Gardner equation for solitary waves in multispecies plasmas

Frank Verheest *(Universiteit Gent, Belgium)* & **Willy Hereman** *(Colorado School of Mines, USA)*

Introduction and goal

Gardner equation is delicate: coefficient B of quadratic nonlinearity should be very small (close to zero), whereas coefficient C of cubic term should be finite

Not all plasma compositions can support this (here: specific dusty plasma model)

Often Sagdeev pseudopotential analysis can be used (but not for soliton interactions)

Goal: compare Sagdeev pseudopotential analysis with reductive perturbation methods (Gardner solitons) to learn how reliable the latter methods are

Korteweg-de Vries family of nonlinear evolution equations

Three fully integrable members :

- Korteweg-de Vries (KdV) equation, with quadratic nonlinearity

$$A\frac{\partial\varphi}{\partial\tau}+B\,\varphi\frac{\partial\varphi}{\partial\xi}+\frac{\partial^3\varphi}{\partial\xi^3}=0$$

- modified KdV equation, with cubic nonlinearity

$$A\frac{\partial\varphi}{\partial\tau}+C\,\varphi^2\frac{\partial\varphi}{\partial\xi}+\frac{\partial^3\varphi}{\partial\xi^3}=0$$

- Gardner or mixed KdV equation, having both such nonlinearities

$$A\frac{\partial\varphi}{\partial\tau}+B\,\varphi\frac{\partial\varphi}{\partial\xi}+C\,\varphi^2\frac{\partial\varphi}{\partial\xi}+\frac{\partial^3\varphi}{\partial\xi^3}=0$$

  Gardner equation is appropriate combination of KdV and mKdV equations

  For consistency, C is of order unity and B should be very small, otherwise quadratic term prevails over cubic term

  Sign of B is irrelevant (can be re-scaled), but sign of C is important, because physically relevant solitons can only be obtained for $C>0$

Sagdeev pseudopotential analysis

- This alternative method (contemporary to reductive perturbation theory in plasmas!) works with nonlinearities in full, but requires all dependent variables be expressible as functions of e.g. electrostatic potential φ
- Moreover, method to generate solitary wave profiles in a co-moving frame requires numerical integration for one profile at the time
- Thus, possible interactions between such solitary waves cannot be studied, whereas for integrable evolution equations elastic scattering properties can be analyzed in theoretical framework, earning them the name "solitons"

Dusty plasma model and expressions

- Model equations :

$$\text{Boltzmann electrons}: \quad n_e=(1-f)\exp[\sigma\varphi]$$
$$\text{Cold negative dust}: \quad n_d=f\left(1+\frac{2\varphi}{V^2}\right)^{-1/2}$$
$$\text{Cairns protons}: \quad n_i=(1+\beta\varphi+\beta\varphi^2)\exp[-\varphi]$$

