# Symbolic Computation of Lax Pairs of Systems of Partial Difference Equations Using Consistency Around the Cube

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# Outline

- What are nonlinear P$\Delta$Es?
- Lax pairs of nonlinear PDEs
- Lax pair of nonlinear P$\Delta$Es
- Algorithm to compute Lax pairs of P$\Delta$Es (Nijhoff 2001, Bobenko & Suris 2001)
- Additional examples
- Conclusions and future work

## What are nonlinear P$\Delta$Es?

- Nonlinear maps with two (or more) lattice points

  Some origins:

  - full discretizations of PDEs
  - discrete dynamical systems in 2 dimensions
  - superposition principle (Bianchi permutability) for Bäcklund transformations between 3 solutions (2 parameters) of a completely integrable PDE

- Example 1: **discrete potential Korteweg-de Vries (pKdV) equation**

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0$$

- $u$ is dependent variable or field (scalar case)

  $n$ and $m$ are lattice points

  $p$ and $q$ are parameters

- **Notation:**

  $$(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (x, x_1, x_2, x_{12})$$

- Alternate notations (in the literature):

  $$(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (u, \tilde{u}, \hat{u}, \hat{\tilde{u}})$$

  $$(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = (u_{00}, u_{10}, u_{01}, u_{11})$$

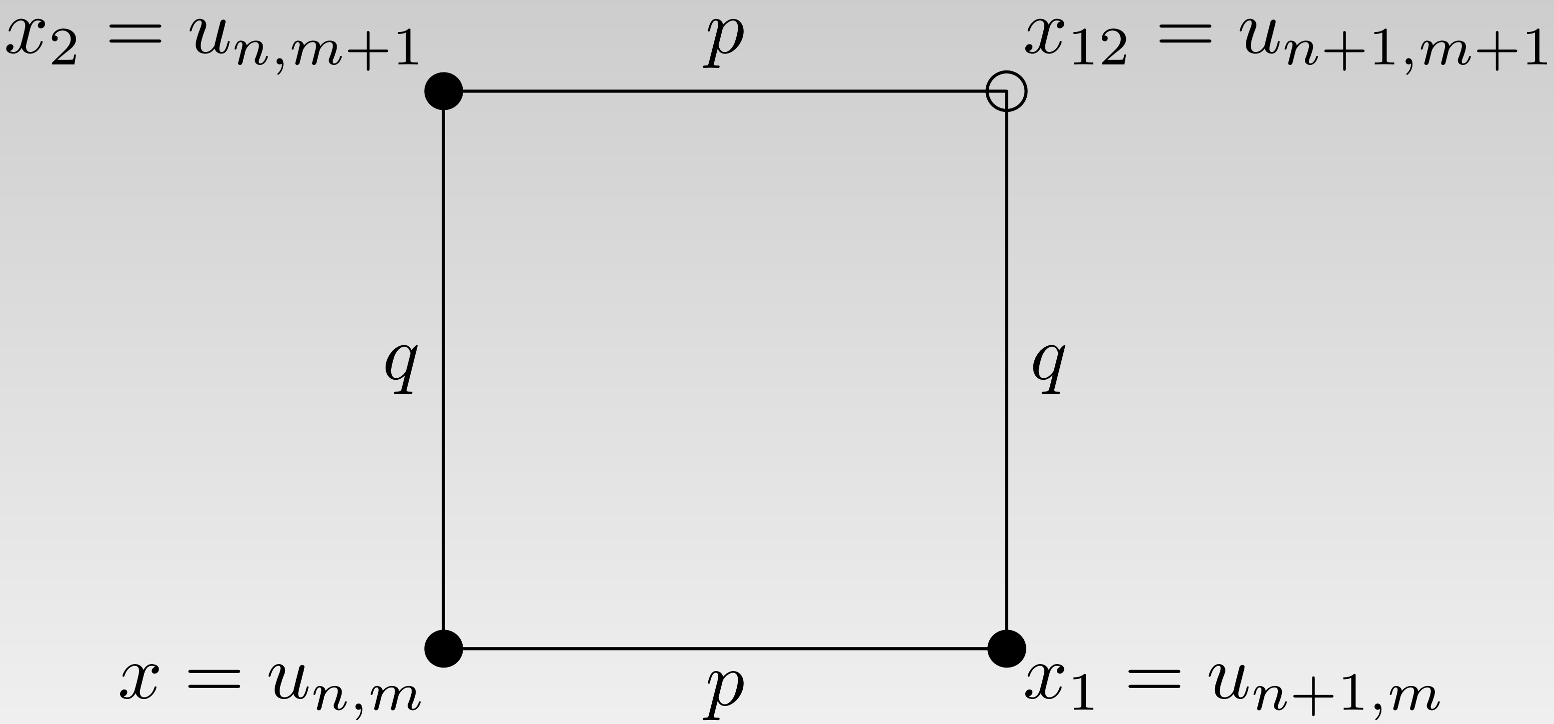
- **Example 1: discrete pKdV equation**

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0$$

Short: $$\boxed{(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0}$$

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - p^2 + q^2 = 0$$

Short: $$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$



# Systems of P$\Delta$Es

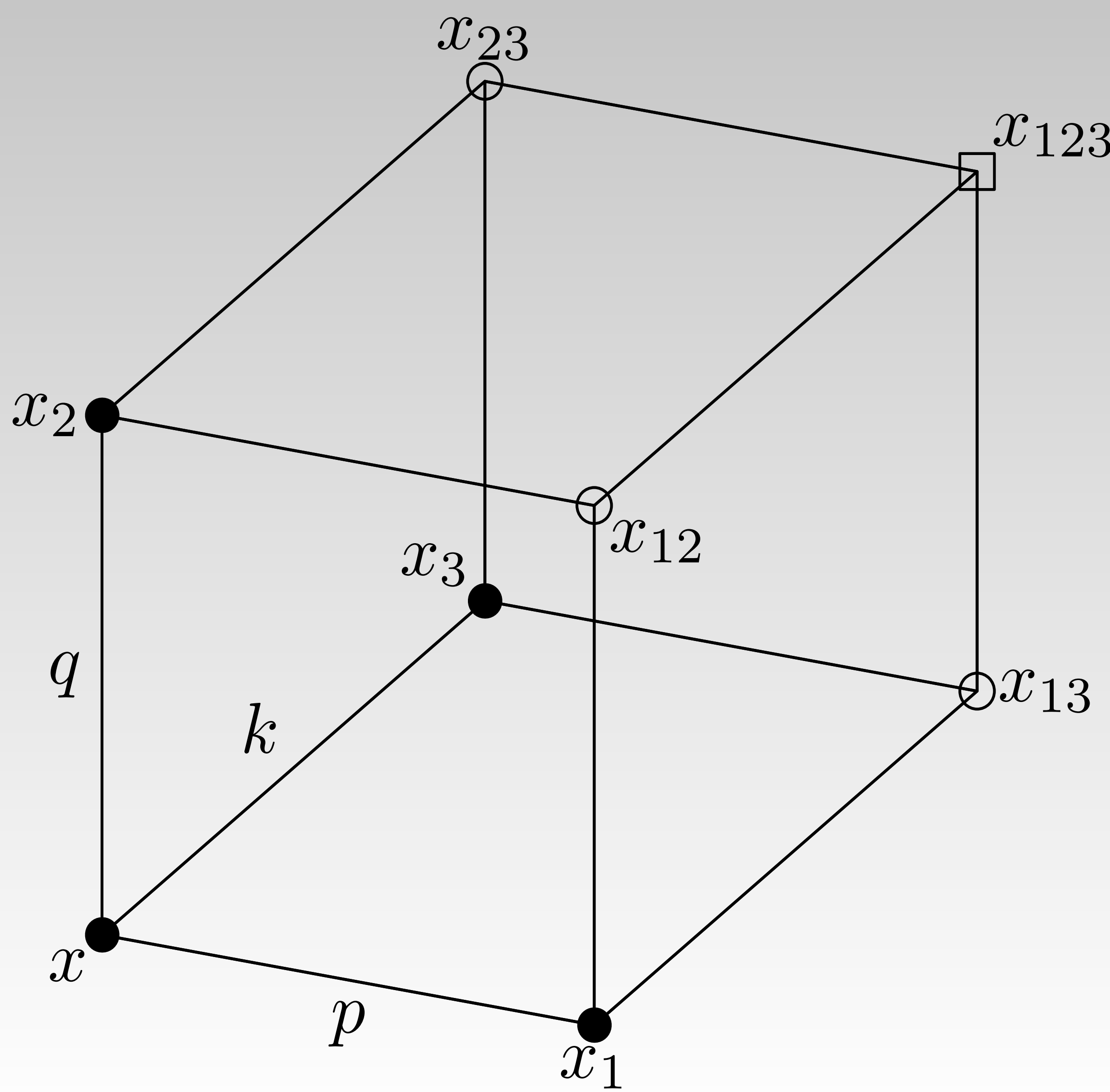
## Example 2: Schwarzian-Boussinesq System

$$
\begin{aligned}
z_1\, y - x_1 + x &= 0 \\
z_2\, y - x_2 + x &= 0 \\
z\, y_{12}(y_1 - y_2) - y(p\, y_2\, z_1 - q\, y_1\, z_2) &= 0
\end{aligned}
$$

- Note: System has two single-edge equations and one full-face equation.

