1)) \\ &\quad (p - \cosh(X - X_1) - \sinh(X - X_1)) \end{aligned}$$

with $x = \delta \cosh(X)$, and, consequently,
 $x_1 = \delta \cosh(X_1)$, $x_2 = \delta \cosh(X_2)$.

- **Example 17:** (Q3) Equation $(\delta) = 0$ (ABS 2003)

$$(q^2 - p^2)(xx_{12} + x_1x_2) + q(p^2 - 1)(xx_1 + x_2x_{12}) - p(q^2 - 1)(xx_2 + x_1x_{12}) = 0$$

- **Lax pair:**

$$L = t \begin{bmatrix} (p^2 - k^2)x_1 + p(k^2 - 1)x & -k(p^2 - 1)xx_1 \\ (p^2 - 1)k & -((p^2 - k^2)x + p(k^2 - 1)x_1) \end{bmatrix}$$

$$M = s \begin{bmatrix} (q^2 - k^2)x_2 + q(k^2 - 1)x & -k(q^2 - 1)xx_2 \\ (q^2 - 1)k & -((q^2 - k^2)x + q(k^2 - 1)x_2) \end{bmatrix}$$

- with $t = \frac{1}{px-x_1}$ and $s = \frac{1}{qx-x_2}$

or $t = \frac{1}{px_1-x}$ and $s = \frac{1}{qx_2-x}$

or $t = \frac{1}{\sqrt{(k^2-1)(p^2-k^2)(px-x_1)(px_1-x)}}$

and $s = \frac{1}{\sqrt{(k^2-1)(q^2-k^2)(qx-x_2)(qx_2-x)}}$.

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{(q^2-1)(px-x_1)(px_1-x)}{(p^2-1)(qx-x_2)(qx_2-x)}.$

- **Example 18:** (α, β) -equation (Quispel 1983)

$$\begin{aligned} & ((p - \alpha)x - (p + \beta)x_1) ((p - \beta)x_2 - (p + \alpha)x_{12}) \\ & - ((q - \alpha)x - (q + \beta)x_2) ((q - \beta)x_1 - (q + \alpha)x_{12}) = 0 \end{aligned}$$

- **Lax pair:**

$$L = t \begin{bmatrix} (p - \alpha)(p - \beta)x + (k^2 - p^2)x_1 & -(k - \alpha)(k - \beta)xx_1 \\ (k + \alpha)(k + \beta) & -((p + \alpha)(p + \beta)x_1 + (k^2 - p^2)x) \end{bmatrix}$$

$$M = s \begin{bmatrix} (q - \alpha)(q - \beta)x + (k^2 - q^2)x_2 & -(k - \alpha)(k - \beta)xx_2 \\ (k + \alpha)(k + \beta) & -((q + \alpha)(q + \beta)x_2 + (k^2 - q^2)x) \end{bmatrix}$$

- with $t = \frac{1}{(\alpha-p)x+(\beta+p)x_1}$ and $s = \frac{1}{(\alpha-q)x+(\beta+q)x_2}$

or $t = \frac{1}{(\beta-p)x+(\alpha+p)x_1}$ and $s = \frac{1}{(\beta-q)x+(\alpha+q)x_2}$

or $t = \frac{1}{\sqrt{(p^2-k^2)((\beta-p)x+(\alpha+p)x_1)((\alpha-p)x+(\beta+p)x_1)}}$

and $s = \frac{1}{\sqrt{(q^2-k^2)((\beta-q)x+(\alpha+q)x_2)((\alpha-q)x+(\beta+q)x_2)}}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{((\beta-p)x+(\alpha+p)x_1)((\alpha-p)x+(\beta+p)x_1)}{((\beta-q)x+(\alpha+q)x_2)((\alpha-q)x+(\beta+q)x_2)}.$

- **Example 19:** (A1) Equation (ABS 2003)

$$p(x+x_2)(x_1+x_{12}) - q(x+x_1)(x_2+x_{12}) - \delta^2 pq(p-q) = 0$$

(Q1) if $x_1 \rightarrow -x_1$ and $x_2 \rightarrow -x_2$

- **Lax pair:**

$$L = t \begin{bmatrix} (k-p)x_1 + kx & -p(\delta^2 k(k-p) + xx_1) \\ p & -((k-p)x + kx_1) \end{bmatrix}$$

$$M = s \begin{bmatrix} (k-q)x_2 + kx & -q(\delta^2 k(k-q) + xx_2) \\ q & -((k-q)x + kx_2) \end{bmatrix}$$

- with $t = \frac{1}{\sqrt{k(k-p)((\delta p+x+x_1)(\delta p-x-x_1))}}$ and

$$s = \frac{1}{\sqrt{k(k-q)((\delta q+x+x_2)(\delta q-x-x_2))}}$$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{q(\delta p+(x+x_1))(\delta p-(x+x_1))}{p(\delta q+(x+x_2))(\delta q-(x+x_2))}.$

Question: Rational choice for t and s ?

- **Example 20:** (A2) Equation (ABS 2003)

$$(q^2 - p^2)(xx_1x_2x_{12} + 1) + q(p^2 - 1)(xx_2 + x_1x_{12}) - p(q^2 - 1)(xx_1 + x_2x_{12}) = 0$$

(Q3) with $\delta = 0$ via Möbius transformation:

$$x \rightarrow x, x_1 \rightarrow \frac{1}{x_1}, x_2 \rightarrow \frac{1}{x_2}, x_{12} \rightarrow x_{12}, p \rightarrow p, q \rightarrow q$$

- **Lax pair:**

$$L = t \begin{bmatrix} k(p^2 - 1)x & -(p^2 - k^2 + p(k^2 - 1)xx_1) \\ p(k^2 - 1) + (p^2 - k^2)xx_1 & -k(p^2 - 1)x_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} k(q^2 - 1)x & -(q^2 - k^2 + q(k^2 - 1)xx_2) \\ q(k^2 - 1) + (q^2 - k^2)xx_2 & -k(q^2 - 1)x_2 \end{bmatrix}$$

- with $t = \frac{1}{\sqrt{(k^2-1)(k^2-p^2)(p-xx_1)(pxx_1-1)}}$

and $s = \frac{1}{\sqrt{(k^2-1)(k^2-q^2)(q-xx_2)(qxx_2-1)}}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{(q^2-1)(p-xx_1)(pxx_1-1)}{(p^2-1)(q-xx_2)(qxx_2-1)}.$

Question: Rational choice for t and s ?

- **Example 21:** Discrete sine-Gordon Equation

$$xx_1x_2x_{12} - pq(xx_{12} - x_1x_2) - 1 = 0$$

(H3) with $\delta = 0$ via extended Möbius transformation:

$$x \rightarrow x, x_1 \rightarrow x_1, x_2 \rightarrow \frac{1}{x_2}, x_{12} \rightarrow -\frac{1}{x_{12}}, p \rightarrow \frac{1}{p}, q \rightarrow q$$

Discrete sine-Gordon equation is **NOT** consistent around the cube, but has a Lax pair!

- **Lax pair:**

$$L = \begin{bmatrix} p & -kx_1 \\ -\frac{k}{x} & \frac{px_1}{x} \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{qx_2}{x} & -\frac{1}{kx} \\ -\frac{x_2}{k} & q \end{bmatrix}$$