Symbolic Computation of Conserved Densities, Generalized Symmetries, and Recursion Operators of Nonlinear Evolution Equations and Lattices

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OUTLINE

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- Publications

- **Purpose**

  Design and implement algorithms to compute polynomial conservation laws and generalized symmetries (later recursion operators) for nonlinear systems of evolution and lattice equations.

- **Motivation**

  – Conservation laws describe the conservation of fundamental physical quantities (linear momentum, energy, etc.).

    Compare with constants of motion (linear momentum, energy) in mechanics.

  – Conservation laws provide a method to study quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures.

  – Conservation laws can be used to test numerical integrators.

  – For PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws or symmetries assures complete **integrability**.

  – Conserved densities and symmetries aid in finding the recursion operator (which guarantees the existence of infinitely many symmetries).

PART I: Evolution Equations (PDEs)

- **System of evolution equations**

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ..., \mathbf{u}_{mx})$$

in a (single) space variable x and time t, and with

$$\mathbf{u} = (u_1, u_2, ..., u_n), \quad \mathbf{F} = (F_1, F_2, ..., F_n).$$

Notation:

$$\mathbf{u}_{mx} = \mathbf{u}^{(m)} = \frac{\partial \mathbf{u}}{\partial x^m}.$$

$\mathbf{F}$ is polynomial in $\mathbf{u}, \mathbf{u}_x, ..., \mathbf{u}_{mx}$.

PDEs of higher order in t should be recast as a first-order system.

- **Examples:**

The Korteweg-de Vries (KdV) equation:

$$u_t + uu_x + u_{3x} = 0.$$

Fifth-order evolution equations with constant parameters (α, β, γ):

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0.$$

Special case. The fifth-order Sawada-Kotera (SK) equation:

$$u_t + 5u^2 u_x + 5u_x u_{2x} + 5uu_{3x} + u_{5x} = 0.$$

The Boussinesq (wave) equation:

$$u_{tt} - u_{2x} + 3uu_{2x} + 3{u_x}^2 + \alpha u_{4x} = 0,$$

written as a first-order system (v auxiliary variable):

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0. \end{aligned}$$

A vector nonlinear Schrödinger equation:

$$\mathbf{B}_t + (|\mathbf{B}|^2\mathbf{B})_x + (\mathbf{B}_0 \cdot \mathbf{B}_x)\mathbf{B}_0 + \mathbf{e} \times \mathbf{B}_{xx} = 0,$$

written in component form, $\mathbf{B}_0 = (a, b)$ and $\mathbf{B} = (u, v)$:

$$u_t + \Big[u(u^2+v^2) + \beta u + \gamma v - v_x\Big]_x = 0,$$
$$v_t + \Big[v(u^2+v^2) + \theta u + \delta v + u_x\Big]_x = 0,$$

$\beta = a^2, \gamma = \theta = ab$, and $\delta = b^2$.

- **Key concept: Dilation invariance.**

Conservation laws, symmetries and recursion operators are invariant under the dilation (scaling) symmetry of the given PDE.

The KdV equation, $u_t + uu_x + u_{3x} = 0$, has scaling symmetry

$$(t, x, u) \to (\lambda^{-3}t, \lambda^{-1}x, \lambda^2 u).$$

u corresponds to two x-derivatives, $u \sim \mathrm{D}_x^2$. Similarly, $\mathrm{D}_t \sim \mathrm{D}_x^3$.

The *weight*, w, of a variable equals the number of x-derivatives the variable carries.

Weights are rational. Weights of dependent variables are nonnegative.

Set $w(\mathrm{D}_x) = 1$.

Due to dilation invariance: $w(u) = 2$ and $w(\mathrm{D}_t) = 3$.

Consequently, $w(x) = -1$ and $w(t) = -3$.

The *rank* of a monomial is its total weight in terms of x-derivatives.

Every monomial in the KdV equation has rank 5.
The KdV equation is *uniform in rank*.

What do we do if equations are not uniform in rank?

Extend the space of dependent variables with parameters carrying weight.

Example: the Boussinesq system

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + u_x - 3uu_x - \alpha u_{3x} &= 0, \end{aligned}$$

is not scaling invariant (u_x and u_{3x} are conflict terms).

Introduce an auxiliary parameter β

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + \beta u_x - 3uu_x - \alpha u_{3x} &= 0, \end{aligned}$$

which has scaling symmetry:

$$(x, t, u, v, \beta) \rightarrow (\lambda x, \lambda^2 t, \lambda^{-2} u, \lambda^{-3} v, \lambda^{-2} \beta).$$

- **CONSERVATION LAWS.**

$$\mathrm{D}_t \rho + \mathrm{D}_x J = 0,$$

with conserved density ρ and flux J.

Both are polynomial in $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \mathbf{u}_{3x}, \ldots$.

$$P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}$$

if J vanishes at infinity.

Conserved densities are equivalent if they differ by a D_x term.

Example: The Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{3x} = 0.$$

Conserved densities:

$$\rho_1 = u, \qquad D_t(u) + D_x(\frac{u^2}{2} + u_{2x}) = 0.$$

$$\rho_2 = u^2, \qquad D_t(u^2) + D_x(\frac{2u^3}{3} + 2uu_{2x} - {u_x}^2) = 0.$$

$$\rho_3 = u^3 - 3{u_x}^2,$$

$$D_t\left(u^3 - 3{u_x}^2\right) + D_x\left(\frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2u_{2x} + 3{u_{2x}}^2 - 6u_xu_{3x}\right) = 0.$$

$$\vdots$$

$$\rho_6 = u^6 - 60u^3{u_x}^2 - 30{u_x}^4 + 108u^2{u_{2x}}^2$$

$$+\frac{720}{7}{u_{2x}}^3 - \frac{648}{7}u{u_{3x}}^2 + \frac{216}{7}{u_{4x}}^2.$$

Time and space dependent conservation law:

$$D_t\left(tu^2 - 2xu\right)$$

$$+ \; D_x\left(\frac{2}{3}tu^3 - xu^2 + 2tuu_{2x} - tu_x^2 - 2xu_{2x} + 2u_x\right) = 0.$$

- **Algorithm for Conservation Laws of PDEs.**

1). Determine weights (scaling properties) of variables and auxiliary parameters.