# Concept of Consistency Around the Cube

Superposition of Bäcklund transformations between 4 solutions $x, x_1, x_2, x_3$ (3 parameters: $p, q, k$)

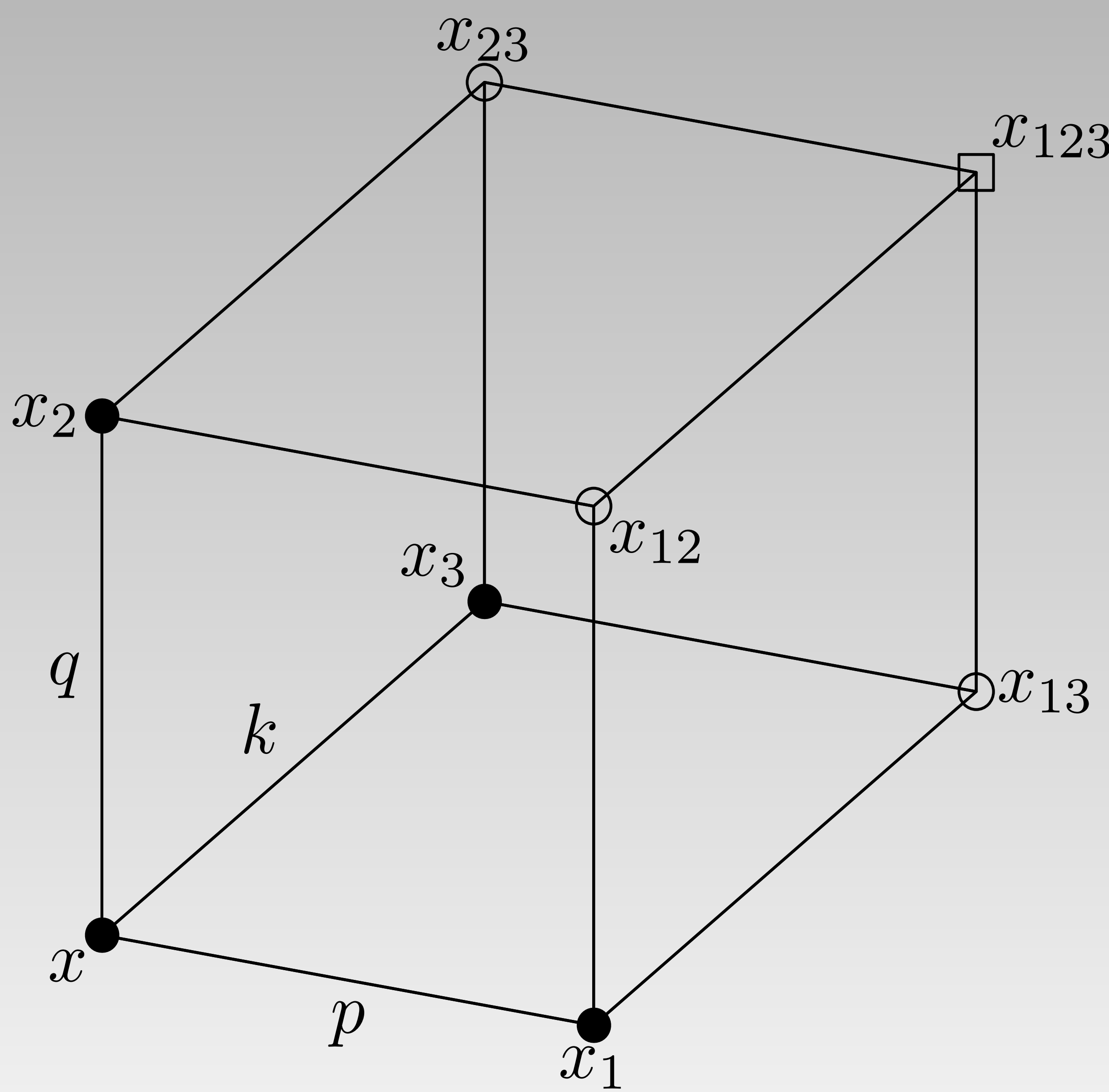


- Introduce a third lattice variable $\ell$
- View $u$ as dependent on three lattice points: $n, m, \ell$. So, $x = u_{n,m} \longrightarrow x = u_{n,m,\ell}$
- Move in three directions:

  $n \to n+1$ over distance $p$

  $m \to m+1$ over distance $q$

  $\ell \to \ell+1$ over distance $k$ (spectral parameter)
- Require that the same lattice holds on the front, bottom, and left faces of the cube
- Require consistency for the computation of $x_{123} = u_{n+1,m+1,\ell+1}$ (3 ways $\to$ same answer)



$x_{23}$
$x_{123}$
$x_2$
$x_{12}$
$x_3$
$q$
$x_{13}$
$k$
$x$
$p$
$x_1$

**Peter D. Lax (1926-)**

Seminal paper: Integrals of nonlinear equations of evolution and solitary waves, Commun. Pure Appl. Math. **21** (1968) 467-490

# Refresher: Lax Pairs of Nonlinear PDEs

- Historical example: Korteweg-de Vries equation

$$\boxed{u_t + \alpha\, u u_x + u_{xxx} = 0} \qquad \alpha \in \mathbb{R}$$

- Key idea: Replace the nonlinear PDE with a compatible linear system (Lax pair):

$$\psi_{xx} + \left(\tfrac{1}{6}\alpha u - \lambda\right)\psi = 0$$

$$\psi_t + 4\psi_{xxx} + \alpha u \psi_x + \tfrac{1}{2}\alpha u_x \psi = 0$$

$\psi$ is eigenfunction; $\lambda$ is constant eigenvalue $(\lambda_t = 0)$ (isospectral)

# Lax Pairs in Operator Form

- Replace a completely integrable nonlinear PDE by a pair of linear equations (called a Lax pair):

$$\mathcal{L}\psi = \lambda\psi \qquad \text{and} \qquad \mathrm{D}_t\psi = \mathcal{M}\psi$$

- Require compatibility of both equations

$$\mathcal{L}_t\psi + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})\psi \doteq 0$$

Lax equation: $\boxed{\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \doteq \mathcal{O}}$

with commutator $[\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}$.

Furthermore, $\mathcal{L}_t\psi = [\mathrm{D}_t, \mathcal{L}]\psi = \mathrm{D}_t(\mathcal{L}\psi) - \mathcal{L}\mathrm{D}_t\psi$

and $\doteq$ means “evaluated on the PDE”

- Example: Lax operators for the KdV equation

$$\boxed{\mathcal{L} = \mathsf{D}_x^2 + \tfrac{1}{6}\alpha u \, \mathbf{I}}$$

$$\boxed{\mathcal{M} = -\left(4\, \mathsf{D}_x^3 + \alpha u \mathsf{D}_x + \tfrac{1}{2}\alpha u_x \, \mathbf{I}\right)}$$

- Note: $\mathcal{L}_t\psi + [\mathcal{L}, \mathcal{M}]\psi = \frac{1}{6}\alpha \left(u_t + \alpha u u_x + u_{xxx}\right) \psi$

## Lax Pairs in Matrix Form (AKNS Scheme)

- Write

$$\psi_{xx} + \left(\tfrac{1}{6}\alpha u - \lambda\right)\psi = 0$$

$$\psi_t + 4\psi_{xxx} + \alpha u \psi_x + \tfrac{1}{2}\alpha u_x \psi = 0$$

  as a first-order system.