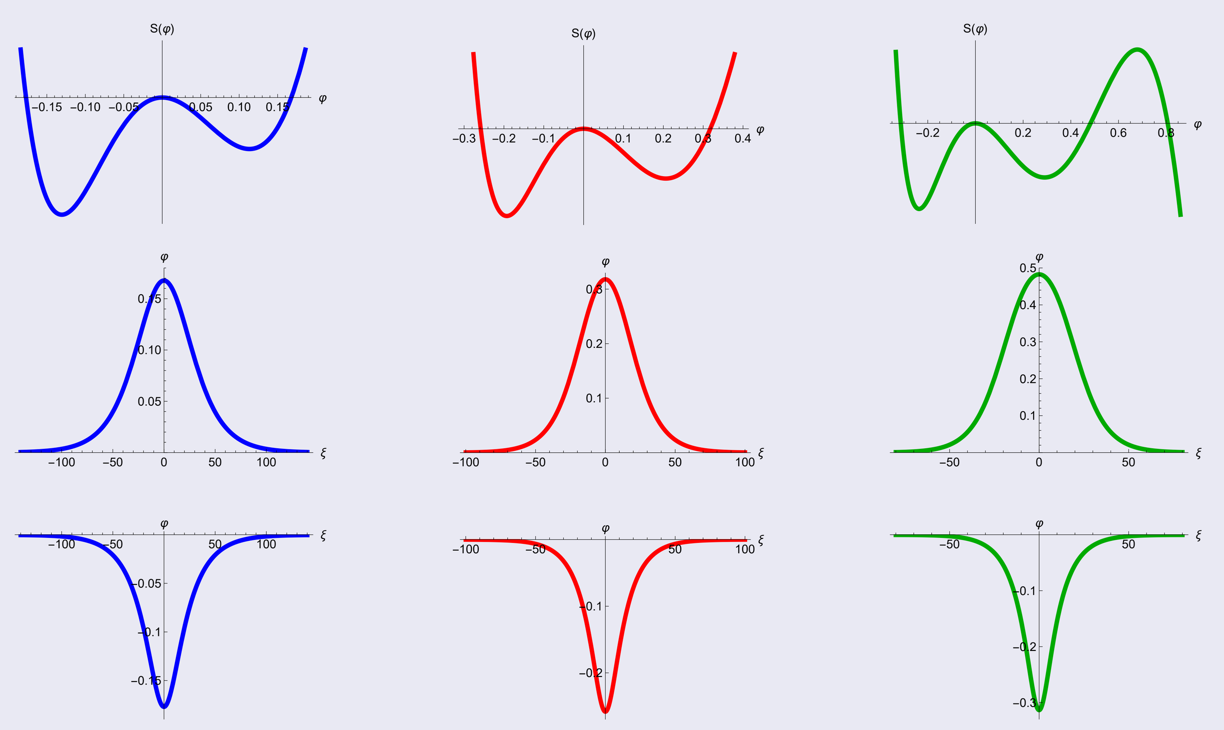
- Poisson's equation leads to conservation law in soliton frame ($\zeta=x-Vt$)

$$\frac{d^2\varphi}{d\zeta^2}=n_e+n_d-n_i \qquad \Longrightarrow \qquad \frac{1}{2}\left(\frac{d\varphi}{d\zeta}\right)^2+S(\varphi)=0$$

- Sagdeev pseudopotential is

$$S(\varphi)=\{1+3\beta-(1+3\beta+3\beta\varphi+\beta\varphi^2)\exp[-\varphi]\} +\frac{1-f}{\sigma}(1-\exp[\sigma\varphi])+fV^2\left(1-\sqrt{1+\frac{2\varphi}{V^2}}\right)$$

Sagdeev potentials, positive and negative soliton profiles



Left column is for $V=1.170$, middle for $V=1.175$ and right for $V=1.180$, all above acoustic velocity $V_a=1.16679$

Figures have been drawn for $f=0.61,\beta=4/7,\sigma=1/20$, chosen so that in Gardner equation $B=0.0116414$ and $C=0.456023$

Gardner equation and its solutions for same model parameters

- Treatment requires stretched variables $\xi=\varepsilon(x-Mt)$ and $\tau=\varepsilon^3 t$, and expansions of densities, momentum, and electrostatic potential

$$\begin{aligned} n_i&=1+\varepsilon n_{i1}+\varepsilon^2 n_{i2}+\varepsilon^3 n_{i3}+\ldots\\ n_e&=1-f+\varepsilon n_{e1}+\varepsilon^2 n_{e2}+\varepsilon^3 n_{e3}+\ldots\\ n_d&=f+\varepsilon n_{d1}+\varepsilon^2 n_{d2}+\varepsilon^3 n_{d3}+\ldots\\ u_d&=\varepsilon u_{d1}+\varepsilon^2 u_{d2}+\varepsilon^3 u_{d3}+\ldots\\ \varphi&=\varepsilon\varphi_1+\varepsilon^2\varphi_2+\varepsilon^3\varphi_3+\ldots \end{aligned}$$

- Standard reductive perturbation theory procedure yields Gardner equation

$$A\frac{\partial\varphi_1}{\partial\tau}+B\,\varphi_1\frac{\partial\varphi_1}{\partial\xi}+C\,\varphi_1^2\frac{\partial\varphi_1}{\partial\xi}+\frac{\partial^3\varphi_1}{\partial\xi^3}=0$$

  with coefficients

$$A=\frac{2f}{M_a^3}\qquad B=1-(1-f)\sigma^2-\frac{3f}{M_a^4}\qquad C=-\frac{1}{2}\left[1+3\beta+(1-f)\sigma^3-\frac{15f}{M_a^6}\right]$$

$$\text{where}\qquad M^2=M_a^2=\frac{f}{1-\beta+(1-f)\sigma}$$

- Solutions for $C>0$ are given by

$$\varphi_1(\xi,\tau)=\frac{6v}{B\left[1+\sqrt{1+\frac{6C}{B^2}v}\,\cosh(\sqrt{v}(\xi-\frac{v\tau}{A}))\right]}$$