2). Construct the form of the density (find monomial building blocks).

3). Determine the constant coefficients.

- **Example:** Density of rank 6 for the KdV equation.

  Step 1: **Compute the weights.**

  Require uniformity in rank. With $w(\mathrm{D}_x) = 1$:

$$w(u) + w(\mathrm{D}_t) = 2w(u) + 1 = w(u) + 3.$$

  Solve the linear system: $w(u) = 2$, $w(\mathrm{D}_t) = 3$.

  Step 2: **Determine the form of the density.**

  List all possible powers of u, up to rank 6 : $[u, u^2, u^3]$.

  Introduce x derivatives to 'complete' the rank.

  u has weight 2, introduce D_x^4.

  u^2 has weight 4, introduce D_x^2.

  u^3 has weight 6, no derivative needed.

  Apply the D_x derivatives.
  Remove terms of the form $D_x u_{px}$, or D_x up to terms kept prior in the list.

$$\begin{aligned} [u_{4x}] &\rightarrow [\,] \quad \text{empty list.} \\ [{u_x}^2, u u_{2x}] &\rightarrow [{u_x}^2] \quad \text{since } u u_{2x} = (u u_x)_x - {u_x}^2. \\ [u^3] &\rightarrow [u^3]. \end{aligned}$$

  Linearly combine the 'building blocks':

$$\rho = c_1 u^3 + c_2 {u_x}^2.$$

Step 3: Determine the coefficients c_i.

Compute $D_t\rho = 3c_1u^2u_t + 2c_2u_xu_{xt}$.

Replace u_t by $-(uu_x + u_{3x})$ and u_{xt} by $-(uu_x + u_{3x})_x$.

Integrate the result, E, with respect to x. To avoid integration by parts, apply the Euler operator (variational derivative)

$$\begin{aligned} L_u &= \sum_{k=0}^{m} (-\mathrm{D}_x)^k \frac{\partial}{\partial u_{kx}} \\ &= \frac{\partial}{\partial u} - \mathrm{D}_x(\frac{\partial}{\partial u_x}) + \mathrm{D}_x^2(\frac{\partial}{\partial u_{2x}}) + \cdots + (-1)^m\mathrm{D}_x^m(\frac{\partial}{\partial u_{mx}}). \end{aligned}$$

to E of order m.

If $L_u(E) = 0$ immediately, then E is a total x-derivative.

If $L_u(E) \neq 0$, the remaining expression must vanish identically.

$$\begin{aligned} D_t\rho &= -D_x[\frac{3}{4}c_1u^4 - (3c_1 - c_2)uu_x^2 + 3c_1u^2u_{2x} \\ &- \ c_2{u_{2x}}^2 + 2c_2u_xu_{3x}] - (3c_1 + c_2){u_x}^3. \end{aligned}$$

The non-integrable term must vanish.

So, $c_1 = -\frac{1}{3}c_2$. Set $c_2 = -3$, hence, $c_1 = 1$.

Result:

$$\rho = u^3 - 3{u_x}^2.$$

Expression $[\ldots]$ yields

$$J = \frac{3}{4}u^4 - 6u{u_x}^2 + 3u^2u_{2x} + 3{u_{2x}}^2 - 6u_xu_{3x}.$$

Example: First few densities for the Boussinesq system:

$$\begin{aligned} \rho_1 &= u, & \rho_2 &= v, \\ \rho_3 &= uv, & \rho_4 &= \beta u^2 - u^3 + v^2 + \alpha{u_x}^2. \end{aligned}$$

(then substitute $\beta = 1$).

- **Application.**

A Class of Fifth-Order Evolution Equations

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$$

where α, β, γ are nonzero parameters.

$$u \sim D_x^2.$$

Special cases:

$$\begin{array}{llll} \alpha = 30 & \beta = 20 & \gamma = 10 & \text{Lax.} \\ \alpha = 5 & \beta = 5 & \gamma = 5 & \text{Sawada}-\text{Kotera.} \\ \alpha = 20 & \beta = 25 & \gamma = 10 & \text{Kaup}-\text{Kupershmidt.} \\ \alpha = 2 & \beta = 6 & \gamma = 3 & \text{Ito.} \end{array}$$

What are the conditions for the parameters α, β and γ so that the equation admits a density of fixed rank?

- **Rank 2:**
  No condition

$$\rho = u.$$

- **Rank 4:**
  Condition: $\beta = 2\gamma$ (Lax and Ito cases)

$$\rho = u^2.$$

– **Rank 6:**

Condition:

$$10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2$$

(Lax, SK, and KK cases)

$$\rho = u^3 + \frac{15}{(-2\beta+\gamma)} {u_x}^2.$$

– **Rank 8:**

1). $\beta = 2\gamma$ (Lax and Ito cases)

$$\rho = u^4 - \frac{6\gamma}{\alpha} u {u_x}^2 + \frac{6}{\alpha} {u_{2x}}^2.$$

2). $\alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45}$ (SK, KK and Ito cases)

$$\rho = u^4 - \frac{135}{2\beta+\gamma} u {u_x}^2 + \frac{675}{(2\beta+\gamma)^2} {u_{2x}}^2.$$

– **Rank 10:**

Condition:

$$\beta = 2\gamma$$

and

$$10\alpha = 3\gamma^2$$

(Lax case)

$$\rho = u^5 - \frac{50}{\gamma} u^2 {u_x}^2 + \frac{100}{\gamma^2} u {u_{2x}}^2 - \frac{500}{7\gamma^3} {u_{3x}}^2.$$

What are the necessary conditions for the parameters α, β and γ so that the equation admits ∞ many polynomial conservation laws?