- Introduce, $\mathbf{\Psi} = \begin{bmatrix} \psi \\ \psi_x \end{bmatrix}$, then, $\mathrm{D}_x \mathbf{\Psi} = \mathbf{X}\mathbf{\Psi}$ with

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ \lambda - \tfrac{1}{6}\alpha u & 0 \end{bmatrix}$$

- Use $\phi_t = \psi_{xt} = \psi_{tx}$ and eliminate $\psi_x, \psi_{xx}, \psi_{xxx}, \ldots$

  to get $\mathrm{D}_t \mathbf{\Psi} = \mathbf{T}\mathbf{\Psi}$ with

$$\mathbf{T} = \begin{bmatrix} \frac{1}{6}\alpha u_x & -4\lambda - \frac{1}{3}\alpha u \\ -4\lambda^2 + \frac{1}{3}\alpha\lambda u + \frac{1}{18}\alpha^2 u^2 + \frac{1}{6}\alpha u_{2x} & -\frac{1}{6}\alpha u_x \end{bmatrix}$$

- Compatibility of

$$\begin{aligned} \mathrm{D}_x \mathbf{\Psi} &= \mathbf{X}\,\mathbf{\Psi} \\ \mathrm{D}_t \mathbf{\Psi} &= \mathbf{T}\,\mathbf{\Psi} \end{aligned}$$

  yields Lax equation (zero-curvature equation):

$$\boxed{\mathrm{D}_t \mathbf{X} - \mathrm{D}_x \mathbf{T} + [\mathbf{X}, \mathbf{T}] \doteq \mathbf{0}}$$

  with commutator $[\mathbf{X}, \mathbf{T}] = \mathbf{X}\mathbf{T} - \mathbf{T}\mathbf{X}$

## Equivalence under Gauge Transformations

- Lax pairs are equivalent under a gauge transformation:

If $(\mathbf{X}, \mathbf{T})$ is a Lax pair then so is $(\tilde{\mathbf{X}}, \tilde{\mathbf{T}})$ with

$$\tilde{\mathbf{X}} = \mathbf{G}\mathbf{X}\mathbf{G}^{-1} + \mathsf{D}_x(\mathbf{G})\mathbf{G}^{-1}$$
$$\tilde{\mathbf{T}} = \mathbf{G}\mathbf{T}\mathbf{G}^{-1} + \mathsf{D}_t(\mathbf{G})\mathbf{G}^{-1}$$

$\mathbf{G}$ is arbitrary invertible matrix and $\tilde{\mathbf{\Psi}} = \mathbf{G}\mathbf{\Psi}$.

Thus,

$$\boxed{\tilde{\mathbf{X}}_t - \tilde{\mathbf{T}}_x + [\tilde{\mathbf{X}}, \tilde{\mathbf{T}}] \doteq \mathbf{0}}$$

- Example: For the KdV equation

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ \lambda - \frac{1}{6}\alpha u & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{X}} = \begin{bmatrix} -ik & \frac{1}{6}\alpha u \\ -1 & ik \end{bmatrix}$$

Here,

$$\tilde{\mathbf{X}} = \mathbf{G}\mathbf{X}\mathbf{G}^{-1} \quad \text{and} \quad \tilde{\mathbf{T}} = \mathbf{G}\mathbf{T}\mathbf{G}^{-1}$$

with

$$\mathbf{G} = \begin{bmatrix} -i\,k & 1 \\ -1 & 0 \end{bmatrix}$$

where $\lambda = -k^2$

# Reasons to Compute a Lax Pair

- Compatible linear system is the starting point for application of the IST and the Riemann-Hilbert method for boundary value problems
- Confirm the complete integrability of the PDE
- Zero-curvature representation of the PDE
- Compute conservation laws of the PDE
- Discover families of completely integrable PDEs

Question: How to find a Lax pair of a completely integrable PDE?

Answer: There is no completely systematic method

# Lax Pair of Nonlinear PΔEs

- Replace the nonlinear PΔE by

$$\psi_1 = \mathrm{L}\,\psi \quad (\text{recall } \psi_1 = \psi_{n+1,m})$$
$$\psi_2 = \mathrm{M}\,\psi \quad (\text{recall } \psi_2 = \psi_{n,m+1})$$

For scalar PΔEs, $\mathrm{L}, \mathrm{M}$ are $2 \times 2$ matrices; $\psi = \begin{bmatrix} f \\ g \end{bmatrix}$

- Express compatibility:

$$\psi_{12} = \mathrm{L}_2\,\psi_2 = \mathrm{L}_2\,\mathrm{M}\,\psi$$
$$\psi_{12} = \mathrm{M}_1\,\psi_1 = \mathrm{M}_1\,\mathrm{L}\,\psi$$

- Lax equation: $\boxed{\mathrm{L}_2\,\mathrm{M} - \mathrm{M}_1\,\mathrm{L} \doteq 0}$

## Equivalence under Gauge Transformations

- Lax pairs are equivalent under a gauge transformation

If $(L, M)$ is a Lax pair then so is $(\mathcal{L}, \mathcal{M})$ with

$$\begin{aligned} \mathcal{L} &= \mathrm{G}_1 \, \mathrm{L} \, \mathrm{G}^{-1} \\ \mathcal{M} &= \mathrm{G}_2 \, \mathrm{M} \, \mathrm{G}^{-1} \end{aligned}$$

where $G$ is non-singular matrix and $\phi = \mathrm{G}\psi$

Proof: Trivial verification that

$$(\mathcal{L}_2 \, \mathcal{M} - \mathcal{M}_1 \, \mathcal{L}) \, \phi \doteq 0 \leftrightarrow (\mathrm{L}_2 \, \mathrm{M} - \mathrm{M}_1 \, \mathrm{L}) \, \psi \doteq 0$$

- Example 1: Discrete pKdV Equation

$$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

- Lax pair:

$$L = tL_c = t \begin{bmatrix} x & p^2 - k^2 - xx_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = sM_c = s \begin{bmatrix} x & q^2 - k^2 - xx_2 \\ 1 & -x_2 \end{bmatrix}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{\mathrm{Det} L_c}} = \frac{1}{\sqrt{k^2 - p^2}}$ and $s = \frac{1}{\sqrt{\mathrm{Det} M_c}} = \frac{1}{\sqrt{k^2 - q^2}}$. Here, $\frac{t_2}{t}\,\frac{s}{s_1} = 1$.

- Example 2: Schwarzian-Boussinesq System

$$\begin{aligned} z_1\, y - x_1 + x &= 0 \\ z_2\, y - x_2 + x &= 0 \\ z\, y_{12}(y_1 - y_2) - y(p\, y_2\, z_1 - q\, y_1\, z_2) &= 0 \end{aligned}$$

- Lax pair:

$$\boxed{L = t L_c = t \begin{bmatrix} -y & 0 & y z_1 \\ \frac{k y y_1}{z} & -\frac{p y z_1}{z} & 0 \\ 0 & 1 & -y_1 \end{bmatrix}}$$

$$M = sM_c = s \begin{bmatrix} -y & 0 & yz_2 \\ \frac{kyy_2}{z} & -\frac{qyz_2}{z} & 0 \\ 0 & 1 & -y_2 \end{bmatrix}$$

with $t = s = \frac{1}{y}$, or $t = \frac{1}{y_1}$ and $s = \frac{1}{y_2}$,

or $t = \sqrt[3]{\frac{z}{y^2 y_1 z_1}}$ and $s = \sqrt[3]{\frac{z}{y^2 y_2 z_2}}$.

Here, $\frac{t_2}{t} \frac{s}{s_1} = \frac{y_1}{y_2}$.

## Lax Pair Algorithm for Scalar PΔEs

(Nijhoff 2001, Bobenko and Suris 2001)

Applies to PΔEs that are consistent around the cube

### Example 1: Discrete pKdV Equation

- Step 1: Verify the consistency around the cube
  - ⋆ Equation on front face of cube:

$$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

Solve for $x_{12} = x - \frac{p^2-q^2}{x_1-x_2} = \frac{x_2 x - x x_1 + p^2 - q^2}{x_2 - x_1}$

Compute $x_{123}$: $\quad x_{12} \longrightarrow x_{123} = x_3 - \frac{p^2-q^2}{x_{13}-x_{23}}$

  - ⋆ Equation on floor of cube:

$$(x - x_{13})(x_1 - x_3) - p^2 + k^2 = 0$$

Solve for $x_{13} = x - \frac{p^2-k^2}{x_1-x_3} = \frac{x_3x - xx_1 + p^2 - k^2}{x_3 - x_1}$

Compute $x_{123}$: $\quad x_{13} \longrightarrow x_{123} = x_2 - \frac{p^2-k^2}{x_{12}-x_{23}}$

⋆ Equation on left face of cube:

$$(x - x_{23})(x_3 - x_2) - k^2 + q^2 = 0$$

Solve for $x_{23} = x - \frac{q^2-k^2}{x_2-x_3} = \frac{x_3x - xx_2 + q^2 - k^2}{x_3 - x_2}$

Compute $x_{123}$: $\quad x_{23} \longrightarrow x_{123} = x_1 - \frac{q^2-k^2}{x_{12}-x_{13}}$

