Symbolic Computation of Soliton Solutions of PDEs through Homogenization

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The Eighth IMACS International Conference on Nonlinear Evolution Equations and Wave Phenomena: Computation and Theory Athens, Georgia, U.S.A.

Monday, March 25, 2013, 11:20 a.m.

Acknowledgements

Willy Malfliet (University of Antwerp, Belgium) Frank Verheest (University of Ghent, Belgium)

Research supported in part by NSF under Grant No. CCF-0830783

This presentation was made in TeXpower

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Motivation

- Hirota's method can be used to find exact solutions of nonlinear PDEs, provided these equations can be brought into a bilinear form.
- Finding bilinear forms for nonlinear PDEs, if they exist at all, is highly nontrivial.
- To circumvent this difficulty, we introduce a simplified version of Hirota's method, and use it to construct solitary and soliton solutions.
- Without bilinear forms, exact solutions can still be constructed straightforwardly by solving a perturbation scheme on the computer, using a symbolic manipulation package.

Simplified Version of Hirota's Method

To illustrate the homogenization method, the well known Korteweg-de Vries (KdV) equation will be used;

$$u_t + b u u_x + u_{xxx} = 0, \tag{1}$$

where b is any real constant.

Write u(x,t) as a Laurent series in the complex plane,

$$u(x,t) = f^{\alpha}(x,t) \sum_{k=0}^{\infty} u_k(x,t) f^k(x,t),$$
 (2)

where f(x,t) is the non-characteristic manifold for the poles, and α is the negative integer which gives the degree of the most singular terms.

Then, by substituting the series into the equation and requiring that the most singular terms vanish, one obtains the values for α and $u_0(x,t)$.

If the next most singular terms are required to vanish, one will obtain the expressions for $u_1(x,t)$, $u_2(x,t)$, etc.

After that, the series will be truncated at the constant level term.

In order to find the leading order α , let

$$u = f^{\alpha}(x, t)u_0(x, t).$$
 (3)

Substituting (3) into (1) yields

$$bf^{2\alpha}u_{0}u_{0x} + \alpha bf^{2\alpha-1}f_{x}u_{0}^{2} + f^{\alpha}\left[u_{03x} + u_{0t}\right] \\ + f^{\alpha-1}\left[3\alpha f_{x}u_{0xx} + 3\alpha f_{xx}u_{0x} + \alpha f_{3x}u_{0} + \alpha f_{t}u_{0}\right] \\ + f^{\alpha-2}\left[3\alpha^{2}f_{x}^{2}u_{0x} - 3\alpha f_{x}^{2}u_{0x} \\ + 3\alpha^{2}f_{x}f_{xx}u_{0} - 3\alpha f_{x}f_{xx}u_{0}\right] \\ + f^{\alpha-3}\left[\alpha^{3}f_{x}^{3}u_{0} - 3\alpha^{2}f_{x}^{3}u_{0} + 2\alpha f_{x}^{3}u_{0}\right] = 0.$$
(4)

From the above equation, you can see that the most singular powers of f are $2\alpha - 1$, and $\alpha - 3$; therefore, by equating these powers, one gets $\alpha = -2$.

Hence, the most singular terms, i.e. the terms in f^{-5} in (4):

$$(-2bf_x u_0^2 - 24f_x^3 u_0)f^{-5}, (5)$$

will vanish if

$$u_0 = -\frac{12f_x^2}{b}.$$
 (6)

Next, to find u_1 let

$$u = f^{-2}(x,t) \left[u_0(x,t) + f(x,t)u_1(x,t) \right]$$
(7)

i.e.

$$u = \frac{u_1}{f} - \frac{12f_x^2}{bf^2},$$
(8)

and substitute (8) for u into (1), and compute, again,

the coefficient of the most singular term (f^{-4} in this case), one finds

$$30bf_x^3u_1 - 360f_x^3f_{xx}.$$
 (9)

To eliminate this term, take

$$u_1 = \frac{12f_{xx}}{b}.$$
 (10)

The next step is to calculate u_2 , for which

$$u = f^{-2}(x,t) \left[u_0(x,t) + f(x,t)u_1(x,t) + f^2(x,t)u_2(x,t) \right]$$

= $\frac{u_0(x,t)}{f^2(x,t)} + \frac{u_1(x,t)}{f(x,t)} + u_2(x,t),$ (11)

is substituted into the given PDE.