– If $\alpha = \frac{3}{10}\gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case).

– If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case).

– If $\alpha = \frac{1}{5}\gamma^2$ and $\beta = \frac{5}{2}\gamma$ then there is a sequence (with gaps!) of conserved densities (KK case).

– If

$$\alpha = -\frac{2\beta^2 - 7\beta\gamma + 4\gamma^2}{45}$$

or

$$\beta = 2\gamma$$

then there is a conserved density of rank 8.

Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case).

SUMMARY: see tables (notice the gaps!)

• GENERALIZED SYMMETRY.

$$\mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, ...)$$

with $\mathbf{G} = (G_1, G_2, ..., G_n)$ is a *symmetry* iff it leaves the PDE invariant for the replacement $\mathbf{u} \to \mathbf{u} + \epsilon\mathbf{G}$ within order ϵ. i.e.

$$\mathrm{D}_t(\mathbf{u} + \epsilon\mathbf{G}) = \mathbf{F}(\mathbf{u} + \epsilon\mathbf{G})$$

must hold up to order ϵ on the solutions of PDE.

Consequently, $\mathbf{G}$ must satisfy the linearized equation

$$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}],$$

where $\mathbf{F}'$ is the Fréchet derivative of $\mathbf{F}$, i.e.,

$$\mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial\epsilon}\mathbf{F}(\mathbf{u} + \epsilon\mathbf{G})|_{\epsilon=0}.$$

Here $\mathbf{u}$ is replaced by $\mathbf{u} + \epsilon\mathbf{G}$, and $\mathbf{u}_{nx}$ by $\mathbf{u}_{nx} + \epsilon\mathrm{D}_x^n\mathbf{G}$.

• Example.

Consider the KdV equation

$$u_t = 6uu_x + u_{3x}.$$

Generalized symmetries:

$$\begin{aligned}
G^{(1)} &= u_x, \quad G^{(2)} = 6uu_x + u_{3x}, \\
G^{(3)} &= 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}, \\
G^{(4)} &= 140u^3u_x + 70u_x^3 + 280uu_xu_{2x} + 70u^2u_{3x} \\
&\quad + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_{5x} + u_{7x}.
\end{aligned}$$

- **Algorithm for Generalized Symmetries of PDEs.**

Consider the KdV equation, $u_t = 6uu_x + u_{3x}$, with $w(u) = 2$.

Step 1: Construct the form of the symmetry.

Compute the form of the symmetry with rank 7.

List all powers in u with rank 7 or less:

$$\mathcal{L} = \{1, u, u^2, u^3\}.$$

For each monomial in $\mathcal{L}$, introduce the needed x-derivatives, so that each term exactly has rank 7. Thus,

$$\begin{aligned} \mathrm{D}_x(u^3) &= 3u^2u_x, \quad \mathrm{D}_x^3(u^2) = 6u_xu_{2x} + 2uu_{3x}, \\ \mathrm{D}_x^5(u) &= u_{5x}, \quad \mathrm{D}_x^7(1) = 0. \end{aligned}$$

Gather the resulting (non-zero) terms

$$\mathcal{R} = \{u^2u_x, u_xu_{2x}, uu_{3x}, u_{5x}\}.$$

The symmetry is a linear combination of these monomials:

$$G = c_1\, u^2u_x + c_2\, u_xu_{2x} + c_3\, uu_{3x} + c_4\, u_{5x}.$$

Step 2: Determine the unknown coefficients c_i.

Compute D_tG and use KdV to remove u_t, u_{tx}, u_{txx}, etc.
Compute the Fréchet derivative.
Equate the resulting expressions.
Group the terms:

$$\begin{aligned} &(12c_1 - 18c_2)u_x^2u_{2x} + (6c_1 - 18c_3)uu_{2x}^2 + (6c_1 - 18c_3)uu_xu_{3x} + \\ &(3c_2 - 60c_4)u_{3x}^2 + (3c_2 + 3c_3 - 90c_4)u_{2x}u_{4x} + (3c_3 - 30c_4)u_xu_{5x} \equiv 0. \end{aligned}$$

Solve the linear system:

$$\begin{aligned}\mathcal{S} &= \{12c_1 - 18c_2 = 0, 6c_1 - 18c_3 = 0, 3c_2 - 60c_4 = 0, \\ &\quad 3c_2 + 3c_3 - 90c_4 = 0, 3c_3 - 30c_4 = 0\}.\end{aligned}$$

Solution: $\frac{c_1}{30} = \frac{c_2}{20} = \frac{c_3}{10} = c_4$.
Setting $c_4 = 1$ one gets: $c_1 = 30, c_2 = 20, c_3 = 10$.
Substitute the result into the symmetry:

$$G = 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}.$$

Note that $u_t = G$ is known as the Lax equation.

- x-t **Dependent symmetries.**

The KdV equation has also symmetries which explicitly depend on x and t.
The same algorithm can be used provided the highest degree of x or t is specified.

Compute the symmetry of rank 2, that is linear in x or t.