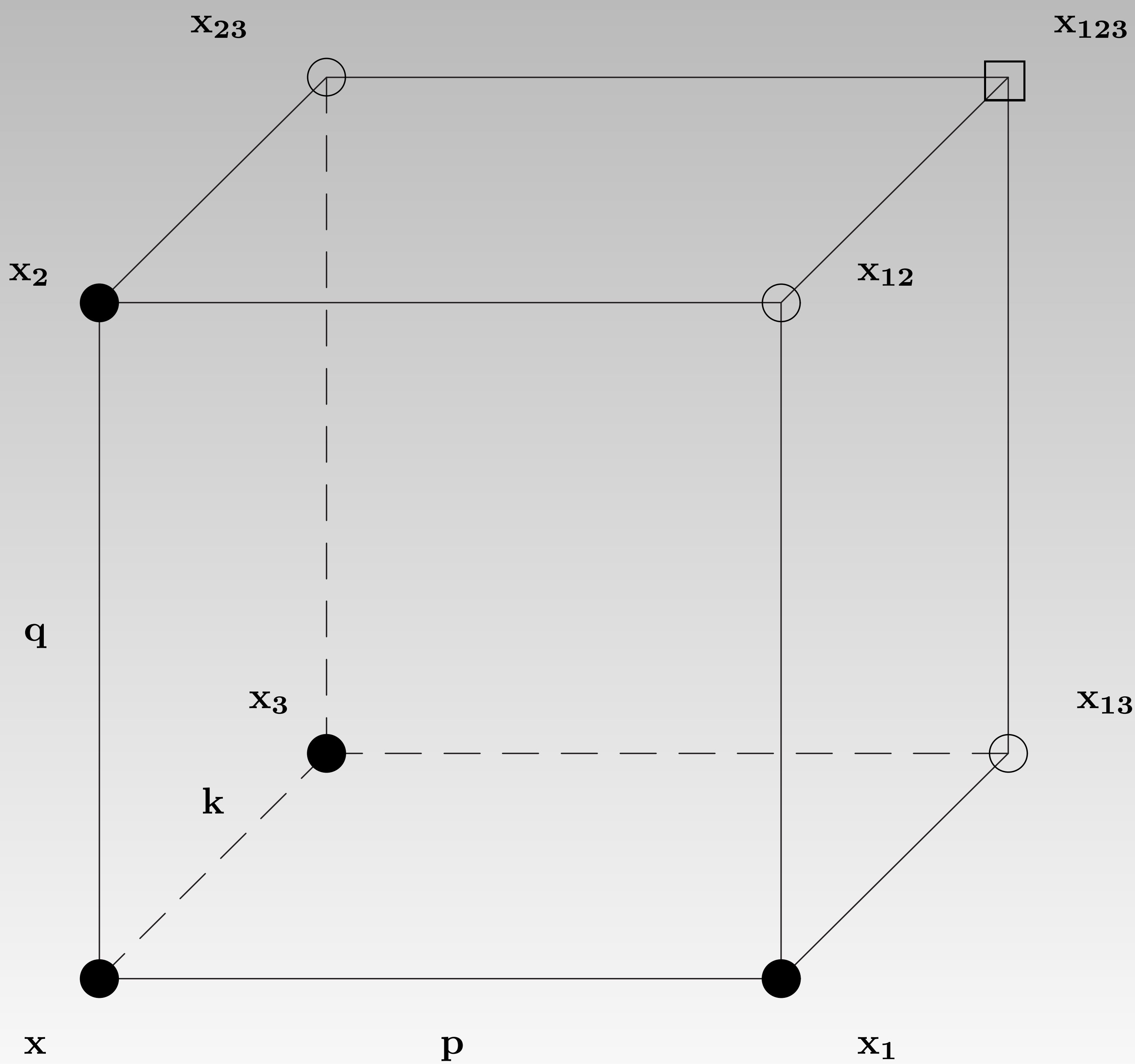
⋆ Verify that all three coincide:

$$x_{123} = x_1 - \frac{q^2 - k^2}{x_{12} - x_{13}} = x_2 - \frac{p^2 - k^2}{x_{12} - x_{23}} = x_3 - \frac{p^2 - q^2}{x_{13} - x_{23}}$$

Upon substitution of $x_{12}, x_{13}$, and $x_{23}$:

$$x_{123} = \frac{p^2 x_1(x_2 - x_3) + q^2 x_2(x_3 - x_1) + k^2 x_3(x_1 - x_2)}{p^2(x_2 - x_3) + q^2(x_3 - x_1) + k^2(x_1 - x_2)}$$

Consistency around the cube is satisfied!


$\mathbf{x_{23}}$
$\mathbf{x_{123}}$
$\mathbf{x_2}$
$\mathbf{x_{12}}$
$\mathbf{q}$
$\mathbf{x_3}$
$\mathbf{x_{13}}$
$\mathbf{k}$
$\mathbf{x}$
$\mathbf{p}$
$\mathbf{x_1}$

- Step 2: Homogenization

⋆ Numerator and denominator of

$x_{13} = \frac{x_3 x - x x_1 + p^2 - k^2}{x_3 - x_1}$ and $x_{23} = \frac{x_3 x - x x_2 + q^2 - k^2}{x_3 - x_2}$

are linear in $x_3$.

⋆ Substitute $x_3 = \frac{f}{g}$ into $x_{13}$ to get

$x_{13} = \frac{xf + (p^2 - k^2 - x x_1) g}{f - x_1 g}$

⋆ On the other hand, $x_3 = \frac{f}{g} \longrightarrow x_{13} = \frac{f_1}{g_1}$.

Thus, $x_{13} = \frac{f_1}{g_1} = \frac{xf + (p^2 - k^2 - x x_1) g}{f - x_1 g}$.

Hence, $f_1 = t \left( xf + (p^2 - k^2 - x x_1) g \right)$ and

$g_1 = t \left( f - x_1 g \right)$

⋆ In matrix form

$$\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = t \begin{bmatrix} x & p^2 - k^2 - xx_1 \\ 1 & -x_1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

Matches $\psi_1 = \mathrm{L}\,\psi$ with $\psi = \begin{bmatrix} f \\ g \end{bmatrix}$

⋆ Similarly, from $x_{23} = \frac{f_2}{g_2} = \frac{xf + (q^2 - k^2 - xx_2)g}{f - x_2 g}$

$$\psi_2 = \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} = s \begin{bmatrix} x & q^2 - k^2 - xx_2 \\ 1 & -x_2 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \mathrm{M}\,\psi.$$

Therefore,

$$L = t\,L_c = t \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = s\,M_c = s \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix}$$

- Step 3: Determine $t$ and $s$

$\star$ Substitute $L = t\,L_c, M = s\,M_c$ into $\mathrm{L}_2\,\mathrm{M} - \mathrm{M}_1\,\mathrm{L} = 0$

$\longrightarrow\ t_2\,s\,(L_c)_2\,M_c - s_1\,t\,(M_c)_1\,L_c = 0$

$\star$ Solve the equation from the (2-1)-element for

$$\frac{t_2}{t}\,\frac{s}{s_1} = f(x, x_1, x_2, p, q, \ldots)$$

Find rational $t$ and $s$.

$\star$ Apply determinant to get

$$\frac{t_2}{t}\,\frac{s}{s_1} = \sqrt{\frac{\det L_c}{\det (L_c)_2}}\,\sqrt{\frac{\det (M_c)_1}{\det M_c}}$$

Solution: $t = \frac{1}{\sqrt{\det L_c}}, \qquad s = \frac{1}{\sqrt{\det M_c}}$

$\longrightarrow$ Always works but introduces roots!

The ratio $\frac{t_2}{t}\frac{s}{s_1}$ is invariant under the change

$$t \to \frac{a_1}{a}\,t, \qquad s \to \frac{a_2}{a}\,s,$$

where $a(x)$ is arbitrary.

Proper choice of $a(x) \implies$ Rational $t$ and $s$.

No roots needed!

## Lax Pair Algorithm – Systems of P$\Delta$Es

(Nijhoff 2001, Bobenko and Suris 2001)

Applies to systems of P$\Delta$Es that are consistent around the cube

**Example 2: Schwarzian-Boussinesq System**

$$
\begin{aligned}
z_1\, y - x_1 + x &= 0 \\
z_2\, y - x_2 + x &= 0 \\
z\, y_{12}(y_1 - y_2) - y(p\, y_2\, z_1 - q\, y_1\, z_2) &= 0
\end{aligned}
$$

- Note: System has two single-edge equations and one full-face equation

- Edge equations require augmentation of system with additional shifted, edge equations

$$
\begin{aligned}
z_{12}\, y_2 - x_{12} + x_2 &= 0 \\
z_{12}\, y_1 - x_{12} + x_1 &= 0
\end{aligned}
$$

- Edge equations will provide additional constraints during homogenization (Step 2).