It turns out that u_2 has to satisfy the original equation,

$$(u_2)_t + bu_2(u_2)_x + (u_2)_{3x} = 0.$$
 (12)

For other equations, the same procedure will be followed. Setting different power terms in f(x,t) equal to zero will allow the finding of u_0 , u_1 , u_2 , etc. The series will always be truncated at the constant level term of f, and the coefficient of that constant level will be set to zero. For the KdV equation, set $u_2 = 0$ to obtain

$$u = \frac{12f_{xx}}{bf} - \frac{12f_{x}^{2}}{bf^{2}} = \frac{12}{b}\frac{\partial^{2}}{\partial x^{2}}\ln f.$$
 (13)

This transformation will allow one to homogenize the equations, once the equation has been "homogenized" it can be readily solved.

First, integrate the equation with respect to x as many times as possible and ignore integration "constants" (i.e., arbitrary functions of t). For the KdV equation only one integration is possible, yielding

$$\int^{x} u_t \, dx + \frac{b}{2} u^2 + u_{2x} = 0. \tag{14}$$

Second, change of dependent variable according to

$$u(x,t) = K \frac{\partial^2}{\partial x^2} \ln f(x,t) = K \frac{(ff_{2x} - f_x^2)}{f^2},$$
 (15)

to transform (1) into a homogeneous equation in f and its derivatives of as low a degree as possible. For the KdV quation, setting $K = \frac{12}{b}$ two terms vanish which allows one to cancel a common factor f^2 . Hence, one obtains a homogeneous equation of second degree,

$$f[f_{xt} + f_{4x}] - f_x f_t - 4f_x f_{3x} + 3f_{2x}^2 = 0.$$
 (16)

The approach below works for "quadratic" equations of the form

$$f \mathcal{L}(f) + \mathcal{N}(f, f) = 0, \qquad (17)$$

where \mathcal{L} denotes a linear differential operator and \mathcal{N} is a bilinear (quadratic) differential operator.

For the KdV equation,

$$\mathcal{L}(f) = f_{xt} + f_{4x}, \qquad (18)$$

and

$$\mathcal{N}(f,g) = -f_x g_t - 4f_x g_{3x} + 3f_{2x} g_{2x}.$$
 (19)

Third, seek a solution of the form

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}(x,t),$$
 (20)

where ϵ serves as a bookkeeping parameter (not a small quantity).

Substitute (20) into (17).

Use Cauchy's product formula,

$$\left(\sum_{r=1}^{\infty} \epsilon^r a_r\right) \left(\sum_{s=1}^{\infty} \epsilon^s b_s\right) = \sum_{n=2}^{\infty} \epsilon^n \sum_{j=1}^{n-1} a_{n-j} b_j, \quad (21)$$

to group the terms in powers of ϵ . Equate to zero the different powers of ϵ :

$$O(\epsilon^{1}) : \mathcal{L}(f^{(1)}) = 0,$$

$$O(\epsilon^{2}) : \mathcal{L}(f^{(2)}) = -\mathcal{N}(f^{(1)}, f^{(1)}),$$
(22)
(23)

$$O(\epsilon^{n}):$$

$$\mathcal{L}(f^{(n)}) = -\left[\sum_{j=2}^{n-1} f^{(n-j)} \mathcal{L}(f^{(j)}) + \sum_{j=1}^{n-1} \mathcal{N}(f^{(n-j)}, f^{(j)})\right],$$

$$n \ge 3.$$
(24)

The latter formula can be written succinctly as

$$\mathcal{L}(f^{(n)}) = -\sum_{j=1}^{n-1} \left[f^{(n-j)} \mathcal{L}(f^{(j)}) + \mathcal{N}(f^{(n-j)}, f^{(j)}) \right], \quad n \ge 3,$$

(25)

provided one uses $\mathcal{L}(f^{(1)}) = 0$.