List all monomials in u, x and t of rank 2 or less:

$$\mathcal{L} = \{1, u, x, xu, t, tu, tu^2\}.$$

For each monomial in $\mathcal{L}$, introduce enough x-derivatives, so that each term exactly has rank 2. Thus,

$$\begin{aligned}&\mathrm{D}_x(xu) = u + xu_x, \quad \mathrm{D}_x(tu^2) = 2tuu_x, \quad \mathrm{D}_x^3(tu) = tu_{3x}, \\ &\mathrm{D}_x^2(1) = \mathrm{D}_x^3(x) = \mathrm{D}_x^5(t) = 0.\end{aligned}$$

Gather the non-zero resulting terms:

$$\mathcal{R} = \{u, xu_x, tuu_x, tu_{3x}\},$$

Build the linear combination

$$G = c_1\, u + c_2\, xu_x + c_3\, tuu_x + c_4\, tu_{3x}.$$

Determine the coefficients c_1 through c_4:

$$G = \frac{2}{3}u + \frac{1}{3}xu_x + 6tuu_x + tu_{3x}.$$

Two symmetries of KdV that explicitly depend on x and t :

$$G = 1 + 6tu_x, \text{ and } G = 2u + xu_x + 3t(6uu_x + u_{3x}),$$

of rank 0 and 2, respectively.

• RECURSION OPERATORS.

A *recursion operator* for a PDE system is the linear operator $\mathbf{\Phi}$ connecting two symmetries $\mathbf{G}$ and $\hat{\mathbf{G}}$:

$$\hat{\mathbf{G}} = \mathbf{\Phi}\mathbf{G}.$$

For n-component systems, $\mathbf{\Phi}$ is an $n \times n$ matrix.

Defining equation for $\mathbf{\Phi}$:

$$\mathrm{D}_t\mathbf{\Phi} + [\mathbf{\Phi}, \mathbf{F}'(u)] = \frac{\partial \mathbf{\Phi}}{\partial t} + \mathbf{\Phi}'[\mathbf{F}] + \mathbf{\Phi} \circ \mathbf{F}'(u) - \mathbf{F}'(u) \circ \mathbf{\Phi} = 0,$$

where [,] means commutator, $\circ$ stands for composition, and $\mathbf{\Phi}'[\mathbf{F}]$ is the variational derivative of $\mathbf{\Phi}$.

• Example.

The recursion operator for the KdV equation (has rank 2)

$$\Phi = \mathrm{D}_x^2 + 2u + 2\mathrm{D}_x u \mathrm{D}_x^{-1} = \mathrm{D}_x^2 + 4u + 2u_x \mathrm{D}_x^{-1},$$

where D_x^{-1} is the integration operator.

For example

$$\begin{aligned}
\Phi u_x &= (\mathrm{D}_x^2 + 2u + 2\mathrm{D}_x u \mathrm{D}_x^{-1}) u_x = 6uu_x + u_{3x}, \\
\Phi(6uu_x + u_{3x}) &= (\mathrm{D}_x^2 + 2u + 2\mathrm{D}_x u \mathrm{D}_x^{-1})(6uu_x + u_{3x}) \\
&= 30u^2 u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}.
\end{aligned}$$

- **Key Observations.**

  The terms in the recursion operator are monomials in $\mathrm{D}_x, \mathrm{D}_x^{-1}, u, u_x, \ldots$

  Recursion operators split naturally in $\Phi = \Phi_0 + \Phi_1$.

  Φ_0 is a differential operator (no D_x^{-1} terms).
  Φ_1 is an integral operator (with D_x^{-1} terms).

  Application of Φ to a symmetry should not leave any integrals.

  For instance, for the KdV equation:

  $\mathrm{D}_x^{-1}(6uu_x + u_{3x}) = 3u^2 + u_{2x}$ is polynomial.

  Use the conserved densities: $\rho^{(1)} = u, \rho^{(2)} = u^2$, $\rho^{(3)} = u^3 - \frac{1}{2}u_x^2$

  $$\begin{aligned}
  &\mathrm{D}_t\rho^{(1)} = \mathrm{D}_t u = u_t = -\mathrm{D}_x J^{(1)},\\
  &\mathrm{D}_t\rho^{(2)} = \mathrm{D}_t u^2 = 2uu_t = -\mathrm{D}_x J^{(2)}, \quad \text{and}\\
  &\mathrm{D}_t\rho^{(3)} = \mathrm{D}_t(u^3 - \frac{1}{2}u_x^2) = \rho^{(3)\prime}(u)[u_t] = (3u^2 - u_x\mathrm{D}_x)u_t = -\mathrm{D}_x J^{(3)},
  \end{aligned}$$

  for polynomial $J^{(i)}$, $i = 1, 2, 3$.

  So, application of D_x^{-1}, or $\mathrm{D}_x^{-1}u$, or $\mathrm{D}_x^{-1}(3u^2 - u_x\mathrm{D}_x)$
  to $6uu_x + u_{3x}$ leads to a polynomial result.

- **Algorithm for Recursion Operators of PDEs.**

  Step 1: **Determine the rank of the recursion operator.**

  Recall: symmetries for the KdV equation, $u_t = 6uu_x + u_{3x}$, are

  $$\begin{aligned}
  &G^{(1)} = u_x, \qquad G^{(2)} = 6uu_x + u_{3x},\\
  &G^{(3)} = 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}.
  \end{aligned}$$

  Hence,

  $R = \text{rank } \Phi = \text{rank } G^{(3)} - \text{rank } G^{(2)} = \text{rank } G^{(2)} - \text{rank } G^{(1)} = 2.$

Step 2: **Construct the form of the recursion operator.**

(i) Determine the pieces of operator Φ_0

List all permutations of type $\mathrm{D}^j u^k$ of rank R, with j and k nonnegative integers.

$$\mathcal{L} = \{\mathrm{D}^2, u\}.$$

(ii) Determine the pieces of operator Φ_1

Combine the symmetries $G^{(j)}$ with D^{-1} and $\rho^{(k)\prime}(u)$, so that every term in

$$\Phi_1 = \sum_j \sum_k G^{(j)} \mathrm{D}^{-1} \rho^{(k)\prime}(u)$$

has rank $\Phi_1 = R$.
The indices j and k are taken so that

$$\text{rank } (G^{(j)}) + \text{rank } (\rho^{(k)\prime}(u)) - 1 = R.$$

List such terms:

$$\mathcal{M} = \{u_x \mathrm{D}^{-1}\}.$$

(iii) Build the operator Φ

Linearly combine the term in

$$\mathcal{R} = \mathcal{L} \bigcup \mathcal{M} = \{\mathrm{D}^2, u, u_x \mathrm{D}^{-1}\}.$$

to get

$$\Phi = c_1 \, \mathrm{D}^2 + c_2 \, u + c_3 \, u_x \mathrm{D}^{-1}.$$

Step 3: **Determine the unknown coefficients.**

Require that

$$\Phi G^{(k)} = G^{(k+1)}, \quad k = 1, 2, 3, ...$$

Solve the linear system:

$\mathcal{S} = \{c_1 - 1 = 0, 18c_1 + c_3 - 20 = 0, 6c_1 + c_2 - 10 = 0, 2c_2 + c_3 - 10 = 0\}$,

Solution: $c_1 = 1, c_2 = 4$, and $c_3 = 2$. So,

$$\Phi = \mathrm{D}^2 + 4\,u + 2\,u_x \mathrm{D}^{-1}.$$

Examples.

The SK equation:

$$u_t = 5u^2 u_x + 5u_x u_{2x} + 5uu_{3x} + u_{5x}.$$

Recursion operator:

$$\begin{aligned}\Phi \;=\;& \mathrm{D}^6 + 3u\mathrm{D}^4 - 3\mathrm{D}u\mathrm{D}^3 + 11\mathrm{D}^2u\mathrm{D}^2 - 10\mathrm{D}^3u\mathrm{D} + 5\mathrm{D}^4u \\ +\;& 12u^2\mathrm{D}^2 - 19u\mathrm{D}u\mathrm{D} + 8u\mathrm{D}^2u + 8\mathrm{D}u\mathrm{D}u + 4u^3 \\ +\;& u_x\mathrm{D}^{-1}(u^2 - 2u_x\mathrm{D}) + G^{(2)}\mathrm{D}^{-1},\end{aligned}$$

with $G^{(2)} = 5u^2 u_x + 5u_x u_{2x} + 5uu_{3x} + u_{5x}$.

For the vector nonlinear Schrödinger system:

$$\begin{aligned}u_t + \Big[u(u^2 + v^2) + \beta u + \gamma v - v_x\Big]_x &= 0,\\ v_t + \Big[v(u^2 + v^2) + \theta u + \delta v + u_x\Big]_x &= 0.\end{aligned}$$

Recursion operator:

$$\mathbf{\Phi} = \begin{pmatrix} \beta - \delta + 2u^2 + 2u_x\mathrm{D}^{-1}u & \theta + 2uv - \mathrm{D} + 2u_x\mathrm{D}^{-1}v \\ \theta + 2uv + \mathrm{D} + 2v_x\mathrm{D}^{-1}u & 2v^2 + 2v_x\mathrm{D}^{-1}v \end{pmatrix}.$$

PART II: Differential-difference (lattice) Equations

- **Systems of lattices equations**

  Consider the system of lattice equations, continuous in time, discretized in (one dimensional) space

$$\dot{\mathbf{u}}_n = \mathbf{F}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$$

  where $\mathbf{u}_n$ and $\mathbf{F}$ are vector dynamical variables.

  $\mathbf{F}$ is polynomial with constant coefficients.

  No restrictions on the level of the shifts or the degree of nonlinearity.

- **CONSERVATION LAW**:

$$\dot{\rho}_n = J_n - J_{n+1}$$

  with density ρ_n and flux J_n.

  Both are polynomials in $\mathbf{u}_n$ and its shifts.

$$\frac{\mathrm{d}}{\mathrm{dt}}(\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1})$$

  if J_n is bounded for all n.

  Subject to suitable boundary or periodicity conditions

$$\sum_n \rho_n = \text{constant}.$$

- **Example.**

Consider the one-dimensional Toda lattice

$$\ddot{y}_n = \exp{(y_{n-1} - y_n)} - \exp{(y_n - y_{n+1})}$$

y_n is the displacement from equilibrium of the nth particle with unit mass under an exponential decaying interaction force between nearest neighbors.

Change of variables:

$$u_n = \dot{y}_n, \qquad v_n = \exp{(y_n - y_{n+1})}$$

yields

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}).$$

Toda system is completely integrable.

The first two density-flux pairs (computed by hand):

$$\rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1}, \quad \text{and} \quad \rho_n^{(2)} = \tfrac{1}{2}u_n^2 + v_n, \quad J_n^{(2)} = u_n v_{n-1}.$$

- **Key concept: Dilation invariance.**

The Toda system as well as the conservation laws and symmetries are invariant under the dilation symmetry

$$(t, u_n, v_n) \rightarrow (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n).$$

Thus, u_n corresponds to one t-derivative: $u_n \sim \frac{\mathrm{d}}{\mathrm{dt}}$.

Similarly, $v_n \sim \frac{\mathrm{d}^2}{\mathrm{dt}^2}$.

Weight, w, of variables are defined in terms of t-derivatives.
Set $w(\frac{\mathrm{d}}{\mathrm{dt}}) = 1$.

Weights of dependent variables are nonnegative, rational, and independent of n.

Due to dilation invariance: $w(u_n) = 1$ and $w(v_n) = 2$.

The *rank* of a monomial is its total weight in terms of t-derivatives.

Require uniformity in rank for each equation to compute the weights: (solve the linear system):

$$w(u_n) + 1 = w(v_n), \quad w(v_n) + 1 = w(u_n) + w(v_n),$$

Solving the linear system yields $w(u_n) = 1$, $w(v_n) = 2$.

- **Equivalence Criterion.**

Define D *shift-down* operator, and U *shift-up* operator, on the set of all monomials in $\mathbf{u}_n$ and their shifts.