- Step 1: Verify the consistency around the cube
  - ⋆ System on the front face:

$$z_1 y - x_1 + x = 0$$

$$z_2 y - x_2 + x = 0$$

$$z_{12} y_2 - x_{12} + x_2 = 0$$

$$z_{12} y_1 - x_{12} + x_1 = 0$$

$$z y_{12}(y_1 - y_2) - y(p y_2 z_1 - q y_1 z_2) = 0$$

Solve for $x_{12}$, $y_{12}$, and $z_{12}$:

$$x_{12} = \frac{x_2 y_1 - x_1 y_2}{y_1 - y_2}$$

$$y_{12} = \frac{(x_2 - x_1)(q y_1 z_2 - p y_2 z_1)}{z(y_1 - y_2)(z_1 - z_2)}$$

$$z_{12} = \frac{x_2 - x_1}{y_1 - y_2}$$

Compute $x_{123}$, $y_{123}$, and $z_{123}$:

$$x_{123} = \frac{x_{23} y_{13} - x_{13} y_{23}}{y_{13} - y_{23}}$$

$$y_{123} = \frac{p y_{23} y_3 z_{13} - q y_{13} y_3 z_{23}}{z_3(y_{13} - y_{23})}$$

$$z_{123} = \frac{x_{23} - x_{13}}{y_{13} - y_{23}}$$

⋆ System on the bottom face:

$$z_1 y - x_1 + x = 0$$

$$z_3 y - x_3 + x = 0$$

$$z y_{13}(y_1 - y_3) - y(p y_3 z_1 - k y_1 z_3) = 0$$

$$z_{13} y_3 - x_{13} + x_3 = 0$$

$$z_{13} y_1 - x_{13} + x_1 = 0$$

Solve for $x_{13}$, $y_{13}$, and $z_{13}$:

$$x_{13} = \frac{x_3 y_1 - x_1 y_3}{y_1 - y_3}$$

$$y_{13} = \frac{(x_2 - x_1)(k y_1 z_3 - p y_3 z_1)}{z(y_1 - y_3)(z_1 - z_2)}$$

$$z_{13} = \frac{x_3 - x_1}{y_1 - y_3}$$

Compute $x_{123}$, $y_{123}$, and $z_{123}$:

$$x_{123} = \frac{x_{23}y_{12} - x_{12}y_{23}}{y_{12} - y_{23}}$$

$$y_{123} = \frac{py_2y_{23}z_{12} - ky_{12}y_2z_{23}}{z_2(y_{12} - y_{23})}$$

$$z_{123} = \frac{x_{23} - x_{12}}{y_{12} - y_{23}}$$

⋆ System on the left face:

$$z_3 y - x_3 + x = 0$$

$$z_2 y - x_2 + x = 0$$

$$z_{23} y_2 - x_{23} + x_2 = 0$$

$$z_{23} y_3 - x_{23} + x_3 = 0$$

$$z y_{23}(y_3 - y_2) - y(p y_2 z_3 - q y_3 z_2) = 0$$

Solve for $x_{23}$, $y_{23}$, and $z_{23}$:

$$x_{23} = \frac{x_3 y_2 - x_2 y_3}{y_2 - y_3}$$

$$y_{23} = \frac{(x_2 - x_1)(k y_2 z_3 - q y_3 z_2)}{z(y_2 - y_3)(z_1 - z_2)}$$

$$z_{23} = \frac{x_3 - x_2}{y_2 - y_3}$$

Compute $x_{123}$, $y_{123}$, and $z_{123}$:

$$x_{123} = \frac{x_{13}y_{12} - x_{12}y_{13}}{y_{12} - y_{13}}$$

$$y_{123} = \frac{qy_1y_{13}z_{12} - ky_1y_{12}z_{13}}{z_1(y_{12} - y_{13})}$$

$$z_{123} = \frac{x_{13} - x_{12}}{y_{12} - y_{13}}$$

Substitute $x_{12}$, $y_{12}$, $y_{12}$, $x_{13}$, $y_{13}$, $z_{13}$, $x_{23}$, $y_{23}$, $z_{23}$ into the above to get

$$x_{123} = \frac{(pz_1(x_3y_2 - x_2y_3) + qz_2(x_1y_3 - x_3y_1) + kz_3(x_2y_1 - x_1y_2)}{(pz_1(y_2 - y_3) + qz_2(y_3 - y_1) + kz_3(y_1 - y_2)}$$

$$y_{123} = \frac{pqy_3z_1(x_2 - x_1) + kpy_2z_1(x_1 - x_3) + kqy_1(x_3z_2 - x_1z_2 + x_1z_3 - x_2z_3)}{z_1(pz_1(y_2 - y_3) + qz_2(y_3 - y_1) + kz_3(y_1 - y_2))}$$

$$z_{123} = \frac{z(z_1 - z_2)(x_3(y_1 - y_2) + x_1(y_2 - y_3) + x_2(y_3 - y_1))}{(x_1 - x_2)(pz_1(y_2 - y_3) + qz_2(y_3 - y_1) + kz_3(y_1 - y_2))}$$

Answer is unique and independent of $x$ and $y$.

Consistency around the cube is satisfied!

- Step 2: Homogenization

⋆ Observed that $x_3$, $y_3$ and $z_3$ appear linearly in numerators and denominators of

$$
\begin{aligned}
x_{13} &= \frac{y_1(x + yz_3) - y_3(x + yz_1)}{y_1 - y_3} \\
y_{13} &= \frac{pyy_3z_1 - kyy_1z_3}{z(y_1 - y_3)} \\
z_{13} &= \frac{y(z_3 - z_1)}{y_1 - y_3} \\
x_{23} &= \frac{y_2(x + yz_3) - y_3(x + yz_2)}{y_2 - y_3} \\
y_{23} &= \frac{qyy_3z_2 - kyy_2z_3}{z(y_2 - y_3)} \\
z_{23} &= \frac{y(z_3 - z_2)}{y_2 - y_3}
\end{aligned}
$$

⋆ Substitute

$$x_3 = \frac{h}{H}, \;\; y_3 = \frac{g}{G}, \;\; \text{and} \;\; z_3 = \frac{f}{F}.$$

⋆ Use constraints (from left face edges)

$$z_1 y \; - \; x_1 + x = 0, \;\; z_2 y - x_2 + x = 0$$

$$\Rightarrow z_3 y - x_3 + x = 0$$

Solve for $x_3 = z_3\, y + x$ (since $x_3$ appears linearly).

Thus, $x_3 = \frac{fy+Fx}{F}$, $y_3 = \frac{g}{G}$, and $z_3 = \frac{f}{F}$.

$\star$ Substitute $x_3,\ y_3,\ z_3$ into $x_{13},\ y_{13},\ z_{13}$:

$$x_{13} = \frac{Fgx - FGxy_1 - fGyy_1 + Fgyz_1}{F(g - Gy_1)}$$

$$y_{13} = \frac{y(fGky_1 - Fgpz_1)}{F(g - Gy_1)z}$$

$$z_{13} = \frac{Gy(Fz_1 - f)}{F(g - Gy_1)}$$

Require that numerators and denominators are linear in $f, F, g$, and $G$. That forces $G = F$.

Hence, $x_3 = \frac{fy+Fx}{F}$, $y_3 = \frac{g}{F}$, and $z_3 = \frac{f}{F}$.

⋆ Compute

$$x_3 = \frac{fy + Fx}{F} \longrightarrow x_{13} = \frac{f_1 y_1 + F_1 x_1}{F_1}$$

$$y_3 = \frac{g}{F} \longrightarrow y_{13} = \frac{g_1}{F_1}$$

$$z_3 = \frac{f}{F} \longrightarrow z_{13} = \frac{f_1}{F_1}$$

Hence,

$$x_{13} = \frac{-fyy_1 + g(x + yz_1) - Fxy_1}{g - Fy_1} = \frac{f_1 y_1 + F_1 x_1}{F_1}$$

$$y_{13} = \frac{y(fky_1 - gpz_1)}{(g - Fy_1)z} = \frac{g_1}{F_1}$$

$$z_{13} = \frac{y(-f + Fz_1)}{g - Fy_1} = \frac{f_1}{F_1}$$

Note that

$$x_{13} = \frac{-fyy_1 + g(x + yz_1) - Fxy_1}{g - Fy_1} = \frac{f_1 y_1 + F_1 x_1}{F_1}$$

is automaticlly satisfied as a result of the relation $x_3 = z_3\, y + x$.

⋆ Write in matrix form:

$$\psi_1 = \begin{bmatrix} f_1 \\ g_1 \\ F_1 \end{bmatrix} = t \begin{bmatrix} -y & 0 & yz_1 \\ \frac{kyy_1}{z} & -\frac{pyz_1}{z} & 0 \\ 0 & 1 & -y_1 \end{bmatrix} \begin{bmatrix} f \\ g \\ F \end{bmatrix} = L\,\psi$$

⋆ Repeat the same steps for $x_{23}, y_{23}, z_{23}$ to obtain

$$\psi_2 = \begin{bmatrix} f_2 \\ g_2 \\ F_2 \end{bmatrix} = t \begin{bmatrix} -y & 0 & yz_2 \\ \frac{kyy_2}{z} & -\frac{qyz_2}{z} & 0 \\ 0 & 1 & -y_2 \end{bmatrix} \begin{bmatrix} f \\ g \\ F \end{bmatrix} = M\,\psi$$

⋆ Therefore,

$$L = tL_c = t \begin{bmatrix} -y & 0 & yz_1 \\ \frac{kyy_1}{z} & -\frac{pyz_1}{z} & 0 \\ 0 & 1 & -y_1 \end{bmatrix}$$

$$M = sM_c = s \begin{bmatrix} -y & 0 & yz_2 \\ \frac{kyy_2}{z} & -\frac{qyz_2}{z} & 0 \\ 0 & 1 & -y_2 \end{bmatrix}$$

- Step 3: Determine $t$ and $s$

⋆ Substitute $L = t\, L_c, M = s\, M_c$ into $\mathrm{L}_2\, \mathrm{M} - \mathrm{M}_1\, \mathrm{L} = 0$

$$\longrightarrow t_2\, s\, (L_c)_2\, M_c - s_1\, t\, (M_c)_1\, L_c = 0$$

⋆ Solve the equation from the (2-1)-element:

$$\frac{t_2}{t}\,\frac{s}{s_1} = \frac{y_1}{y_2}.$$

Thus, $t = s = \frac{1}{y}$, or $t = \frac{1}{y_1}$ and $s = \frac{1}{y_2}$,

or $t = \sqrt[3]{\frac{z}{y^2 y_1 z_1}}$ and $s = \sqrt[3]{\frac{z}{y^2 y_2 z_2}}$.