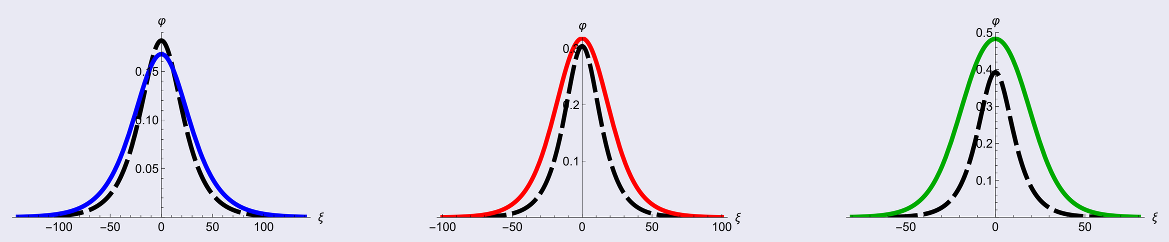
  where v is soliton velocity in (ξ,τ)-coordinates

Solutions and comparison with Sagdeev's approach

- Use Gardner equation with numerical coefficients corresponding to compositional parameters

$$0.768044\,\frac{\partial\varphi_1}{\partial\tau}+0.0116414\,\varphi_1\frac{\partial\varphi_1}{\partial\xi}+0.456023\,\varphi_1^2\frac{\partial\varphi_1}{\partial\xi}+\frac{\partial^3\varphi_1}{\partial\xi^3}=0$$

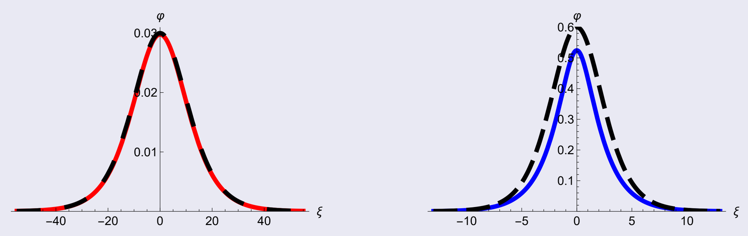
- Analytic solutions are depicted graphically (dashed lines) and compared to Sagdeev numerical counterparts for $V=1.170$, $V=1.175$ and $V=1.180$



- Observations :
  - For V slightly above acoustic speed, Gardner soliton is slightly higher, but as V increases the fully nonlinear solitons prevail
  - Amplitudes increase when V or v increase while their widths decrease, where link is $v=V-V_a=V-M_a$
  - Overall, Sagdeev profiles are taller *and* wider than Gardner solitons

What about simple KdV ion-acoustic solitons?

Same exercise, done for simplest ion-acoustic solitons based on KdV model, indicates that KdV solitons (dashed lines) are taller than fully nonlinear Sagdeev ones, and no longer match from $v=0.03$ onwards (left: $v=0.01$, right: $v=0.20$)



Conclusions

- Comparison between Gardner equation and Sagdeev pseudopotential analysis for same dusty plasma model shows interesting analogies and differences
- For $\varphi\leqslant 0.3$, Gardner solutions are reliable; for $\varphi\geqslant 0.3$, Sagdeev profiles dominate both in amplitudes as in widths, and nonlinear effects can no longer be limited to cubic terms
- Comparison with simple ion-acoustic waves leads to different conclusions: for $v>0.3$, Gardner solitons become too tall
- Results seem model-dependent and are also affected by conditions ($B\ll C$, with $C>0$ and finite) for Gardner model
- Sagdeev analysis indicates for same model parameters and velocities positive and negative solutions, but with different amplitudes. Negative solitons in Gardner analysis do not match amplitudes or boundary conditions