For a monomial m :

$$Dm = m|_{n \to n-1}, \quad \text{and} \quad Um = m|_{n \to n+1}.$$

For example

$$Du_{n+2}v_n = u_{n+1}v_{n-1}, \qquad Uu_{n-2}v_{n-1} = u_{n-1}v_n.$$

Compositions of D and U define an *equivalence relation*. All shifted monomials are *equivalent*.

For example

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}.$$

Equivalence criterion:

Two monomials m_1 and m_2 are equivalent, $m_1 \equiv m_2$, if

$$m_1 \; = \; m_2 + [M_n - M_{n+1}]$$

for some polynomial M_n.

For example, $u_{n-2}u_n \; \equiv \; u_{n-1}u_{n+1}$ since

$u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}]$.

Main representative of an equivalence class is the monomial with label n on u (or v).

For example, $u_n u_{n+2}$ is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3}$, etc.

Use lexicographical ordering to resolve conflicts.

For example, $u_n v_{n+2}$ (not $u_{n-2}v_n$) is the main representative of the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4}$, etc.

- **Steps of the Algorithm for Lattices.**

Three-step algorithm to find conserved densities:

1). Determine the weights.

2). Construct the form of density.

3). Determine the coefficients.

Example: Density of rank 3 or the Toda lattice,

$$\dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}).$$

Step 1: **Compute the weights.**

Here $w(u_n) = 1$ and $w(v_n) = 2$.

Step 2: **Construct the form of the density.**

List all monomials in u_n and v_n of rank 3 or less:

$$\mathcal{G} = \{u_n^3, u_n^2, u_n v_n, u_n, v_n\}.$$

For each monomial in $\mathcal{G}$, introduce enough t-derivatives to obtain weight 3. Use the lattice to remove $\dot{u}_n$ and $\dot{v}_n$:

$$\frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n^3) = u_n^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,$$
$$\frac{\mathrm{d}}{\mathrm{dt}}(u_n^2) = 2u_n v_{n-1} - 2u_n v_n,$$
$$\frac{\mathrm{d}}{\mathrm{dt}}(v_n) = u_n v_n - u_{n+1} v_n,$$
$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n.$$

Gather the resulting terms in a set

$$\mathcal{H} = \{u_n^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n\}.$$

Replace members in the same equivalence class by their main representatives .

For example, $u_n v_{n-1} \equiv u_{n+1} v_n$ are replaced by $u_n v_{n-1}$.

Linearly combine the monomials in

$$\mathcal{I} = \{u_n^3, u_n v_{n-1}, u_n v_n\}$$

to obtain

$$\rho_n = c_1\, u_n^3 + c_2\, u_n v_{n-1} + c_3\, u_n v_n.$$

Step 3: **Determine the coefficients.**

Require that $\dot{\rho}_n = J_n - J_{n+1}$, holds.

Compute $\dot{\rho}_n$ and use the lattice to remove $\dot{u}_n$ and $\dot{v}_n$.

Group the terms

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n + (c_3 - c_2)v_{n-1}v_n \\ &\quad + c_2 u_{n-1}u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2.\end{aligned}$$

Use the equivalence criterion to modify $\dot{\rho}_n$.

Replace $u_{n-1}u_n v_{n-1}$ by $u_n u_{n+1} v_n + [u_{n-1}u_n v_{n-1} - u_n u_{n+1} v_n]$.
Introduce the main representatives. Thus

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n \\ &\quad + (c_3 - c_2)v_n v_{n+1} + [(c_3 - c_2)v_{n-1}v_n - (c_3 - c_2)v_n v_{n+1}] \\ &\quad + c_2 u_n u_{n+1} v_n + [c_2 u_{n-1}u_n v_{n-1} - c_2 u_n u_{n+1} v_n] \\ &\quad + c_2 v_n^2 + [c_2 v_{n-1}^2 - c_2 v_n^2] - c_3 u_n u_{n+1} v_n - c_3 v_n^2.\end{aligned}$$

Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom.

Rearrange the terms to match the pattern $[J_n - J_{n+1}]$.

Hence

$$\begin{aligned}\dot{\rho}_n &= (3c_1 - c_2)u_n^2 v_{n-1} + (c_3 - 3c_1)u_n^2 v_n \\ &\quad + (c_3 - c_2)v_n v_{n+1} + (c_2 - c_3)u_n u_{n+1} v_n + (c_2 - c_3)v_n^2 \\ &\quad + [\{(c_3 - c_2)v_{n-1}v_n + c_2 u_{n-1}u_n v_{n-1} + c_2 v_{n-1}^2\} \\ &\quad - \{(c_3 - c_2)v_n v_{n+1} + c_2 u_n u_{n+1} v_n + c_2 v_n^2\}].\end{aligned}$$

The terms inside the square brackets determine:

$$J_n = (c_3 - c_2)v_{n-1}v_n + c_2 u_{n-1}u_n v_{n-1} + c_2 v_{n-1}^2.$$

The terms outside the square brackets must vanish, thus

$$\mathcal{S} = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}.$$

The solution is $3c_1 = c_2 = c_3$, so choose $c_1 = \frac{1}{3}$, and $c_2 = c_3 = 1$:

$$\rho_n = \tfrac{1}{3}\, u_n^3 + u_n(v_{n-1} + v_n), \qquad J_n = u_{n-1}u_n v_{n-1} + v_{n-1}^2.$$

Analogously, conserved densities of rank ≤ 5:

$$\rho_n^{(1)} = u_n \qquad \rho_n^{(2)} = \tfrac{1}{2}{u_n}^2 + v_n$$

$$\rho_n^{(3)} = \tfrac{1}{3}{u_n}^3 + u_n(v_{n-1} + v_n)$$

$$\rho_n^{(4)} = \tfrac{1}{4}{u_n}^4 + {u_n}^2(v_{n-1} + v_n) + u_n u_{n+1} v_n + \tfrac{1}{2}{v_n}^2 + v_n v_{n+1}$$

$$\begin{aligned}\rho_n^{(5)} &= \tfrac{1}{5}{u_n}^5 + {u_n}^3(v_{n-1} + v_n) + u_n u_{n+1} v_n(u_n + u_{n+1}) \\ &\quad + u_n v_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_n v_n(v_{n-1} + v_n + v_{n+1}).\end{aligned}$$