# Additional Examples

- Example 3: Lattice due to Hietarinta (2011)

$$x_1 z - y_1 - x = 0$$

$$x_2 z - y_2 - x = 0$$

$$z_{12} - \frac{y}{x} - \frac{1}{x}\left(\frac{p x_1 - q x_2}{z_1 - z_2}\right) = 0$$

Note: System has two single-edge equations and one full-face equation.

- Lax pair:

$$L = t \begin{bmatrix} \frac{yz}{x} & \frac{k}{x} & \frac{kx - p x_1 z - y z z_1}{x} \\ -x_1 z & z_1 & x z_1 \\ z & 0 & -z z_1 \end{bmatrix}$$

and

$$M = s \begin{bmatrix} \frac{yz}{x} & \frac{k}{x} & \frac{kx - qx_2 z - yzz_2}{x} \\ -x_2 z & z_2 & xz_2 \\ z & 0 & -zz_2 \end{bmatrix},$$

where $t = s = \frac{1}{z}$, or $t = \frac{1}{z_1}$, $s = \frac{1}{z_2}$,

or $t = \sqrt[3]{\frac{x}{x_1 z^2 z_1}}$, $s = \sqrt[3]{\frac{x}{x_2 z^2 z_2}}$.

Here, $\frac{t_2}{t} \frac{s}{s_1} = \frac{z_1}{z_2}$.

- Example 4: Discrete Boussinesq System (Tongas and Nijhoff 2005)

$$
\begin{aligned}
z_1 - x x_1 + y &= 0 \\
z_2 - x x_2 + y &= 0 \\
(x_2 - x_1)(z - x x_{12} + y_{12}) - p + q &= 0
\end{aligned}
$$

- Lax pair:

$$
\boxed{L = t \begin{bmatrix} -x_1 & 1 & 0 \\ -y_1 & 0 & 1 \\ p - k - x y_1 + x_1 z & -z & x \end{bmatrix}}
$$

$$\boxed{M = s\begin{bmatrix} -x_2 & 1 & 0 \\ -y_2 & 0 & 1 \\ q-k-xy_2+x_2z & -z & x \end{bmatrix}}$$

with $t = s = 1$, or $t = \frac{1}{\sqrt[3]{p-k}}$ and $s = \frac{1}{\sqrt[3]{q-k}}$.

Note: $x_3 = \frac{f}{F}$, $y_3 = \frac{g}{F}$, and $\psi = \begin{bmatrix} f \\ F \\ g \end{bmatrix}$, and $\frac{t_2}{t}\,\frac{s}{s_1} = 1$.

- Example 5: System of pKdV Lattices (Xenitidis and Mikhailov 2009)

$$\begin{aligned}(x - x_{12})(y_1 - y_2) - p^2 + q^2 &= 0\\ (y - y_{12})(x_1 - x_2) - p^2 + q^2 &= 0\end{aligned}$$

- Lax pair:

$$L = \begin{bmatrix} 0 & 0 & tx & t(p^2 - k^2 - xy_1) \\ 0 & 0 & t & -ty_1 \\ Ty & T(p^2 - k^2 - x_1 y) & 0 & 0 \\ T & -Tx_1 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & sx & s(q^2 - k^2 - xy_2) \\ 0 & 0 & s & -sy_2 \\ Sy & S(q^2 - k^2 - x_2 y) & 0 & 0 \\ S & -Sx_2 & 0 & 0 \end{bmatrix}$$

with $t = s = T = S = 1$,

or $t\,T = \frac{1}{\sqrt{\mathrm{Det} L_c}} = \frac{1}{p-k}$ and $s\,S = \frac{1}{\sqrt{\mathrm{Det} M_c}} = \frac{1}{q-k}$.

Note: $\frac{t_2}{T}\,\frac{S}{s_1} = 1$ and $\frac{T_2}{t}\,\frac{s}{S_1} = 1$,

or $\frac{\mathcal{T}_2}{\mathcal{T}}\,\frac{\mathcal{S}}{\mathcal{S}_1} = 1$, with $\mathcal{T} = t\,T$, $\mathcal{S} = s\,S$.

Here, $x_3 = \frac{f}{F}$, $y_3 = \frac{g}{G}$, and $\psi = [f \;\; F \;\; g \;\; G]^{\mathrm{T}}$.

- Example 6: Discrete NLS System (Xenitidis and Mikhailov 2009)

$$
\begin{aligned}
y_1 - y_2 - y\left((x_1 - x_2)y + p - q\right) &= 0 \\
x_1 - x_2 + x_{12}\left((x_1 - x_2)y + p - q\right) &= 0
\end{aligned}
$$

- Lax pair:

$$
\boxed{L = t\begin{bmatrix} -1 & x_1 \\ y & k - p - yx_1 \end{bmatrix}}
$$

$$
\boxed{M = s\begin{bmatrix} -1 & x_2 \\ y & k - q - yx_2 \end{bmatrix}}
$$

with $t = s = 1$, or $t = \frac{1}{\sqrt{\mathrm{Det} L_c}} = \frac{1}{\sqrt{\alpha - k}}$ and $s = \frac{1}{\sqrt{\beta - k}}$.

Note: $x_3 = \frac{f}{F}$, $\psi = \begin{bmatrix} f \\ F \end{bmatrix}$, and $\frac{t_2}{t}\frac{s}{s_1} = 1$.

- Example 7: Schwarzian Boussinesq Lattice (Nijhoff 1999)

$$\begin{aligned} z_1\, y - x_1 + x &= 0 \\ z_2\, y - x_2 + x &= 0 \\ z\, y_{12}\,(y_1 - y_2) - y(p\, y_2\, z_1 - q\, y_1\, z_2) &= 0 \end{aligned}$$

- Lax pair:

$$\boxed{L = t \begin{bmatrix} y & 0 & -yz_1 \\ -\frac{kyy_1}{z} & \frac{pyz_1}{z} & 0 \\ 0 & -1 & y_1 \end{bmatrix}}$$

$$\boxed{M = s\begin{bmatrix} y & 0 & -yz_2 \\ -\frac{kyy_2}{z} & \frac{pyz_2}{z} & 0 \\ 0 & -1 & y_2 \end{bmatrix}}$$

with $t = s = \frac{1}{y}$, or $t = \frac{1}{y_1}$ and $s = \frac{1}{y_2}$,

or $t = \sqrt[3]{\frac{z}{y^2 y_1 z_1}}$ and $s = \sqrt[3]{\frac{z}{y^2 y_2 z_2}}$.

Note: $x_3 = \frac{fy+Fx}{F},\ y_3 = \frac{g}{F},\ z_3 = \frac{f}{F},\ \psi = \begin{bmatrix} f \\ g \\ F \end{bmatrix}$,

and $\frac{t_2}{t}\frac{s}{s_1} = \frac{y_1}{y_2}$.

- Example 9: Toda modified Boussinesq System (Nijhoff 1992)

$$y_{12}(p-q+x_2-x_1)-(p-1)y_2+(q-1)y_1=0$$

$$y_1y_2(p-q-z_2+z_1)-(p-1)yy_2+(q-1)yy_1=0$$

$$y(p+q-z-x_{12})(p-q+x_2-x_1)-(p^2+p+1)y_1 + (q^2+q+1)y_2=0$$

- Lax pair:

$$L=t\begin{bmatrix} k+p-z & \frac{1+k+k^2}{y} & \frac{-k^2y-y_1-p^2(y_1-y)-ky(x_1-z)+yzx_1}{y} - \frac{p(y_1+yx_1+yz)}{y} \\ 0 & p-1 & (1-k)y_1 \\ 1 & 0 & p-k-x_1 \end{bmatrix}$$

$$M = s\begin{bmatrix} k+q-z & \frac{1+k+k^2}{y} & \frac{-k^2y-y_2-q^2(y_2-y)-ky(x_2-z)+yzx_2}{y} \\ & & -\frac{q(y_2+yx_2+yz)}{y} \\ 0 & q-1 & (1-k)y_2 \\ 1 & 0 & q-k-x_2 \end{bmatrix}$$

with $t = s = 1$, or $t = \sqrt[3]{\frac{y_1}{y}}$ and $s = \sqrt[3]{\frac{y_2}{y}}$.