For example, application of (24) with n = 3 gives $\mathcal{L}(f^{(3)}) = -\left[f^{(1)}\mathcal{L}(f^{(2)}) + \mathcal{N}(f^{(2)}, f^{(1)}) + \mathcal{N}(f^{(1)}, f^{(2)})\right].$ (26)

The N-soliton solution of the KdV is then generated from

$$f^{(1)} = \sum_{i=1}^{N} \exp(\theta_i) = \sum_{i=1}^{N} \exp(k_i x - \omega_i t + \delta_i), \quad (27)$$

where k_i, ω_i and δ_i are constants. Substitution of (27) into (22) yields the dispersion relation $P(k_i, \omega_i) = 0$.

For the KdV equation,

$$P(k_i, \omega_i) = k_i (k_i^3 - \omega_i), \qquad i = 1, 2, ..., N.$$
(28)

The dispersion law is thus $\omega_i = k_i^3$. Using (27), we compute the right hand side of (23):

$$-\sum_{i,j=1}^{N} 3k_i k_j^2 (k_i - k_j) \exp(\theta_i + \theta_j) = \sum_{1 \le i < j \le N} 3k_i k_j (k_i - k_j)^2 \exp(\theta_i + \theta_j).$$
(29)

Observe that the terms in $\exp(2\theta_i)$ are missing, which is typical for quadratic equations in f which admit solitons. Furthermore, (29) determines the form of $f^{(2)}$:

$$f^{(2)} = \sum_{1 \le i < j \le N} a_{ij} \exp(\theta_i + \theta_j) =$$

$$\sum_{1 < i < j \le N} a_{ij} \exp[(k_i + k_j)x - (\omega_i + \omega_j)t + (\delta_i + \delta_j)].$$
(30)

With (18), (28), and (30), the left hand side of (23) is readily computed

$$\mathcal{L}(f^{(2)}) = \sum_{1 \le i < j \le N} P(k_i + k_j, \omega_i + \omega_j) a_{ij} \exp(\theta_i + \theta_j)$$
$$= \sum_{1 \le i < j \le N} 3k_i k_j (k_i + k_j)^2 a_{ij} \exp(\theta_i + \theta_j).$$
(31)

Equating (29) and (31), we have

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \qquad 1 \le i < j \le N.$$
 (32)

Proceeding in a similar way with (26) leads to the explicit form of $f^{(3)}$.

For example, for N = 3 we have

$$f^{(3)} = b_{123} \exp(\theta_1 + \theta_2 + \theta_3)$$

= $b_{123} \exp[(k_1 + k_2 + k_3)x - (\omega_1 + \omega_2 + \omega_3)t + (\delta_1 + \delta_2 + \delta_3)],$ (33)

with

$$b_{123} = a_{12} a_{13} a_{23} = \frac{\left(k_1 - k_2\right)^2 \left(k_1 - k_3\right)^2 \left(k_2 - k_3\right)^2}{\left(k_1 + k_2\right)^2 \left(k_1 + k_3\right)^2 \left(k_2 + k_3\right)^2}.$$
 (34)

For N = 3, one can show by the computation of the rest of the scheme that $f^{(n)} = 0$ for n > 3. Therefore the expansion in (20) truncates, and

$$f = 1 + \exp \theta_1 + \exp \theta_2 + \exp \theta_3 + a_{12} \exp(\theta_1 + \theta_2) + a_{13} \exp(\theta_1 + \theta_3) + a_{23} \exp(\theta_2 + \theta_3) + b_{123} \exp(\theta_1 + \theta_2 + \theta_3),$$
(35)

where we set $\epsilon = 1$.

Note that f has no terms in $\exp(2\theta_1)$, $\exp(2\theta_2)$, $\exp(2\theta_1 + \theta_2)$ and $\exp(2\theta_2 + \theta_1)$, ...

Upon substitution of (35) into (15) the well-known three-soliton solution of the KdV equation follows.

Conclusion

Hirota's direct method for solving completely integrable PDEs is algorithmic and implementable in the language of any computer algebra system.

The main advantages of the simplified version of Hirota's method is that the bilinear representation for the equation becomes superfluous.

Therefore, the method can be applied to equations for which the bilinear form is not known or does not exist.

Software Demonstration

Software packages in Mathematica

Codes are available via the Internet: URL: http://inside.mines.edu/~whereman/

Thank You