• GENERALIZED SYMMETRIES

A vector function $\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ is a *symmetry* if the infinitesimal transformation $\mathbf{u}_n \to \mathbf{u}_n + \epsilon\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ leaves the lattice system invariant within order ϵ.
Consequently, $\mathbf{G}$ must satisfy the linearized equation

$$\mathrm{D}_t\mathbf{G} = \mathbf{F}'(\mathbf{u}_n)[\mathbf{G}],$$

where $\mathbf{F}'$ is the Fréchet derivative of $\mathbf{F}$, i.e.,

$$\mathbf{F}'(\mathbf{u}_n)[\mathbf{G}] = \frac{\partial}{\partial \epsilon}\mathbf{F}(\mathbf{u}_n + \epsilon\mathbf{G})|_{\epsilon=0}.$$

Here, $\mathbf{u}_n \to \mathbf{u}_n + \epsilon\mathbf{G}(..., \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, ...)$ means that $\mathbf{u}_{n+k}$ is replaced by $\mathbf{u}_{n+k} + \epsilon\mathbf{G}_{|n\to n+k}$.

• Example

Consider the Toda lattice

$$\dot{u}_n = v_{n-1} - v_n, \qquad \dot{v}_n = v_n(u_n - u_{n+1}).$$

Higher-order symmetry of rank $(3, 4)$:

$$\begin{aligned} G_1 &= v_n(u_n + u_{n+1}) - v_{n-1}(u_{n-1} + u_n), \\ G_2 &= v_n(u_{n+1}^2 - u_n^2) + v_n(v_{n+1} - v_{n-1}). \end{aligned}$$

- **Algorithm for Generalized Symmetries of DDEs.**

Consider the Toda system with $w(u_n) = 1$ and $w(v_n) = 2$. Compute the form of the symmetry of ranks $(3, 4)$, i.e. the first component of the symmetry has rank 3, the second rank 4.

Step 1: **Construct the form of the symmetry.**

List all monomials in u_n and v_n of rank 3 or less:

$$\mathcal{L}_1 = \{u_n^3, u_n^2, u_n v_n, u_n, v_n\},$$

and of rank 4 or less:

$$\mathcal{L}_2 = \{u_n^4, u_n^3, u_n^2 v_n, u_n^2, u_n v_n, u_n, v_n^2, v_n\}.$$

For each monomial in $\mathcal{L}_1$ and $\mathcal{L}_2$, introduce enough t-derivatives, so that each term exactly has rank 3 and 4, respectively.

Using the lattice equations, for the monomials in $\mathcal{L}_1$:

$$\begin{aligned}
&\frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n^3) = u_n^3, \qquad \frac{\mathrm{d}^0}{\mathrm{dt}^0}(u_n v_n) = u_n v_n,\\
&\frac{\mathrm{d}}{\mathrm{dt}}(u_n^2) = 2u_n \dot{u}_n = 2u_n v_{n-1} - 2u_n v_n,\\
&\frac{\mathrm{d}}{\mathrm{dt}}(v_n) = \dot{v}_n = u_n v_n - u_{n+1} v_n,\\
&\frac{\mathrm{d}^2}{\mathrm{dt}^2}(u_n) = \frac{\mathrm{d}}{\mathrm{dt}}(\dot{u}_n) = \frac{\mathrm{d}}{\mathrm{dt}}(v_{n-1} - v_n)\\
&\quad = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n.
\end{aligned}$$

Gather the resulting terms:

$$\mathcal{R}_1 = \{u_n^3, u_{n-1} v_{n-1}, u_n v_{n-1}, u_n v_n, u_{n+1} v_n\}.$$

$$\begin{aligned}\mathcal{R}_2 &= \{u_n^4, u_{n-1}^2 v_{n-1}, u_{n-1}u_n v_{n-1}, u_n^2 v_{n-1}, v_{n-2}v_{n-1}, v_{n-1}^2, u_n^2 v_n, \\ &\quad u_n u_{n+1} v_n, u_{n+1}^2 v_n, v_{n-1}v_n, v_n^2, v_n v_{n+1}\}.\end{aligned}$$

Linearly combine the monomials in $\mathcal{R}_1$ and $\mathcal{R}_2$

$$\begin{aligned}G_1 &= c_1\, u_n^3 + c_2\, u_{n-1}v_{n-1} + c_3\, u_n v_{n-1} + c_4\, u_n v_n + c_5\, u_{n+1}v_n, \\ G_2 &= c_6\, u_n^4 + c_7\, u_{n-1}^2 v_{n-1} + c_8\, u_{n-1}u_n v_{n-1} + c_9\, u_n^2 v_{n-1} \\ &\quad + c_{10}\, v_{n-2}v_{n-1} + c_{11}\, v_{n-1}^2 + c_{12}\, u_n^2 v_n + c_{13}\, u_n u_{n+1} v_n \\ &\quad + c_{14}\, u_{n+1}^2 v_n + c_{15}\, v_{n-1}v_n + c_{16}\, v_n^2 + c_{17}\, v_n v_{n+1}.\end{aligned}$$

Step 2: Determine the unknown coefficients.

Require that the symmetry condition holds.