Note: $x_3 = \frac{f}{F}$, $y_3 = \frac{g}{F}$, $\psi = \begin{bmatrix} f \\ g \\ F \end{bmatrix}$

Here, $\frac{t_2}{t}\,\frac{s}{s_1} = 1$.

# Conclusions and Future Work

- *Mathematica* code works for scalar $\mathbf{P\Delta E}s$ in 2D defined on quad-graphs (quadrilateral faces).
- *Mathematica* code has been extended to systems of $\mathbf{P\Delta E}s$ in 2D defined on quad-graphs.
- Code can be used to test (i) consistency around the cube and compute or test (ii) Lax pairs.
- Consistency around cube $\Longrightarrow$ $\mathrm{P\Delta E}$ has Lax pair.
- $\mathrm{P\Delta E}$ has Lax pair $\not\Rightarrow$ consistency around cube. Indeed, there are $\mathrm{P\Delta E}$s with a Lax pair that are not consistent around the cube.
  Example: discrete sine-Gordon equation.

- Avoid the determinant method to avoid square roots! Factorization plays an essential role!
- Hard cases: $Q4$ equation (elliptic curves, Weierstraß functions) (Nijhoff 2001) and $Q5$
- Future Work: Extension to more complicated systems of $\mathbf{P}\Delta\mathrm{E}s$.
- $\mathbf{P}\Delta\mathrm{E}s$ in 3D: Lax pair will be expressed in terms of tensors. Consistency around a "hypercube". Examples: discrete Kadomtsev-Petviashvili (KP) equations.

Thank You

# Additional Examples

- Example 9: (H1) Equation (ABS classification)

$$(x - x_{12})(x_1 - x_2) + q - p = 0$$

- Lax pair:

$$L = t \begin{bmatrix} x & p - k - x x_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} x & q - k - x x_2 \\ 1 & -x_2 \end{bmatrix}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{k-p}}$ and $s = \frac{1}{\sqrt{k-q}}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = 1.$

- Example 10: (H2) Equation (ABS 2003)

$$(x-x_{12})(x_1-x_2)+(q-p)(x+x_1+x_2+x_{12})+q^2-p^2=0$$

- Lax pair:

$$L = t\begin{bmatrix} p-k+x & p^2-k^2+(p-k)(x+x_1)-xx_1 \\ 1 & -(p-k+x_1) \end{bmatrix}$$

$$M = s\begin{bmatrix} q-k+x & q^2-k^2+(q-k)(x+x_2)-xx_2 \\ 1 & -(q-k+x_2) \end{bmatrix}$$

with $t = \frac{1}{\sqrt{2(k-p)(p+x+x_1)}}$ and $s = \frac{1}{\sqrt{2(k-q)(q+x+x_2)}}$

Note: $\frac{t_2}{t}\frac{s}{s_1} = \frac{p+x+x_1}{q+x+x_2}$.

- Example 11: (H3) Equation (ABS 2003)

$$p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) + \delta(p^2 - q^2) = 0$$

- Lax pair:

$$L = t \begin{bmatrix} kx & -\left(\delta(p^2 - k^2) + pxx_1\right) \\ p & -kx_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} kx & -\left(\delta(q^2 - k^2) + qxx_2\right) \\ q & -kx_2 \end{bmatrix}$$

with $t = \frac{1}{\sqrt{(p^2-k^2)(\delta p+xx_1)}}$ and $s = \frac{1}{\sqrt{(q^2-k^2)(\delta q+xx_2)}}$

Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{\delta p+xx_1}{\delta q+xx_2}$.

- Example 12: (H3) Equation ($\delta = 0$) (ABS 2003)

$$p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) = 0$$

- Lax pair:

$$L = t \begin{bmatrix} kx & -pxx_1 \\ p & -kx_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} kx & -qxx_2 \\ q & -kx_2 \end{bmatrix}$$

with $t = s = \frac{1}{x}$ or $t = \frac{1}{x_1}$ and $s = \frac{1}{x_2}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{x\,x_1}{x\,x_2} = \frac{x_1}{x_2}$.

- Example 13: (Q1) Equation (ABS 2003)

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta^2 pq(p-q)=0$$

- Lax pair:

$$L = t\begin{bmatrix} (p-k)x_1+kx & -p\left(\delta^2 k(p-k)+xx_1\right) \\ p & -\left((p-k)x+kx_1\right) \end{bmatrix}$$

$$M = s\begin{bmatrix} (q-k)x_2+kx & -q\left(\delta^2 k(q-k)+xx_2\right) \\ q & -\left((q-k)x+kx_2\right) \end{bmatrix}$$

with $t=\frac{1}{\delta p\pm(x-x_1)}$ and $s=\frac{1}{\delta q\pm(x-x_2)}$,
or $t=\frac{1}{\sqrt{k(p-k)\big((\delta p+x-x_1)(\delta p-x+x_1)\big)}}$ and

$$s=\frac{1}{\sqrt{k(q-k)\big((\delta q+x-x_2)(\delta q-x+x_2)\big)}}$$

Note: $\frac{t_2}{t}\,\frac{s}{s_1}=\frac{q\big(\delta p+(x-x_1)\big)\big(\delta p-(x-x_1)\big)}{p\big(\delta q+(x-x_2)\big)\big(\delta q-(x-x_2)\big)}$.

- Example 14: (Q1) Equation $(\delta = 0)$ (ABS 2003)

$$p(x - x_2)(x_1 - x_{12}) - q(x - x_1)(x_2 - x_{12}) = 0$$

which is the cross-ratio equation

$$\frac{(x - x_1)(x_{12} - x_2)}{(x_1 - x_{12})(x_2 - x)} = \frac{p}{q}$$

- Lax pair:

$$L = t \begin{bmatrix} (p-k)x_1 + kx & -pxx_1 \\ p & -\left((p-k)x + kx_1\right) \end{bmatrix}$$

$$M = s \begin{bmatrix} (q-k)x_2 + kx & -qxx_2 \\ q & -\left((q-k)x + kx_2\right) \end{bmatrix}$$

Here, $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{q(x-x_1)^2}{p(x-x_2)^2}$. So, $t = \frac{1}{x-x_1}$ and $s = \frac{1}{x-x_2}$

or $t = \frac{1}{\sqrt{k(k-p)}(x-x_1)}$ and $s = \frac{1}{\sqrt{k(k-q)}(x-x_2)}$.

- Example 15: (Q2) Equation (ABS 2003)

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+pq(p-q)(x+x_1+x_2+x_{12})-pq(p-q)(p^2-pq+q^2)=0$$

- Lax pair:

$$L=t\begin{bmatrix}(k-p)(kp-x_1)+kx & -p\left(k(k-p)(k^2-kp+p^2-x-x_1)+xx_1\right) \\ p & -\left((k-p)(kp-x)+kx_1\right)\end{bmatrix}$$

$$M=s\begin{bmatrix}(k-q)(kq-x_2)+kx & -q\left(k(k-q)(k^2-kq+q^2-x-x_2)+xx_2\right) \\ q & -\left((k-q)(kq-x)+kx_2\right)\end{bmatrix}$$

- with

$$t = \frac{1}{\sqrt{k(k-p)\big((x-x_1)^2-2p^2(x+x_1)+p^4\big)}}$$

and

$$s = \frac{1}{\sqrt{k(k-q)\big((x-x_2)^2-2q^2(x+x_2)+q^4\big)}}$$

Note:

$$\begin{aligned} \frac{t_2}{t}\,\frac{s}{s_1} &= \frac{q\left((x-x_1)^2-2p^2(x+x_1)+p^4\right)}{p\left((x-x_2)^2-2q^2(x+x_2)+q^4\right)} \\ &= \frac{p\left((X+X_1)^2-p^2\right)\left((X-X_1)^2-p^2\right)}{q\left((X+X_2)^2-q^2\right)\left((X-X_2)^2-q^2\right)} \end{aligned}$$

with $x = X^2$, and, consequently, $x_1 = X_1^2$, $x_2 = X_2^2$.