Solution:

$$\begin{gathered}c_1 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0, \\ -c_2 = -c_3 = c_4 = c_5 = -c_{12} = c_{14} = -c_{15} = c_{17}.\end{gathered}$$

Therefore, with $c_{17} = 1$, the symmetry of rank $(3, 4)$ is:

$$\begin{aligned}G_1 &= u_n v_n - u_{n-1}v_{n-1} + u_{n+1}v_n - u_n v_{n-1}, \\ G_2 &= u_{n+1}^2 v_n - u_n^2 v_n + v_n v_{n+1} - v_{n-1}v_n.\end{aligned}$$

Analogously, the symmetry of rank $(4, 5)$ reads

$$\begin{aligned}G_1 &= u_n^2 v_n + u_n u_{n+1} v_n + u_{n+1}^2 v_n + v_n^2 + v_n v_{n+1} - u_{n-1}^2 v_{n-1} \\ &\quad - u_{n-1}u_n v_{n-1} - u_n^2 v_{n-1} - v_{n-2}v_{n-1} - v_{n-1}^2, \\ G_2 &= u_{n+1}v_n^2 + 2u_{n+1}v_n v_{n+1} + u_{n+2}v_n v_{n+1} - u_n^3 v_n + u_{n+1}^3 v_n \\ &\quad - u_{n-1}v_{n-1}v_n - 2u_n v_{n-1}v_n - u_n v_n^2.\end{aligned}$$

- **Example: Nonlinear Schrödinger (NLS) equation.**

Ablowitz and Ladik discretization of the NLS equation:

$$i\,\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n^* u_n (u_{n+1} + u_{n-1}).$$

u_n^* is the complex conjugate of u_n.

Treat u_n and $v_n = u_n^*$ as independent variables and add the complex conjugate equation. Absorb i in the scale on t :

$$\begin{aligned} \dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}), \\ \dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

Since $v_n = u_n^*$, $w(v_n) = w(u_n)$.

No uniformity in rank! Introduce an auxiliary parameter α with weight.

$$\begin{aligned} \dot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}), \\ \dot{v}_n &= -\alpha(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \end{aligned}$$

Uniformity in rank leads to

$$\begin{aligned} w(u_n) + 1 &= w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n), \\ w(v_n) + 1 &= w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n). \end{aligned}$$

which yields

$$w(u_n) = w(v_n) = \frac{1}{2}, w(\alpha) = 1.$$

Uniformity in rank is essential for steps 1 and 2.
After Step 2, set $\alpha = 1$. Step 3 leads to the result:

$$\rho_n^{(1)} = c_1 u_n v_{n-1} + c_2 u_n v_{n+1}, \quad \text{etc.}$$

PART III: Software

- **Scope and Limitations of Algorithms.**
  - Systems of evolution equations or lattice equations must be polynomial in dependent variables. No *explicitly* dependencies on the independent variables.
  - Only one space variable (continuous or discretized) is allowed.
  - Program only computes polynomial conservation laws and generalized symmetries (no recursion operators yet).
  - Program computes conservation laws and symmetries that explicitly depend on the independent variables, if the highest degree is specified.
  - No limit on the number of equations in the system. In practice: time and memory constraints.
  - Input systems may have (nonzero) parameters. Program computes the compatibility conditions for parameters such that conservation laws and symmetries (of a given rank) exist.
  - Systems can also have parameters with (unknown) weight. This allows one to test evolution and lattice equations of non-uniform rank.
  - For systems where one or more of the weights is free, the program prompts the user for info.
  - Fractional weights and ranks are permitted.
  - Complex dependent variables are allowed.
  - PDEs and lattice equations must be of first-order in t.

- **Conclusions and Future Research**
  - Implement the recursion operator algorithm for PDEs.
  - Design an algorithm for recursion operators of DDEs.
  - Improve software, compare with other packages.
  - Add tools for parameter analysis (Gröbner basis).
  - Generalization towards broader classes of equations (e.g. u_{xt}).
  - Generalization towards more space variables (e.g. Kadomtsev-Petviashvili equation).
  - Conservation laws with time and space dependent coefficients.
  - Conservation laws with n dependent coefficients.
  - Exploit other symmetries in the hope to find conserved densities. of non-polynomial form
  - Application: test models for integrability.
  - Application: study of classes of nonlinear PDEs or DDEs.
  - Compute constants of motion for dynamical systems (e.g. Lorenz and Hénon-Heiles systems)

- **Implementation in Mathematica – Software**
  - Ü. Göktaş and W. Hereman, The software package *InvariantsSymmetries.m* and the related files are available at http://www.mathsource.com/cgi-bin/msitem?0208-932. *MathSource* is an electronic library of *Mathematica* material.
  - Software: available via FTP, ftp site *mines.edu*
    in

    pub/papers/math_cs_dept/software/condens
    pub/papers/math_cs_dept/software/diffdens

    or via the Internet

    URL: http://www.mines.edu/fs_home/whereman/

- **Publications**

1). Ü. Göktaş and W. Hereman, Symbolic computation of conserved densities for systems of nonlinear evolution equations, J. Symbolic Computation, 24 (1997) 591–621.

2). Ü. Göktaş, W. Hereman, and G. Erdmann, Computation of conserved densities for systems of nonlinear differential-difference equations, Phys. Lett. A, 236 (1997) 30–38.

3). Ü. Göktaş and W. Hereman, Computation of conserved densities for nonlinear lattices, Physica D, 123 (1998) 425–436.

4). Ü. Göktaş and W. Hereman, Algorithmic computation of higher-order symmetries for nonlinear evolution and lattice equations, Advances in Computational Mathematics 11 (1999), 55-80.

5). W. Hereman and Ü. Göktaş, Integrability Tests for Nonlinear Evolution Equations. In: *Computer Algebra Systems: A Practical Guide*, Ed.: M. Wester, Wiley and Sons, New York (1999) Chapter 12, pp. 211-232.

6). W. Hereman, Ü. Göktaş, M. Colagrosso, and A. Miller, Algorithmic integrability tests for nonlinear differential and lattice equations, Computer Physics Communications 115 (1998) 428–446.