- Example 16: (Q3) Equation (ABS 2003)

$$(q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12})$$
$$-p(q^2-1)(xx_2+x_1x_{12})-\frac{\delta^2}{4pq}(p^2-q^2)(p^2-1)(q^2-1)=0$$

- Lax pair:

$$L=t\begin{bmatrix} -4kp\left(p(k^2-1)x+(p^2-k^2)x_1\right) & -(p^2-1)(\delta^2k^2-\delta^2k^4-\delta^2p^2+\delta^2k^2p^2-4k^2pxx_1) \\ -4k^2p(p^2-1) & 4kp\left(p(k^2-1)x_1+(p^2-k^2)x\right) \end{bmatrix}$$

$$M=s\begin{bmatrix} -4kq\left(q(k^2-1)x+(q^2-k^2)x_2\right) & -(q^2-1)(\delta^2k^2-\delta^2k^4-\delta^2q^2+\delta^2k^2q^2-4k^2qxx_2) \\ -4k^2q(q^2-1) & 4kq\left(q(k^2-1)x_2+(q^2-k^2)x\right) \end{bmatrix}$$

- with

$$t=\frac{1}{2k\sqrt{p(k^2-1)(k^2-p^2)\left(4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2\right)}}$$

and

$$s=\frac{1}{2k\sqrt{q(k^2-1)(k^2-q^2)\left(4q^2(x^2+x_2^2)-4q(1+q^2)xx_2+\delta^2(1-q^2)^2\right)}}.$$

Note:

$$\frac{t_2}{t}\,\frac{s}{s_1}$$

$$= \frac{q(q^2-1)\left(4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x^2+x_2^2)-4q(1+q^2)xx_2+\delta^2(1-q^2)^2\right)}$$

$$= \frac{q(q^2-1)\left(4p^2(x-x_1)^2-4p(p-1)^2xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x-x_2)^2-4q(q-1)^2xx_2+\delta^2(1-q^2)^2\right)}$$

$$= \frac{q(q^2-1)\left(4p^2(x+x_1)^2-4p(p+1)^2xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x+x_2)^2-4q(q+1)^2xx_2+\delta^2(1-q^2)^2\right)}$$

where

$$
\begin{aligned}
&4p^2(x^2+x_1^2)-4p(1+p^2)xx_1+\delta^2(1-p^2)^2 \\
&= \delta^2(p-\mathrm{e}^{X+X_1})(p-\mathrm{e}^{-(X+X_1)})(p-\mathrm{e}^{X-X_1})(p-\mathrm{e}^{-(X-X_1)}) \\
&= \delta^2(p-\cosh(X+X_1)+\sinh(X+X_1)) \\
&\qquad (p-\cosh(X+X_1)-\sinh(X+X_1)) \\
&\qquad (p-\cosh(X-X_1)+\sinh(X-X_1)) \\
&\qquad (p-\cosh(X-X_1)-\sinh(X-X_1))
\end{aligned}
$$

with $x = \delta\cosh(X)$, and, consequently,

$x_1 = \delta\cosh(X_1),\ x_2 = \delta\cosh(X_2)$.

- Example 17: (Q3) Equation $(\delta)=0$ (ABS 2003)

$$(q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12})$$
$$-p(q^2-1)(xx_2+x_1x_{12})=0$$

- Lax pair:

$$L=t\begin{bmatrix}(p^2-k^2)x_1+p(k^2-1)x & -k(p^2-1)xx_1\\ (p^2-1)k & -\left((p^2-k^2)x+p(k^2-1)x_1\right)\end{bmatrix}$$

$$M=s\begin{bmatrix}(q^2-k^2)x_2+q(k^2-1)x & -k(q^2-1)xx_2\\ (q^2-1)k & -\left((q^2-k^2)x+q(k^2-1)x_2\right)\end{bmatrix}$$

- with $t = \frac{1}{px - x_1}$ and $s = \frac{1}{qx - x_2}$

  or $t = \frac{1}{px_1 - x}$ and $s = \frac{1}{qx_2 - x}$

  or $t = \frac{1}{\sqrt{(k^2-1)(p^2-k^2)(px-x_1)(px_1-x)}}$

  and $s = \frac{1}{\sqrt{(k^2-1)(q^2-k^2)(qx-x_2)(qx_2-x)}}$.

  Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{(q^2-1)(px-x_1)(px_1-x)}{(p^2-1)(qx-x_2)(qx_2-x)}$.

- Example 18: $(\alpha,\beta)$-equation (Quispel 1983)

$$\big((p-\alpha)x-(p+\beta)x_1\big)\big((p-\beta)x_2-(p+\alpha)x_{12}\big)$$
$$-\big((q-\alpha)x-(q+\beta)x_2\big)\big((q-\beta)x_1-(q+\alpha)x_{12}\big)=0$$

- Lax pair:

$$L=t\begin{bmatrix}(p-\alpha)(p-\beta)x+(k^2-p^2)x_1 & -(k-\alpha)(k-\beta)xx_1\\ (k+\alpha)(k+\beta) & -\big((p+\alpha)(p+\beta)x_1+(k^2-p^2)x\big)\end{bmatrix}$$

$$M=s\begin{bmatrix}(q-\alpha)(q-\beta)x+(k^2-q^2)x_2 & -(k-\alpha)(k-\beta)xx_2\\ (k+\alpha)(k+\beta) & -\big((q+\alpha)(q+\beta)x_2+(k^2-q^2)x\big)\end{bmatrix}$$

- with $t = \frac{1}{(\alpha-p)x+(\beta+p)x_1}$ and $s = \frac{1}{(\alpha-q)x+(\beta+q)x_2}$

  or $t = \frac{1}{(\beta-p)x+(\alpha+p)x_1}$ and $s = \frac{1}{(\beta-q)x+(\alpha+q)x_2}$

  or $t = \frac{1}{\sqrt{(p^2-k^2)\big((\beta-p)x+(\alpha+p)x_1\big)\big((\alpha-p)x+(\beta+p)x_1\big)}}$

  and $s = \frac{1}{\sqrt{(q^2-k^2)\big((\beta-q)x+(\alpha+q)x_2\big)\big((\alpha-q)x+(\beta+q)x_2\big)}}$

  Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{\big((\beta-p)x+(\alpha+p)x_1\big)\big((\alpha-p)x+(\beta+p)x_1\big)}{\big((\beta-q)x+(\alpha+q)x_2\big)\big((\alpha-q)x+(\beta+q)x_2\big)}$.

- Example 19: (A1) Equation (ABS 2003)

$$p(x+x_2)(x_1+x_{12})-q(x+x_1)(x_2+x_{12})-\delta^2pq(p-q)=0$$

(Q1) if $x_1 \to -x_1$ and $x_2 \to -x_2$

- Lax pair:

$$L = t\begin{bmatrix} (k-p)x_1+kx & -p\left(\delta^2k(k-p)+xx_1\right) \\ p & -\left((k-p)x+kx_1\right) \end{bmatrix}$$

$$M = s\begin{bmatrix} (k-q)x_2+kx & -q\left(\delta^2k(k-q)+xx_2\right) \\ q & -\left((k-q)x+kx_2\right) \end{bmatrix}$$

- with $t = \frac{1}{\sqrt{k(k-p)\Big((\delta p+x+x_1)(\delta p-x-x_1)\Big)}}$ and

  $s = \frac{1}{\sqrt{k(k-q)\Big((\delta q+x+x_2)(\delta q-x-x_2)\Big)}}$

  Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{q\Big(\delta p+(x+x_1)\Big)\Big(\delta p-(x+x_1)\Big)}{p\Big(\delta q+(x+x_2)\Big)\Big(\delta q-(x+x_2)\Big)}$.

  Question: Rational choice for $t$ and $s$?

- Example 20: (A2) Equation (ABS 2003)

$$(q^2-p^2)(xx_1x_2x_{12}+1)+q(p^2-1)(xx_2+x_1x_{12})$$
$$-p(q^2-1)(xx_1+x_2x_{12})=0$$

(Q3) with $\delta=0$ via Möbius transformation:

$$x\to x, x_1\to \frac{1}{x_1}, x_2\to \frac{1}{x_2}, x_{12}\to x_{12}, p\to p, q\to q$$

- Lax pair:

$$L=t\begin{bmatrix} k(p^2-1)x & -\left(p^2-k^2+p(k^2-1)xx_1\right) \\ p(k^2-1)+(p^2-k^2)xx_1 & -k(p^2-1)x_1 \end{bmatrix}$$

$$M=s\begin{bmatrix} k(q^2-1)x & -\left(q^2-k^2+q(k^2-1)xx_2\right) \\ q(k^2-1)+(q^2-k^2)xx_2 & -k(q^2-1)x_2 \end{bmatrix}$$

- with $t = \frac{1}{\sqrt{(k^2-1)(k^2-p^2)(p-xx_1)(pxx_1-1)}}$

  and $s = \frac{1}{\sqrt{(k^2-1)(k^2-q^2)(q-xx_2)(qxx_2-1)}}$

  Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{(q^2-1)(p-xx_1)(pxx_1-1)}{(p^2-1)(q-xx_2)(qxx_2-1)}$.

  Question: Rational choice for $t$ and $s$?

- Example 21: Discrete sine-Gordon Equation

$$x x_1 x_2 x_{12} - pq(x x_{12} - x_1 x_2) - 1 = 0$$

(H3) with $\delta = 0$ via extended Möbius transformation:

$$x \to x, x_1 \to x_1, x_2 \to \frac{1}{x_2}, x_{12} \to -\frac{1}{x_{12}}, p \to \frac{1}{p}, q \to q$$

Discrete sine-Gordon equation is NOT consistent around the cube, but has a Lax pair!

- Lax pair:

$$L = \begin{bmatrix} p & -kx_1 \\ -\frac{k}{x} & \frac{px_1}{x} \end{bmatrix} \qquad M = \begin{bmatrix} \frac{qx_2}{x} & -\frac{1}{kx} \\ -\frac{x_2}{k} & q \end{bmatrix}$$